Relating threshold tolerance graphs to other graph classes

Tiziana Calamoneri[1]* and Blerina Sinaimeri[2]

[1] Sapienza University of Rome
via Salaria 113, 00198 Roma, Italy.
[2] INRIA and Université de Lyon
Université Lyon 1, LBBE, CNRS UMR558, France.

Abstract. A graph $G = (V, E)$ is a *threshold tolerance* if it is possible to associate weights and tolerances with each node of G so that two nodes are adjacent exactly when the sum of their weights exceeds either one of their tolerances. Threshold tolerance graphs are a special case of the well-known class of tolerance graphs and generalize the class of threshold graphs which are also extensively studied in literature. In this note we relate the threshold tolerance graphs with other important graph classes. In particular we show that threshold tolerance graphs are a proper subclass of co-strongly chordal graphs and strictly include the class of co-interval graphs. To this purpose, we exploit the relation with another graph class, *min leaf power graphs (mLPGs).*

Keywords: threshold tolerance graphs, strongly chordal graphs, leaf power graphs, min leaf power graphs.

1 Introduction

In the literature, there exist hundreds of graph classes (for an idea of the variety and the extent of this, see [2]), each one introduced for a different reason, so that some of them have been proven to be in fact the same class only in a second moment. It is the case of threshold graphs, that have been introduced many times with different names and different definitions (the interested reader can refer to [13]). Threshold graphs play an important role in graph theory and they model constraints in many combinatorial optimization problems [8, 12, 17]. In this paper we consider one of their generalizations, namely threshold tolerance graphs.

A graph $G = (V, E)$ is a *threshold tolerance* graph if it is possible to associate weights and tolerances with each node of G so that two nodes are adjacent exactly when the sum of their weights exceeds either of their tolerances. More formally, there are positive real-valued functions, weights g and tolerances t on V such that $\{x, y\} \in E$ if and only if $g(x) + g(y) \geq \min(t(x), t(y))$. In the following we denote by *TT* the class

* This work was supported in part by the Italian Ministry of Education, University, and Research (MIUR) under PRIN 2012C4E3KT national research project "AMANDA' - Algorithmics for MAssive and Networked DAta"

of threshold tolerance graphs and indicate with $G = (V, E, g, t)$ a graph in this class. Threshold tolerance graphs have been introduced in [14] as a generalization of threshold graphs (we refer to this class by *Thr*). Indeed, threshold graphs constitute a proper subclass, and can be obtained by defining the tolerance function as a constant [15]. Specifically, a graph $G = (V, E)$ is a *threshold* graph if there is a real number t and for every vertex v in V there is a real weight a_v such that: $\{v, w\}$ is an edge if and only if $a_v + a_w \geq t$ ([15, 13]).

A *chord* of a cycle is an edge between two non consecutive vertices x, y of the cycle. A chord between two vertices x, y in an even cycle C is odd, when the distance in C between x and y is odd. A graph is *chordal* if every cycle of length at least 4 has a chord.

A graph is *strongly chordal* if it is chordal and every cycle of even length at least 6 has an odd chord.

Strongly chordal graphs can be also characterized in terms of excluding subgraphs. A k-sun (also known as trampoline), for $k \geq 3$, is the graph on $2k$ vertices obtained from a clique $\{c_1, \ldots, c_k\}$ on k vertices and an independent set $\{s_1, \ldots, s_k\}$ on k vertices and edge $\{s_i, c_i\}, \{s_i, c_{i+1}\}$ for all $1 \leq i < k$, and $\{s_k, c_k\}, \{s_k, c_1\}$.

A graph is strongly chordal if and only if it does not contain either a cycle on at least 4 vertices or a k-sun as an induced subgraph [10]. Strongly chordal graphs are a widely studied class of graphs that is characterized by several equivalent definitions that the interested reader can find in [2]. We will call *SC* the class of strongly chordal graphs.

A graph is *co-strongly chordal* if its complement is a strongly chordal graph.

It is known [15] that every threshold tolerance graph is co-strongly chordal but it is not known whether there exist graphs that are co-strongly chordal but not threshold tolerance; in other words, it is not known whether the inclusion is strict or not. In fact, in ISGCI (Information System on Graph Classes ands their Inclusions) [9] it is conjectured that these two classes could be possibly equal.

We provide a graph that belongs to the class of co-strongly chordal graphs but not to the class of threshold tolerance graphs, so proving that these two classes do not coincide.

A graph is a *tolerance graph* [11] if to every node v can be assigned a closed interval I_v on the real line and a tolerance t_v such that x and y are adjacent if and only if $|I_x \cap I_y| \geq \min\{t_x, t_y\}$, where $|I|$ is the length of the interval I. We will call *Tol* the class of tolerance graphs.

A graph is an *interval graph* if it has an intersection model consisting of intervals on a straight line. Clearly interval graphs are included in tolerance graphs and can be obtained by fixing a constant tolerance function. We will call *Int* the class of interval graphs.

It is known that co-*Tol* includes *TT* and that *TT* includes co-*Int*; while it can be easily derived that co-*Tol* properly includes *TT*, it is not known whether the other inclusion is strict or not (again, in ISGCI [9] it is conjectured that *TT* could be possibly equal to co-*Int*). We prove that both the inclusions are proper.

2 Preliminaries

In order to prove that *TT* is properly included in co-*SC*, we need to introduce the classes of leaf power graphs (*LPG*) and min leaf power graphs (*mLPG*).

A graph $G(V,E)$ is a *leaf power graph* [16] if there exists a tree T, a positive edge weight function w on T and a nonnegative number d_{max} such that there is an edge $\{u,v\}$ in E if and only if for their corresponding leaves in T, l_u, l_v, we have $d_{T,w}(l_u,l_v) \leq d_{max}$, where $d_{T,w}(l_u,l_v)$ is defined as the sum of the weights of the edges of T on the (unique) path between l_u and l_v. In symbols, we will write $G = LPG(T,w,d_{max})$.

A *t-caterpillar* is a tree in which all the nodes are within distance 1 of a central path, called spine, constituted of t nodes.

Although there has been a lot of work on this class of graphs (for a survey on this topic see e.g. [1]), a complete description of leaf power graphs is still unknown.

The following result is particularly relevant for our reasoning.

Fact 1 *[1] LPG is a proper subclass of SC. Furthermore, the graph in Figure 1 is a strongly chordal graph and not a leaf power graph.*

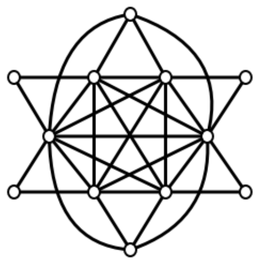


Fig. 1: A strongly chordal graph which is not in LPG [1].

The class mLPG is defined similarly to the class of leaf power graphs reversing the inequality in the definition. Formally, a graph $G = (V,E)$ is a *min leaf power graph (mLPG)* [4] if there exists a tree T, a positive edge weight function w on T and an integer d_{min} such that there is an edge (u,v) in E if and only if for their corresponding leaves in T l_u, l_v we have $d_{T,w}(l_u,l_v) \geq d_{min}$; in symbols, $G = mLPG(T,w,d_{min})$.

In [3] it is proved that $LPG \cap mLPG$ is not empty, and that neither of the classes LPG and mLPG is contained in the other. Furthermore, a number of papers deal with this class with a special focus on the intersection with LPGs (e.g. see [5–7]).

The next result will be useful in the following.

Fact 2 *[3] The class co-LPG coincides with mLPG and, vice-versa, the class co-mLPG coincides with LPG.*

The main issue related to LPG and mLPG is to prove that a certain class belongs to them by providing a constructive method that, given a graph, defines tree T, edge-weight function w and value d_{min} or d_{max}.

In the next section we will prove that threshold tolerance graphs are mLPGs and use this fact to separate the class TT from other graph classes.

3 Threshold tolerance graphs are mLPGs

Before proving the main result of this section, i.e. that threshold tolerance graphs are mLPGs, we need to demonstrate a preliminary lemma stating that, when we deal with threshold tolerance graphs, w.l.o.g. we can restrict ourselves to the case when g and t take only positive integer values.

Lemma 1. *A graph $G = (V, E)$ is a threshold tolerance if and only if there exist two functions $g, t : V \to N^+$ such that (V, E, g, t) is threshold tolerance.*

Proof. Clearly if f, g exist then by definition G is a threshold tolerance graph. Suppose now G is a threshold tolerance graph which weight and tolerance functions g and t are both defined from V to $\mathbb{R}^+$. We show that nevertheless, it is not restrictive to assume that $g, t : V \to \mathbb{Q}^+$ in view of the density of rational numbers among real numbers. So, we can assume that, for each $v \in V$, $t(v) = n_v/d_v$. Let m be the minimum common multiple of all the numbers d_v, $v \in V$. So we can express $t(v)$ as $t(v) = \frac{n_v \cdot m/d_v}{m}$ where m/d_v is an integer.

Define now the new functions g' and t' as $g'(v) = g(v) \cdot m$ and $t'(v) = t(v) \cdot m$, $v \in V$. Clearly, it holds that $g' : V \to \mathbb{Q}^+$ while $t' : V \to \mathbb{N}^+$.

In order to prove the claim, it remains to prove that g' and t' define the same graph defined by t and g. This descends from the fact that $g'(x) + g'(y) = (g(x) + g(y)) \cdot m \geq \min(t(x), t(y)) \cdot m = \min(t'(x), t'(y))$ if and only if $g(x) + g(y) \geq \min(t(x), t(y))$. □

Theorem 1. *Threshold tolerance graphs are mLPGs.*

Proof. Let $G = (V, E, g, t)$ be a threshold tolerance graph. Let $K = \max_v t(v)$. In view of Lemma 1, it is not restrictive to assume that $g : V \to \mathbb{N}^+$, so we split the nodes of G in groups $S_1, \ldots, S_K$ such that $S_i = \{v \in V(G) : t(v) = i\}$. Observe that for some values of i the set S_i can be empty. We associate to G a caterpillar T as in Figure 2.

The spine of the caterpillar is formed by K nodes, $x_1, \ldots, x_K$, and each node x_i is connected to the leaves l_v corresponding to nodes v in S_i. The weights w of the edges of T are defined as follows:

- For each edge of the spine $w(x_i, x_{i+1}) = 0.5$ for $0 \leq i \leq K - 1$.
- For each leaf l_v connected to the spine through node x_i we assign a weight $w(v, x_i) = g(v) + \frac{K - t(v)}{2}$.

We show that $G = mLPG(T, w, K)$. To this purpose consider two nodes u and v in G. By construction, in T we have that l_u is connected to $x_{t(u)}$ and l_v to $x_{t(v)}$, where $t(u)$ and $t(v)$ are not necessary distinct. Clearly, w.l.o.g we can assume $t(v) \geq t(u)$, i.e. $t(u) = \min(t(u), t(v))$. We have that

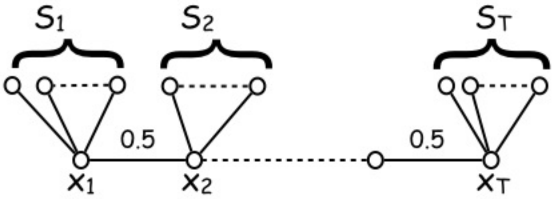


Fig. 2: The caterpillar used in the proof of Theorem 1 to prove that threshold tolerance graphs are in mLPG.

$$
\begin{aligned}
d_T(l_u, l_v) &= w(l_u, x_{t(u)}) + \frac{t(v) - t(u)}{2} + w(l_v, x_{t(v)}) \\
&= g(u) + \frac{K - t(u)}{2} + \frac{t(v) - t(u)}{2} + g(v) + \frac{K - t(v)}{2} \\
&= g(u) + g(v) + K - t(u)
\end{aligned}
$$

Clearly, $d_T(l_u, l_v) \geq K$ if and only if $g(u) + g(v) \geq t(u) = \min(t(u), t(v))$ and this proves the assertion. □

Now we are ready to prove the following Theorem.

Theorem 2. *$TT \subsetneq$ co-SC.*

Proof. Observe that according to the previous facts, TT are included in *mLPG* which in turn is strictly included in co-strongly chordal class of graphs. This proves the claim. □

(a) (b)

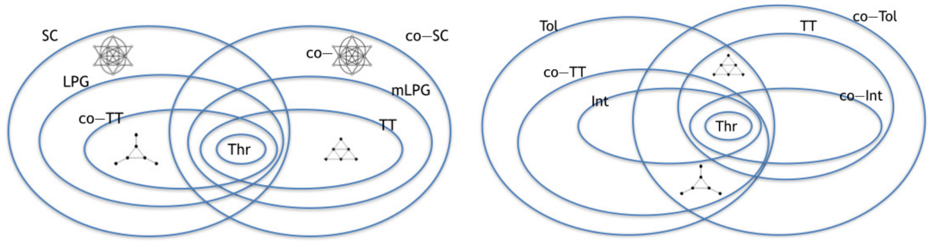
Fig. 3: (a) S_3 (b) $\bar{S_3}$

4 Separating threshold tolerance graphs from other graph classes

It is known [15] that co-$Tol \subseteq TT$. This inclusion is in fact proper; indeed, the sun of dimension 3, S_3, shown in Figure 3(a), is a tolerance graph but not a co-threshold tolerance graph [15]. It follows that its complement, $\bar{S_3}$, shown in Figure 3(b), is a co-tolerance graph but not a threshold tolerance graph, so proving co-$Tol \subset TT$.

It is easy to see that the graph S_3 belongs to the class of split antimatchings that are provably included in LPG but not in mLPG [4]. Furthermore, $\bar{S_3}$ is a co-threshold tolerance graph [15], i.e. S_3 is a threshold tolerance graph;finally, $\bar{S_3}$ is a split matching, and hence included in mLPG but not in LPG [4].

These inclusions prove the following:



(a) (b)

Fig. 4: Graphical representation of the known inclusions among the classes handled in this paper.

Theorem 3. *The classes $TT \setminus LPG$ and co-TT $\setminus$mLPG are not empty.*

Collecting these results and those reported in [4, 15] about S_3 and $\bar{S_3}$ we can finally conclude that $S_3 \in (Tol \cap TT) \setminus$ (co-$TT \cup$ co-*Int*) while $\bar{S_3} \in$ (co-$TT \cap$ co-Tol) $\setminus$ ($TT \cup$ *Int*). The results obtained are depicted in Fig. 4.

5 Conclusions and Open Problems

In this paper, we clarified the relation between some classes of graphs. In particular, we have been able to position some special graphs, in order to prove that some class inclusions are strict. In particular, we proved that threshold tolerance graphs are strictly included in co-strongly chordal graphs, so confuting a conjecture reported in [9]. In order to do this, we exploit mLPGs deducing as a side effect that threshold tolerance graphs are mLPGs.

We summarize the obtained results in the two diagrams of Figure 4, from which it naturally arises an interesting open problem: how are related tolerance graphs and leaf power graphs (and, analogously, co-tolerance and min leaf power graphs)?

References

1. A. Brandstädt, On Leaf Powers, Technical report, University of Rostock, (2010).
2. A. Brandstädt, V. B. Le, J. Spinrad, Graph classes: a survey, SIAM Monographs on discrete mathematics and applications (1999).
3. T. Calamoneri, E. Montefusco, R. Petreschi, B. Sinaimeri, Exploring Pairwise Compatibility Graphs, Theoretical Computer Science , 468 (2013) 23–36.

4. T. Calamoneri, R. Petreschi, B. Sinaimeri, On relaxing the constraints in pairwise compatibility graphs, In: Md. S. Rahman and S.-i. Nakano (Eds.), WALCOM 2012, LNCS vol. 7157, Springer, Berlin (2012) 124–135.
5. T. Calamoneri, R. Petreschi, B. Sinaimeri, On the Pairwise Compatibility Property of Some Superclasses of Threshold Graphs, Discrete Mathematics, Algorithms and Applications, 5(2) (2013).
6. T. Calamoneri, R. Petreschi, On pairwise compatibility graphs having Dilworth number two, Theoretical Computer Science 524 (2014) 34–40.
7. T. Calamoneri, R. Petreschi, On Dilworth k Graphs and Their Pairwise Compatibility. WALCOM 2014, LNCS, Springer, Berlin (2014) 213–224.
8. V. Chvátal, P. L. Hammer, Set-packing and threshold graphs, Res.Report, Comp.Sci. Dept. Univ. of Waterloo, Ontario, (1973).
9. H.N. de Ridder et al. Information System on Graph Classes and their Inclusions (ISGCI), `http://www.graphclasses.org`.
10. M. Farber, Characterizations of strongly chordal graphs, Discrete Mathematics 43 (1983) 173–189.
11. M. C. Golumbic, C. L. Monma, W. T. Trotter Jr., Tolerance graphs, Discrete Applied Mathematics, 9(2), (1984) 157–170,
12. P. B. Henderson, Y. Zalcstein, A Graph-Theoretic Characterization of the PV-chunk Class of Synchronizing Primitives. SIAM J.Comput 6(1), (1977) 88–108.
13. N.V.R. Mahadev, U.N. Peled, Threshold Graphs and Related Topics, Elsevier, (1995).
14. C.L. Monma, B. Reed, W.T. Trotter Jr., A generalization of threshold graphs with Tolerance, Congressus Numerantium 55 (1986) 187–197.
15. C.L. Monma, B. Reed, W.T. Trotter Jr., Threshold Tolerance Graphs, J. Graph Theory 12 (1988) 343–362.
16. N. Nishimura, P. Ragde, D. M. Thilikos, On graph powers for leaf-labeled trees, J. Algorithms 42 (2002) 69–108.
17. E. T. Ordman, Minimal threshold separators and memory requirements for synchronization, SIAM Journal on Computing, Vol. 18, (1989) 152–165.