Logics for Collective Reasoning

Daniele Porello

Laboratory for Applied Ontology, ISTC-CNR, Trento, Italy

Abstract. In this paper, we discuss the approach based on Social Choice Theory and Judgment Aggregation to the definition of collective reasoning. We shall make explicit the aggregative nature of the notion of collective reasoning that is defined in the Judgment Aggregation account and we shall stress that the notion of logical coherence plays a fundamental role in defining collective attitudes. Unfortunately, as several results in Judgment Aggregation show, coherence is not compatible with fair aggregation procedures. On closer inspection, the notion of coherence that is jeopardized by Judgment Aggregation is based on classical logic. In this work, we propose to revise the standard view of rationality of Judgment Aggregation by exploring the realm of non-classical logics. in particular, we will present possibility results for substructural logics. Those logics, we argue, provide a viable notion of collective reasoning.

1 Introduction

In the past decades, Social Choice Theory and Judgment Aggregation have been extensively studied in philosophy, welfare economics, and also in AI and multiagent systems, in order to provide a principled definition of the aggregation of individual attitudes into a social or collective attitude. The view of Judgment Aggregation (JA) [11] of collective attitudes is an *aggregative* view: a number of agents provides their propositional attitudes (belief, judgments, preference, etc.) to a centralized mechanism that aggregates the individual possibly divergent attitudes into a collective set of attitudes [2]. The collective outcome is supposed to satisfy a number of rationality conditions, usually expressed as constraints in classical logic. In particular, JA studies the procedures to aggregate individual attitudes that satisfy a number of normative desiderata, such as impartiality between agents and rationality of the collective outcome. By exploiting the insightful methodology of social choice theory [4], JA provides a principled way to understand collective attitudes and collective rationality, how they can be defined from a number of possibly conflicting individual attitudes, and how to provide a formal modeling of them. Unfortunately, the main formal result in Judgment Aggregation is a negative result. In particular, it is an impossibility theorem that states that there is no aggregation procedure that can guarantee at the same time both the rationality of the outcome and the fairness of the aggregation [9]. The impossibility theorem, far from being a mere mathematical curiosity, is instantiated by situations that have actually occurred in the deliberative practice of important collegial court such as the US Supreme Court [8]. Those cases of collective irrationality have been labelled *doctrinal paradoxes* and *discursive dilemmas*. A simple example is the following. Take a committee composed by three members, who have to decide whether to implement a policy B: "we should increase workers' salaries" on the grounds of A: "low salaries cause crisis" and the material implication $A \to B$: "if low salaries cause crisis, then we should increase workers' salaries". The information at issue is here represented by means of propositional logic and every agent is supposed to be rational, that is, agents respect the rules of logic. Now suppose members hold different opinions, as follows.

	A	$A \to B$	B
1	yes	yes	yes
2	no	yes	no
3	yes	no	no

Agent 1 accepts the three propositions, 2 does not accept A, therefore she rejects B, and 3 agrees on the premise A but she thinks that it is not sufficient to conclude B. If individuals vote by majority, then the collective accepts A, which is voted by 1 and 3, it accepts $A \to B$, voted by 1 and 2, but it does reject B. If we assume that rejecting a proposition is equivalent to accepting its negation, then, even if each individual opinion is logically consistent, the collective set $\{A, A \to B, \neg B\}$ is inconsistent, that is, contradictory. This toy example of collective inconsistency is a case of *discursive dilemma* that has received increasing attention in the past decade and has provided the seminal results in the formal theory of *judgment aggregation* [9,11]. Discursive dilemmas show that the problem of ascribing reasoning capabilities to groups requires a careful examination of which procedures that aggregate individual attitudes ensure (at least) consistency. That is, the definition of collective attitudes is delicate.

Consistency is a crucial feature for defining collective attitudes. One reason is that consistency is strictly related to agency [10]. By enforcing consistency, we can talk about the behavior of a *single* collective agent, who is supposed to be the bearer of collective attitudes, and distinguish it from the behavior of a number of agents, for which contradictory positions are simply expressions of possible disagreement. That is, at least a minimal form of (synchronical) consistency is viewed as a necessary condition for defining a collective agent.[1] For those reasons, an important concept in judgment aggregation is the concept of *collective rationality* that can be viewed as a preservation property: the collectivity has to be as rational as its members (i.e. if the individuals are consistent, then the collectivity has to be consistent). One way of viewing judgment aggregation impossibility results is that they show that individual rationality is not preserved by means of fair aggregation procedures [7]. For instance, discursive dilemmas show that the majority rule does not preserve consistency. Therefore, we cannot talk about a collective agent whose attitudes are defined by majority. This is a drawback, provided we want to ascribe collective agency to many real representative assemblies. A weaker notion of consistency has been used to define collective agents in [15]. This notion does not rule out group agents defined by majority, however it still admits possible (complex) inconsistencies.

Two assumptions are usually endorsed by the Judgment Aggregation model. Firstly, that individual and collective rationality can be modeled by means of a logic that is

[1] For the role of consistency in defining collective agency and for the relationship between consistency and the notion of coherence, we refer to [10] and [12].

mostly "classical". [2] Secondly, that individual and collective rationality are of the same logical type. In this work, we start questioning those two assumptions. Concerning the first, we ask whether there exist a well-behaved logic that is consistent wrt the propositions that are obtained by means of a fair aggregation procedure. In order to simplify the presentation, we focus in this paper on the majority rule, and we ask whether there is a logic that is consistent with respect to the output obtained by majority. We will carefully analyze the inference rules that are allowed at individual and collective level and, for that reason, we shall use an inferential view of logic based on proof-theory. Quite surprisingly, we will show that there is a logic that guarantees consistency for the majority rule. Concerning the second assumption, we will show that by letting individual reason by classical logic, as in the standard Judgment Aggregation setting, and by interpreting collective reasoning in substructural logic, we can provide again possibility results for consistent collective reasoning.

A strong conclusion one could draw is that the impossibility results in Judgment Aggregation are threatening only a notion of collective reasoning that is based on a classical view of logic. There do exist logics of collective reasoning for which consistency is guaranteed for any possible agenda of proposition, that means that, for those logics, we have a general possibility result.

The remainder of this paper is organized as follows. In Section 2, we present the relevant background on proof-theory and substructural logics. Section 3 analyses discursive dilemmas by investigating the reasoning steps that are responsible of collective inconsistencies. Then, we informally discuss the possible way out provided by substructural logics. Section 4 presents the formal model for discussing Judgment Aggregation in a number of non-classical logics and Section 5 presents the main possibility results. The technical results of this paper are mainly based on the model provided in [14]. Section 6 concludes and indicates possible future work.

2 Background on Sequent Calculi and Substructural Logics

Gentzen's sequent calculi [17] provides an important theory in logic that allows for investigating properties of inference rules, deduction, and reasoning. Besides providing a fine-grained tool to analyse reasoning, sequent calculus can be used to model a number of logics in an elegant way. Sequents are expressions of the form $\Gamma \vdash \Delta$, where Γ, the premises of the sequent, and Δ, the conclusions of the sequent, are made out of formulas in a given logic. The intuitive meaning of a sequent expression is that the conjunction of the formulas in Γ entails the disjunction of the formulas in Δ. A sequent calculus is then specified by two classes of rules, cf Table 1. *Structural rules* determine the structure of the sequent, for instance, they entail that in classical logic Γ and Δ are *sets* of formulas. *Logical rules* define the behavior of the logical connectives. Sequent calculus then is capable of characterizing validities in a number of logics, by defining rules that combine proofs according to the meaning of the logical connectives in a particular logic.

[2] That means that the consequence operator of those logics is standard in the sense of Tarski. Logics used by Judgment Aggregation have a standard consequence operator, including many modal logics and the "general logic" discussed in [1]. As we shall see, one crucial aspect of standard consequences is monotonicity.

A fundamental intuition concerning logic is due to the tradition of substructural logics and in particular to Jean Yves Girard's Linear Logic [5]: The structural rules of the sequent calculus determine the behavior of logical connectives. For instance, classical logic is imposed, by assuming *weakening* (W), *contraction* (C), and *exchange* (E) (cf. Table 1). Intuitively, (W) corresponds to the monotonicity of the entailment, (C) amounts to assuming that identical occurrences of a formula do not matter for deduction, and (E) entails the commutativity of conjunction and disjunction. We shall use the standard notation, e.g. $A \wedge B$ and $A \vee B$, when we intend to refer to connectives in classical logic (CL), namely, in case we assume that the structural rules hold.

By disabling one or more structural rules, we enter the realm of *substructural logics*. We shall focus here on Linear Logic [5,18] since it provides the tools that we need to analyze collective reasoning and to state our possibility results.

The motivation for investigating substructural logics is usually related to the idea of modeling a constructive notion of reasoning that accounts for the computational resources that are required by deduction. Constructivity is neglected by the principles of classical logic. For instance, contraction (C) allows for an unbounded number of duplications in proof-search. Note that Gentzen's surprising insight that shows that intuitionistic logic can be captured by simply forcing the right-hand side of the sequent to contain at most one formula can also be interpreted in terms of a substructural approach to logic. This restriction has been interpreted as a manner to disable structural rules locally [5], that is, on the right-hand side of the sequent. This is sufficient to provide a sound and complete calculus that captures intuitionistic (i.e. constructive) reasoning.

Linear Logic (LL) rejects the global validity of (W) and (C) both on the left and on the right hand side of the sequent.[3] In Table 1, we present two ways of defining logical rules: an *additive* version and a *multiplicative* version. They depend on how we handle premises in the deduction: in the case of conjunctions, additives force to share the same premises, whereas multiplicatives combine different premise. The two formulations are redundant in classical logic and that is due to structural rules: by assuming (W) and (C), additive and multiplicative versions are equivalent. If we drop weakening and contraction, additives and multiplicatives are no longer equivalent, hence we need to account for two different types of conjunctions and disjunctions. This operators are somehow invisible in classical logic, because of the structural rules.

Therefore, the rules that define logical connectives in the sequent calculus of LL have to be split into two classes. Accordingly, in LL there are two different types of conjunction, a multiplicative conjunction $\otimes$ (tensor) and an additive conjunction $\&$ (with), and two types of disjunctions, multiplicative $⅋$ (parallel) and additive $\oplus$ (plus). Implications can be defined by means of disjunctions and negations as usual. For example, LL implication is $A \multimap B \equiv \neg A ⅋ B$. Intuitively, LL captures resource bounded reasoning and non-monotonic inferences. For example, suppose the proper axiom $e \vdash c$ represents the inference "if I have one euro (e), then I buy one coffee (c)". In classical logic, one can infer by means of contraction $e \vdash e \wedge c$, namely, that I still have one euro, besides having the coffee. By dropping contraction, LL captures a form of causality: the antecedent has to be *consumed* during the inferential process.

[3] For lack of space, we cannot discuss the rule of exchange (E) here. By disabling it, we are lead to consider non-commutative logics.

Given a set of propositional atoms $\mathcal{A}$, the language of LL is defined as follows.[4]

$$\mathcal{L}_{LL} ::= \mathcal{A} \mid \neg L \mid L \otimes L \mid L \parr L \mid L \oplus L \mid L \,\&\, L$$

The sequent calculus for LL is presented in Table 1. A sequent is an expression $\Gamma \vdash \Delta$, where Γ and Δ denote, in the case of linear logic, *multisets* of occurrences of formulas.[5] LL is sound and complete wrt to its semantics, Moreover, LL enjoys cut elimination. [6].

3 An inferential analysis of discursive dilemmas

We want to discuss the reasoning steps that are involved in the famous case of doctrinal paradox emerged in the U.S. Supreme Court case *Arizona v Fulminante* [8]. We simplify the exposition by reporting the votes of three of the nine judges. Judges i_1, i_2, and i_3 have to decide whether to revise a trial on the ground of the legal doctrine that entails that a trial must be revised if and only if both the following conditions hold:

a : the confession was coerced.
b : the confession affected the outcome of the trial.

The matters that the judges had to decide can be represented by means of the agenda of propositions: $\{a, b, a \wedge b, \neg a, \neg b, \neg(a \wedge b)\}$, where negations are used to represent individuals' dissent concerning a given matter. The way the judges voted can be fairly represented by the following profile.

	a	$a \wedge b$	b	$\neg a$	$\neg(a \wedge b)$	$\neg b$
i_1	1	1	1	0	0	0
i_2	1	0	0	0	1	1
i_3	0	0	1	1	1	0
maj.	**1**	0	**1**	0	**1**	0

Individuals are assumed to be rational and accordingly each judge has a consistent set of propositions. However, by majority, the collective set $\{a, b, \neg(a \wedge b)\}$ is not consistent. The fact that $\{a, b, \neg(a \wedge b)\}$ is not consistent means (proof-theoretically) that we can

[4] We focus on the multiplicative-additive fragment of LL. Another important part of LL is given by the *exponentials*, that allow for retrieving the usual classical inferences in a controlled way. Therefore, instead of being yet another non-classical logic, Linear Logic is motivated at least as an analysis of proofs in classical and intuitionistic logic. We leave a discussion of the exponential for future work.

[5] Without W and C, also negation behaves differently. For example, the *ex falso quodlibet* principle is no longer globally valid in linear logic. For sake of simplicity of presentation, we shall use a single notation for negation. That is, linear logic has a *paraconsistent* flavor. We have decided not to discuss the case of paraconsistent logics because we are here assuming that a possiblity results has to provide a consistent model of collective reasoning. By means of paraconsistency, we would still have an inconsistent notion of collective reasoning, and we may cope with that by means of paraconsistent logics. A closer inspection of paraconsistent logics is left for future work.

actually infer a contradiction from the assumption that a, b and $\neg(a \wedge b)$ hold. We can infer the contradiction by reasoning in classical logic as follows. Assume that the proposition of the agenda that are elected by majority can be used as non-logical axioms (as assumptions) of the deduction.

$$\cfrac{\cfrac{\cfrac{}{\vdash a}\text{ majority} \quad \cfrac{}{\vdash b}\text{ majority}}{\vdash a \wedge b}\ R\wedge^{\star} \qquad \cfrac{}{\vdash \neg(a \wedge b)}\text{ majority}}{\vdash (a \wedge b) \wedge \neg(a \wedge b)}\ R\wedge$$

Proof 1

Starting from the assumptions, we introduce the conjunction $a \wedge b$ and that contradicts the assumption $\vdash \neg(A \wedge B)$. On closer inspection, there is an important difference that regards the way in which we collectively obtain $a \wedge b$ and $\neg(a \wedge b)$: the former is inferred from other propositions that have been accepted by majority, that is a and b, whereas the latter is immediately accepted by majority. Moreover, $a \wedge b$ is inferred form two propositions that have been elected by means of two different coalitions of agents, that is $\{i_1, i_2\}$ for a and $\{i_1, i_3\}$ for b, whereas $\neg(a \wedge b)$ holds at the collective level by virtue of a single winning coalition that supports it, i.e. $\{i_2, i_3\}$.

By means of linear logic, we can keep track of such a distinction, by choosing the tensor conjunction to model the introduction of the conjunction (cf. step $R\wedge^{\star}$) in collective reasoning and the additive conjunction to model the assumption $\vdash \neg(a \,\&\, b)$. The multiplicative conjunction $\otimes$ joins two proofs that have different contexts, i.e. that have been obtained by exploiting different assumptions, whereas the additive conjunction would force the assumptions to coincide. In order to illustrate this point, we use sequent calculus instantiated by non-logical axioms of the following form. We view winning coalitions of voters as premises of sequents, that act as a sort of ground for making a proposition collectively true. Then, rules of inference can be instantiated as follows.

$$\cfrac{\cfrac{}{\{i_1, i_2\} \vdash a}\text{ majority} \qquad \cfrac{}{\{i_1, i_3\} \vdash b}\text{ majority}}{\{i_1, i_2\}, \{i_1, i_3\} \vdash a \wedge b}\ R\otimes$$

Classical inferences cannot *per se* keep track of the fact that $a \wedge b$ in Proof 1 is inferred by two propositions that have been obtained by different winning coalitions. In linear logic, we can write the following proof that interprets the introduction of the conjunction multiplicatively.

$$\cfrac{\cfrac{\cfrac{}{\vdash a}\text{ majority} \quad \cfrac{}{\vdash b}\text{ majority}}{\vdash a \otimes b}\ R\otimes \qquad \cfrac{}{\vdash \neg(a \,\&\, b)}\text{ majority}}{\vdash (a \otimes b) \otimes \neg(a \,\&\, b)}\ R\&$$

Proof 2

Proof 1 was simply showing how to derive the inconsistency of the discursive dilemma. By contrast, Proof 2 is a legitimate proof of linear logic, that is, $(a \otimes b) \otimes \neg(a \,\&\, b)$ is *not* inconsistent in linear logic. In the coalitional interpretation of sequent calculus that we suggested above, the conclusion of Proof 2 simply means that there are different winning coalitions for a and b and that does not contradict the fact that there is a also a single winning coalition for $\neg(a \,\&\, b)$. Moreover, from the assumptions above (i.e. non-logical axioms of Proof 2) it is not possible in linear logic to derive $a \,\&\, b$, that is, it is not possible to infer that there is a single winning coalition supporting a and b. Therefore, the set of formulas $\{a, b, \neg(a \,\&\, b)\}$ is not inconsistent in linear logic. Hence, the interpretation that we have sketched shows that there is room in linear logic for a consistent view of (majoritarian) collective reasoning. As we shall see, this fact does not depend on the conjunction. It relies on the distinction between additives and multiplicatives. Consider a case of disjunctive discursive dilemma.

	a	$a \vee b$	b	$\neg a$	$\neg(a \vee b)$	$\neg b$
i_1	1	1	0	0	0	1
i_2	0	1	1	1	0	0
i_3	0	0	0	1	1	1
maj.	0	**1**	0	**1**	0	**1**

Again, each individual is consistent, whereas the collective set $\{\neg a, \neg b, a \vee b\}$ is not consistent in classical logic.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \neg a}\text{majority} \quad \dfrac{}{\vdash \neg b}\text{majority}}{\vdash \neg a \wedge \neg b}R\wedge \qquad \dfrac{}{\neg a \wedge \neg b \vdash \neg(a \vee b)}\star}{\vdash \neg(a \vee b)}\text{cut} \qquad \dfrac{}{\vdash a \vee b}\text{majority}}{\vdash \neg(a \vee b) \wedge (a \vee b)}R\wedge$$

Proof 3

Inference step $\star$ is justified by De Morgan duality. Again, we can choose how to interpret the logical operators by means of linear logic.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \neg a}\text{majority} \quad \dfrac{}{\vdash \neg b}\text{majority}}{\vdash \neg a \otimes \neg b}R\otimes \qquad \dfrac{}{\neg a \otimes \neg b \vdash \neg(a \parr b)}\star}{\vdash \neg(a \parr b)}\text{cut} \qquad \dfrac{}{\vdash a \oplus b}\text{majority}}{\vdash \neg(a \parr b) \otimes (a \oplus b)}R\otimes$$

Proof 4

Step $\star$ holds because of De Morgan for multiplicatives in Linear Logic.[6] Again, by using the distinction between additives and multiplicatives, we obtain a consistent collective

[6] Duality holds between operators of the same type, i.e. between the multiplicative conjunction and disjunction, and between the additive conjunction and disjunction. Additives and multiplicatives are independent.

set: since $\neg(a \parr b) \otimes (a \oplus b)$ is consistent in linear logic, also the set $\{\neg a, \neg b, a \oplus b\}$ is consistent.

4 The model

We proposed to abandon two hypotheses of the standard model of JA, that is, classical logic as a model of rationality and the homogeneity of individual and collective rationality. In this section, we restate the model of Judgment Aggregation [11,3] in proof-theoretical terms, we allow for modeling rationality in non-classical logics, and for evaluating individual and collective reasoning with respect to different logics. The formal treatment is based on [14]

Let N be a (finite) set of agents. An *agenda* $\mathcal{X}_{\mathcal{L}}$ is a (finite) set of propositions in the language $\mathcal{L}_L$ of a given logic L that is closed under complements. i.e. non-double negations. Moreover, we shall assume that the agenda does not contain tautologies or contradictions. We slightly rephrase the usual rationality conditions on judgment sets in terms of sequents derivability.

A *judgement set* J is a subset of $\mathcal{X}_L$ such that J is (wrt L) *consistent* ($J \nvdash_L \emptyset$), *complete* (for all $\phi \in \mathcal{X}_L$, $\phi \in J$ or $\neg\phi \in J$) and *deductive closed* (if $J \vdash_L \phi$ and $\phi \in \mathcal{X}_L$, $\phi \in J$). Denote $J(\mathcal{X}_L)$ the set of all judgement sets on $\mathcal{X}_L$. A *profile* of judgements sets **J** is a vector $(J_1, \ldots, J_n)$, where $n = |N|$.

We discuss agendas defined in a number of languages and logics. We intend to model aggregators that take profiles of judgments sets that are rational according to a given logic L and return a set of judgement which can be evaluated with respect to a (possibly) different logic L'. In case L and L' are the same, we are in the standard situation in JA.

In case the languages of L and L' are different, we need to define a translation function from the language of L into the language of L'. In general, a *translation* is just a function that maps formulas of one language into the other $t : \mathcal{L}_L \to \mathcal{L}_{L'}$. An *aggregator* is then a function $F : J(\mathcal{X}_L)^n \to J(\mathcal{X}'_{L'})$ such that F is the composition of an aggregator in the standard JA sense ($F' : L(\mathcal{X})^n \to \mathcal{P}(\mathcal{X})$) with a function $T : \mathcal{P}(\mathcal{X}_L) \to \mathcal{P}(\mathcal{X}'_{L'})$ that lifts t to sets of propositions: for $J \subset \mathcal{X}_L$, $T(J) = \{t(\phi) \mid \phi \in J\} \subset \mathcal{X}'_{L'}$. Thus, we have that $F(\mathbf{J}) = T(F'(\mathbf{J})) \subseteq \mathcal{X}_{\mathbf{L}'}$. For example, the majority rule $M : J(\mathcal{X})^n \to J(\mathcal{X}'_{L'})$ is defined as follows. Let $N_\phi = \{i \mid \phi \in J_i\}$, define $M' : J(\mathcal{X}_L)^n \to \mathcal{P}(\mathcal{X}_L)$ such that $M'(\mathbf{J}) = \{\phi \in \mathcal{X}_L \mid |N_\phi| > n/2\}$; then, given a translation t, $M(\mathbf{J}) = T(M'(\mathbf{J}))$.

Note that our definition allows for aggregators that return sets of judgments that are inconsistent wrt L and that may not to be inconsistent wrt L'. That is why the codomain of F' is defined by the powerset $\mathcal{P}(\mathcal{X}_L)$.

We shall concentrate on the following *additive translation* of CL into LL: $\textsc{add} : \mathcal{L}_{CL} \to \mathcal{L}_{LL}$. ADD is defined as follows: for a atomic, $\textsc{add}(a) = a$ and $\textsc{add}(\neg a) = \neg a$; for A in $\mathcal{L}_{CL}$, $\textsc{add}(\neg A) = \neg(\textsc{add}(A))$, $\textsc{add}(A \wedge B) = \textsc{add}(A) \,\&\, \textsc{add}(B)$, $\textsc{add}(A \vee B) = \textsc{add}(A) \oplus \textsc{add}(B)$ (i.e. we replace each classical connective with its additive counterpart). The translation reflects our interpretation of LL reasoning as coalitional reasoning. In particular, $M : J(\mathcal{X}_{CL})^n \to J(\mathcal{X}_{LL})$ defined by $M(\mathbf{J}) = \textsc{add}(M'(\mathbf{J}))$ embeds classical formulas that are accepted according to the majority rule into LL by viewing them as additive formulas (i.e. they are collectively accepted because of a single winning coalition that supports them). Thus, additives are modeling propositions accepted because of a winning coalition, whereas multiplicatives can be used to reason about propositions accepted by different winning coalitions. The general notion of collective rationality can be stated in our framework as follows.

Definition 1 (Collective rationality). *An aggregation procedure $F : J(\mathcal{X}_L)^n \rightarrow \mathcal{P}(\mathcal{X}'_{L'})$ is collectively rational wrt to the logic L' iff for every agenda $\mathcal{X}_L$ and every profile $\boldsymbol{J} \in J(\mathcal{X}_L)^n$, $F(\boldsymbol{J})$ is consistent, complete and deductively closed wrt L'.*

The standard notion of collective rationality [11,3] states that a procedure is collectively rational iff for every agenda defined in classical logic and every profile, the output of the aggregation procedure is consistent, complete, and deductively closed wrt classical logic. The standard definition is just an instantiation of Definition 1.

5 Logics for collective reasoning

The seminal impossibility theorem in [9] proves that there is no fair aggregation procedure that is collectively rational wrt classical logic. In particular, it follows that the majority rule is not collectively rationally wrt classical logic, as the profile in the discursive dilemma shows. More precisely, the theorem applies to every agenda that violates the so called *median property* [11] that can be restated in our setting as follows: for every subset S of the agenda $\mathcal{X}_L$ that is minimally inconstant wrt L, S has to be of cardinality at most 2. For example, the agenda of the discursive dilemma violates the median property, as it includes $\{a, b, \neg(a \wedge b)\}$ that is a set of cardinality 3 that is minimally inconsistent wrt classical logic.

Theorem 1. *If the agenda $\mathcal{X}_{CL}$ does not satisfy the median property, then the majority rule is not collectively rational wrt CL.*

Unfortunately, the median property basically amounts to disabling classical inferences in collective reasoning, it is a severe language restriction that does not correspond to the reasoning capabilities of classical logic. For instance an agenda such as $\{a \wedge b, \neg(a \wedge b)\}$ satisfies the median property, however we cannot reason on such conjunctive statement by using the rules that define the conjunction, on pain of inconsistency.

By moving to intuitionistic logic (IL), we do not observe any substantial difference from the point of view of collective rationality.

Theorem 2. *If the agenda $\mathcal{X}_{IL}$ does not satisfy the median property, the majority rule is not collectively rational wrt IL.*

In order to see that, it is enough to consider that Proof 1 in Section 3 is also a proof in intuitionistic logic, therefore the (conjunctive) discursive dilemma provides a counter example to the collective rationality of the majority rule wrt to IL. Hence, the non-constructivity of classical reasoning is not the cause of paradoxes in judgment aggregation. A general possibility result requires substructural logics and the distinction between multiplicatives and additives. Define *Additive Linear Logic* (ALL) as the fragment of linear logic defined by the negation and by additive connectives ($\neg$, $\&$, $\oplus$). We can prove the following result.

Theorem 3. *For every agenda $\mathcal{X}_{ALL}$, the majority rule is collectively rational wrt ALL.*

The proof of this result can be easily adapted from the proof of Theorem 2 in [14]. The argument relies on an interesting property of additive linear logic, that is, every provable sequent in ALL must contain at most two formulas. That precisely corresponds to the median property. Note that in case of ALL, it is not a severe language restriction that guarantees the possibility results, it is the very nature of the inference rules of additive linear logic. That is, the restriction is grounded on the permitted inferences.[7]

We propose now to evaluate collective reasoning by means of linear logic and let individuals reason by means of classical logic, in order to provide a possibility result for the standard JA setting. As a corollary of Theorem 1, we can prove the following result.

Theorem 4. *For every agenda $\mathcal{X}_{CL}$, the (deductive closure of the) additive translation of the majority rule ($cl(\text{ADD}(M'(\boldsymbol{J})))$) is collectively rational wrt LL.*

The theorem depends on the additive translation, that is, on the interpretation we adopted in Section 3. The result shows that the distinction between additives and multiplicatives at the level of collective reasoning is capable of guaranteeing collective rationality wrt linear logic. The aggregation procedure $\text{ADD}(M'(\mathbf{J}))$ translates classical formulas accepted by majority into additive formulas, then it deductively closes the collective set of propositions by means of full linear logic reasoning. It is worth noticing that, besides providing a possibility result, Theorem 4 proposes a consistent model of discursive dilemmas, that is, it provides a logical diagnosis of discursive dilemmas that reflects the interpretation we provided in Section 3.

6 In favor of non-classical collective rationality

We discussed the importance of consistency and collective rationality for an aggregative view of collective reasoning. We have seen that a view of rationality based on classical logic is not capable of defining a viable notion of collective reasoning. The challenge of this paper is to give up one assumption that is usually endorsed, that is, instead of dropping normative properties of the aggregation procedure, we decided to give up classical logic. We have seen that there do exist well-behaved logics, such as linear logic, for which majoritarian collective rationality is guaranteed. We claim that linear logics are a good candidate for developing a theory of collective reasoning. Future work has to be done in order to fully motivate this idea. In particular, one direction is to develop a logical model of the relationship between individual and collective rationality. A way to approach this problem is to develop a non-classical account of *coalition logic* [13] that is based on linear logic. In particular, by interpreting the axiom $[C_1]A \wedge [C_2]B \rightarrow [C_1 \cup C_2](A \wedge B)$ of coalition logic by using the distinction between multiplicative and additive conjunctions, we shall provide a logical foundation of the additive translation and of the interpretation we provided in Section 3. A fist step

[7] Note that the proof-theoretical analysis pinpoints what inference rule are responsible for deriving collective inconsistency. For example, by allowing weakening (W) in collective reasoning, we lose collective rationality also for ALL.

for developing a linear coalition logic has been proposed in [16]. Moreover, another direction is to investigate the rich framework of semantics of linear logic, such as game semantics, in order to provide an accurate semantic description of collective paradoxes. Finally, another direction has to investigate whether there are other non-classical logics for which possibility results can be achieved, a good candidate seems to be relevance logics.

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Identities

$$\frac{}{A \vdash A}\text{ ax} \qquad \frac{\Gamma, A \vdash \Delta \qquad \Gamma' \vdash A, \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\text{ cut}$$

Structural Rules

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}\text{ E} \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\text{ E}$$

$$\frac{\Gamma, A, A, \vdash \Delta}{\Gamma, A \vdash \Delta}\text{ C} \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\text{ C}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\text{ W} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\text{ W}$$

Negation

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ L\neg \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ R\neg$$

Multiplicative presentation of logical connectives

$$\wedge\text{R}\ \frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge\text{L}$$

$$\parr\text{L}\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \parr B \vdash \Delta, \Delta'} \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta}\ \parr\text{R}$$

Additive presentation of logical connectives

$$\&\text{R}\ \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \& B, \Delta} \qquad \frac{\Gamma, A_i \vdash \Delta}{\Gamma, A_0 \& A_1 \vdash \Delta}\ \&\text{L}$$

$$\oplus\text{L}\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} \qquad \frac{\Gamma \vdash A_i, \Delta}{\Gamma \vdash A_0 \oplus A_1, \Delta}\ \oplus\text{R}$$

Multiplicatives in Linear Logic

$$\otimes\text{R}\ \frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'} \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}\ \otimes\text{L}$$

$$\parr\text{L}\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \parr B \vdash \Delta, \Delta'} \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta}\ \parr\text{R}$$

Additives in Linear Logic

$$\&\text{R}\ \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \& B, \Delta} \qquad \frac{\Gamma, A_i \vdash \Delta}{\Gamma, A_0 \& A_1 \vdash \Delta}\ \&\text{L}$$

$$\oplus\text{L}\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} \qquad \frac{\Gamma \vdash A_i, \Delta}{\Gamma \vdash A_0 \oplus A_1, \Delta}\ \oplus\text{R}$$

Table 1. Sequent calculi for Classical and Linear Logic