Simulations of Opinion Formation in Multi-Agent Systems using Kinetic Theory

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***Abstract*—In this paper we formulate the problem of opinion formation using a physical metaphore. We consider a multi-agent system where each agent is associated with an opinion and interacts with any other agent. Interpreting the agents as the molecules of a gas, we model the evolution of opinion in the system according to a kinetic model based on the analysis of interactions among agents. From a microscopic description of each interaction between two agents, we derive the stationary profiles of the opinion under given assumption. Results show that, depending on the average opinion and on the parameters of the model, different profiles can be found, but all stationary profiles are characterized by the presence just of one or two maxima. Analytic results are confirmed by simulations shown in the last part of the paper.**

I. Introduction

In this paper we describe a model for opinion formation among agents. We assume that each agent is associated with an opinion $v \in I \subseteq \mathbb{R}$ and that it can change its opinion each time it interacts with another agent. In the literature, various approaches that study opinion evolution in a society of agents have been proposed, and many of them are based on *Cellular Automata* (*CA*) because CA describe well global effects of local phenomena (see, e.g., [1]). In order to overcome the synchronism that CA assume, in recent years the use of microscopic models inspired from physics has been introduced to describe asynchronous social interactions among agents in a society [2]. Such models are based on the idea that the laws of kinetic theory, which are typically used to describe the effects of interactions between two molecules of a gas, can also be used to model interactions between two agents.

Statistical mechanics and kinetic theory describe the details of each interaction between two molecules in a gas but they also allow finding classic laws which describe macroscopic properties of gases [3]. Analogously, from the microscopic laws which describe the details of each interaction between two agents, collective behaviour can be described from a macroscopic point of view [4]. All the research challenges related to the application of kinetic and statistical formalisms to describe multi-agent systems gave rise to new disciplines known as *econophysics* and *sociophysics* [5]. Such new disciplines have been used to describe, e.g., wealth evolution [6] and market economy [7], and they have also been adopted to characterize opinion evolution in a society [8].

In this paper, we focus on a model for opinion formation based on kinetic theory of gases and we analyze the evolution of the opinion in a society of agents. In particular, we analyze numerically the opinion evolution according to a model introduced in [9], which mandates that each agent can change its own opinion because of two different reasons [10]. The first reason is related to *compromise* between interacting agents. More precisely, the model assumes that both agents involved in an interaction can change their respective opinions in favor of that of the other agent. The second reason only involves each single agent and it is related to the fact that an agent can change its opinion autonomously, giving rise to a process known as *diffusion*.

This paper is organized as follows. In Section II, we describe the considered kinetic model from an analytic point of view. In Section III, we derive a stationary profile of opinion in a specific case. In Section IV, we show relevant simulation results. Section V concludes the paper.

II. Kinetic Formulation of Opinion Formation

Kinetic theory of gases describes, from a microscopic point of view, the effects of interactions among molecules in gases. By reinterpreting the molecules of a gas as agents, one can use the kinetic framework to describe social interactions among agents. While molecules are typically associated with relevant physical properties, like their velocities, agents can be associated with attributes that represent some of their characteristics. In particular, since in this work we are interested in modeling the evolution of the opinion, we assume that each agent is associated with a scalar parameter v which represents its opinion and which is defined in a given interval I. In the following, we consider $I = [-1, 1]$, where ± 1 represent extremal opinions.

In order to use the kinetic approach, we need to define a function, denoted as $f(w, t)$, which represents the density of opinion w at time t, and which is defined for each opinion $w \in I$ and for each time $t \geq 0$. According to such a definition,

$$\int_I f(w, t) \mathrm{d}w = 1. \quad (1)$$

In order to formulate the problem of opinion evolution in kinetic terms, we assume that the function $f(w, t)$ evolves on the basis of the Boltzmann equation, which, under our assumptions, can be written as

$$\frac{\partial f}{\partial t} = \mathcal{Q}(f, f)(w, t) \quad (2)$$

where $\mathcal{Q}$ is denoted as *collisional operator*. According to (2), the temporal evolution of the opinion density is governed by the collisional operator $\mathcal{Q}$, whose explicit formulation depends on the details of binary interactions between any pairs of agents. Before deriving a formula for $\mathcal{Q}$, let us describe the effects of each binary interaction.

Denoting as (w, v) the opinions of two agents before their interaction, we assume that the following model holds

$$\begin{cases} w' = w + \gamma(v - w) + \eta D(|w|) \\ v' = v + \gamma(w - v) + \eta_* D(|v|) \end{cases} \tag{3}$$

where: (w', v') are the post-interaction opinions of the two agents; γ is a constant defined in $(0, \frac{1}{2})$; η and η_* are two independent random variables with the same statistics; and $D(\cdot)$ is a function that describes the impact of diffusion in the considered interaction [9]. From (3) it can be observed that the post-interaction opinions of the two agents are obtained by adding to their pre-interaction opinions two terms: the first one is related to compromise, while the second one is related to diffusion through function $D(\cdot)$.

Observe that the contribution of the compromise is proportional to the difference between the two pre-interaction opinions. Taking, for instance, the first equation in (3), we can conclude that the second addend on the right hand side of (3) is positive if $v > w$, so that the opinion of the considered agent (whose pre-interaction opinion is w) increases if it interacts with an agent with greater opinion. At the opposite, if $w > v$ the second addend is negative, so that the contribution of compromise decreases the opinion of the considered agent (towards that of the agent it interacts with). Observe that the contribution of compromise is negligible if $\gamma \simeq 0$, while it becomes relevant as γ increases.

Concerning the diffusion term, we assume that function $D(\cdot)$ depends on the absolute value of the opinion, meaning that the propensity of changing opinion is symmetrical with respect to 0 (namely, with respect to the middle point of I). Moreover, we assume that $D(\cdot)$ is non increasing with respect to the absolute value of the opinion, coherently with the fact that, typically, extremal opinions are more difficult to change. Finally, we assume that $0 \leq D(|w|) \leq 1$ for all $w \in I$. According to such assumptions, the contribution of diffusion can be either positive or negative depending on the value of η and η_*. In the following, we denote the probability density function of η and η_* as $\vartheta(\cdot)$ and we assume that

$$\begin{aligned} \int \eta \vartheta(\eta) \mathrm{d}\eta = \int \eta_* \vartheta(\eta_*) \mathrm{d}\eta_* = 0 \\ \int \eta^2 \vartheta(\eta) \mathrm{d}\eta = \int \eta_*^2 \vartheta(\eta_*) \mathrm{d}\eta_* = \sigma^2. \end{aligned} \tag{4}$$

Such a choice corresponds to considering η and η_* as 0 mean random variable with standard deviation σ.

The effects of diffusion in the opinion evolution are taken into account through the *transition rate*, which is defined as

$$W(w, v, w', v') = \vartheta(\eta)\vartheta(\eta_*)\chi_I(w')\chi_I(v') \tag{5}$$

where χ_I is the indicator function of set I (equals to 1 if its argument belongs to I, and to 0 otherwise). The indicator function in (5) is meant to impose that post-interaction opinions still belong to interval I [9].

Now that we have completed the definition of the law that describe each single interaction, we can write the explicit expression of the collisional operator $\mathcal{Q}$ used in (2), which is given by

$$Q(f, f) = \int_{\mathbb{B}^2} \int_I \left[{}'W \frac{1}{J} f('w) f('v) - W f(w) f(v) \right] \mathrm{d}v \mathrm{d}\eta \mathrm{d}\eta_*$$

where $\mathbb{B}$ is the support of ϑ, $'w$ and $'v$ are the pre-interaction variables which generate w and v, respectively, $'W$ is the transition rate and J is the Jacobian of the transformation of $('w, 'v)$ in (w, v). The two addends in the previous equation represent the gain and the loss of agents in $\mathrm{d}w$, respectively [9].

In order to study the opinion evolution, we need to introduce the weak form of the Boltzmann equation. Generally speaking, the weak form of a differential equation is obtained by multiplying both sides by a test function, namely a smooth function with compact support, and integrating. The weak form of the Boltzmann equation can then be found by multiplying both sides of (2) by a test function $\phi(w)$ and integrating with respect to w. Using a proper change of variable in the integral, the weak form of the Boltzmann equation can be written as:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_I f(w, t)\phi(w) \mathrm{d}w = \\ &\int_{\mathbb{B}^2} \int_{I^2} W f(w) f(v) (\phi(w') - \phi(w)) \mathrm{d}w \mathrm{d}v \mathrm{d}\eta \mathrm{d}\eta_*. \end{aligned} \tag{6}$$

Setting $\phi(w) = 1$ in (6) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_I f(w, t) \mathrm{d}w = 0. \tag{7}$$

Such an equality corresponds to the fact that the number of agents is time invariant. This property is also found in classic kinetic theory and it corresponds to mass conservation.

Considering $\phi(w) = w$ as a test function and using (3) in (6) gives

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_I f(w, t) w \mathrm{d}w = \\ &\gamma \int_{\mathbb{B}^2} \int_{I^2} W f(w) f(v)(v - w) \mathrm{d}w \mathrm{d}v \mathrm{d}\eta \mathrm{d}\eta_* \\ &+ \int_{\mathbb{B}^2} \int_{I^2} W f(w) f(v) \eta D(|w|) \mathrm{d}w \mathrm{d}v \mathrm{d}\eta \mathrm{d}\eta_*. \end{aligned} \tag{8}$$

Denoting as $u(t)$ the average value of the opinion at time t, namely

$$u(t) = \int_I f(w, t) w \, \mathrm{d}w \tag{9}$$

the left hand side of (8) corresponds to the derivative $\dot{u}(t)$ of the average opinion. Moreover, observe that the right hand side term of (8) is 0. As a matter of fact, the first integral is 0 for symmetry reasons, while the second integral is 0 because, according to (4), the average value of ϑ is 0. From (8) it can then be concluded that $\dot{u}(t) = 0$, and, therefore, the average opinion is conserved, namely $u(t) = u(0) = u$. This property corresponds to the conservation of momentum in kinetic theory.

We are now interested in studying the asymptotic behaviour of the distribution function $f(w, t)$. For this reason, in order to simplify notation, let us define a new temporal variable

$$\tau = \gamma t \tag{10}$$

where γ is the coefficient which appears in (3) and it is related to compromise. Assuming that $\gamma \to 0$, namely that each interaction causes small changes of opinions, the function

$$g(w,\tau) = f(w,t) \tag{11}$$

describes the asymptotic behaviour of $f(w,t)$. In [9] it is shown that by substituting $f(w,t)$ with $g(w,\tau)$ in (6) and using a Taylor series expansion of $\phi(w)$ around w in (6) the following equation of g can be derived

$$\frac{\mathrm{d}g}{\mathrm{d}\tau} = \frac{\lambda}{2}\frac{\partial^2}{\partial w^2}(D(|w|)^2 g) + \frac{\partial}{\partial w}((w-u)g) \tag{12}$$

where

$$\lambda = \sigma^2/\gamma. \tag{13}$$

Equation (12) is known in the literature as the weak form of the Fokker-Planck equation [11].

We are now interested in studying stationary solutions of this equation, namely those which satisfy

$$\frac{\mathrm{d}g}{\mathrm{d}\tau} = 0. \tag{14}$$

In the following, we denote such solutions as g_∞. In next section we analyze such solutions for different diffusion functions $D(|w|)$ and for different values of the parameter λ.

III. Results

In this section we derive relevant stationary profiles for the opinion density g. Such profiles are defined as solutions of (14) and, therefore, they depend on the parameters u and λ, which represent the average opinion and the ratio σ^2/γ, respectively, and on the choice of the diffusion function D. Recalling the initial assumptions on D, we develop our results considering

$$D(|w|) = 1 - |w| \qquad w \in I \tag{15}$$

which is symmetrical with respect to 0 and decreasing in $|w|$.

With this choice of the diffusion function, equations (3) become

$$\begin{cases} w' = w - \gamma(w-v) + \eta(1-|w|) \\ v' = v - \gamma(v-w) + \eta_*(1-|v|). \end{cases} \tag{16}$$

In order to guarantee that post-collisional opinions still belong to the considered interval I, we need to set the support of ϑ to

$$\mathbb{B} = (-(1-\gamma), 1-\gamma). \tag{17}$$

As a matter of fact, from (16)

$$\begin{aligned} |w'| &\le (1-\gamma)|w| + \gamma|v| + |\eta|(1-|w|) \\ &\le (1-\gamma)|w| + \gamma + |\eta|(1-|w|) \end{aligned} \tag{18}$$

and if, according to (17), $|\eta| \le (1-\gamma)$ then

$$|w'| \le (1-\gamma)|w| + \gamma + (1-\gamma)(1-|w|) = 1 \tag{19}$$

Hence, we can conclude that if $\eta \in \mathbb{B}$, then $w' \in I$. Analogous results can be derived for v'.

Now, if we substitute the expression of D defined in (15) in (12) the stationary solution g_∞ can then be found, according to (14), by solving the following partial differential equation.

$$\frac{\lambda}{2}\frac{\partial}{\partial w}\big((1-|w|)^2 g\big) + (w-m)g = C \tag{20}$$

where C is a constant which is necessarily 0. As a matter of fact, by integrating (20) one obtains

$$\frac{\lambda}{2}\int_{-v_1}^{v_2}\frac{\partial}{\partial w}\big((1-|w|)^2 g\big)\mathrm{d}w + \int_{-v_1}^{v_2}(w-m)g\mathrm{d}w = C(v_2+v_1)$$

from which, assuming that $v_1 \to 1$ and $v_2 \to 1$ one obtains

$$0 + m - m = 2C$$

which corresponds to $C = 0$.

Let us start by considering $w > 0$, so that (20) can be written as

$$\frac{\lambda}{2}(1-w)^2\frac{\partial g}{\partial w} + [(w-m) + \lambda(w-1)]\, g = 0. \tag{21}$$

Dividing both sides by g one obtains

$$\frac{g'}{g} = \frac{2}{(1-w)} + \frac{2(m-w)}{\lambda(1-w)^2} \tag{22}$$

and observing that

$$\frac{2(m-w)}{\lambda(1-w)^2} = \frac{\mathrm{d}}{\mathrm{d}w}\left(-\frac{2}{\lambda}\log(1-w) + \frac{2(m-1)}{\lambda(1-w)}\right) \tag{23}$$

equation (22) can be written as

$$(\log g(w))' = \left(\log(1-w)^{-2-\frac{2}{\lambda}} + \frac{2(m-1)}{\lambda(1-w)}\right)' \tag{24}$$

where we have used the facts that

$$\begin{aligned} \frac{g'}{g} &= (\log g(w))' \\ \frac{2}{(1-w)} &= -2(\log(1-w))'. \end{aligned} \tag{25}$$

Integrating (24) and applying the exponential function, one finally obtains the following expression for the stationary profile

$$g_\infty(w) = \tilde{c}_{u,\lambda}(1-|w|)^{-2-\frac{2}{\lambda}}\exp\left(\frac{2(m-1)}{\lambda(1-|w|)}\right) \tag{26}$$

where $\tilde{c}_{u,\lambda}$ is a normalization constant that depends on the average opinion u and on λ.

Let us now consider $w < 0$ so that equation (20) becomes

$$\frac{\lambda}{2}(1+w)^2\frac{\partial g}{\partial w} + [(w-m) + \lambda(w+1)]\, g = 0. \tag{27}$$

Dividing both sides by g leads to

$$\frac{g'}{g} = -\frac{2}{(1+w)} + \frac{2(m-w)}{\lambda(1+w)^2} \tag{28}$$

and by applying analogous calculation to the case with $w > 0$ one obtains

$$(\log g(w))' = \left(\log(1+w)^{-2-\frac{2}{\lambda}} - \frac{2(m+1)}{\lambda(1+w)}\right)'. \tag{29}$$

Integrating (29) leads to the following formula for the stationary profile

$$g_\infty(w) = \hat{c}_{u,\lambda}(1-|w|)^{-2-\frac{2}{\lambda}}\exp\left(\frac{-2(m+1)}{\lambda(1-|w|)}\right) \tag{30}$$

where w has been substituted by $-|w|$ and $\hat{c}_{u,\lambda}$ is a normalization constant.

Since g_∞ is the solution of a differential equation it must be continuous. From (26) and (30) it is evident that g_∞ is continuous for $w > 0$ and $w < 0$. Imposing that g_∞ is also continuous in $w = 0$, the following equality needs to hold

$$\tilde{c}_{u,\lambda} \exp\left(\frac{2u}{\lambda}\right) = \hat{c}_{u,\lambda} \exp\left(\frac{-2u}{\lambda}\right). \tag{31}$$

Finally, the solution of (20) is

$$g_\infty(w) = c_{u,\lambda}(1-|w|)^{-2-\frac{2}{\lambda}} \exp\left[-\frac{2(1-uw)}{\lambda(1-|w|)}\right] \tag{32}$$

where $c_{u,\lambda}$ is the quantity in (31) and it needs to be determined in order to ensure

$$\int_I g_\infty(w) = 1. \tag{33}$$

Observe that g_∞ is piecewise $\mathcal{C}^1$ and it is non-differentiable in $w = 0$ (as the function D). Moreover, the solution is symmetric if we change w and u with $-w$ and $-u$, namely

$$g_\infty(w; u, \lambda) = g_\infty(-w; -u, \lambda). \tag{34}$$

If $u = 0$, from (34) we can conclude that g_∞ is an even function. Moreover, using a change of variable for negative values of w, the integral of g_∞ can be written as

$$\int_{-1}^{1} g_\infty(w)\mathrm{d}w = 2c_{0,\lambda}\int_0^1 (1-w)^{-2-\frac{2}{\lambda}} \exp\left(\frac{-2}{\lambda(1-w)}\right)\mathrm{d}w$$

and, using the change of variable $t = \frac{-2}{\lambda(1-w)}$, the previous integral can be expressed as

$$2c_{0,\lambda}\left(\frac{\lambda}{2}\right)^{\frac{2}{\lambda}+1}\int_{\frac{2}{\lambda}}^{+\infty} t^{\frac{2}{\lambda}} e^{-t}\mathrm{d}t. \tag{35}$$

Finally, introducing the incomplete gamma function defined as

$$\Gamma(x, a) = \int_a^{+\infty} t^{x-1}\mathrm{e}^{-t}\mathrm{d}t, \tag{36}$$

the value of $c_{0,\lambda}$ which satisfies (33) is then

$$c_{0,\lambda} = \left[2\left(\frac{\lambda}{2}\right)^{\frac{2}{\lambda}+1}\Gamma\left(\frac{2}{\lambda}+1, \frac{2}{\lambda}\right)\right]^{-1}. \tag{37}$$

The case with $u = 0$ is the only one where the value of $c_{u,\lambda}$ can be found analytically. Other cases can be studied numerically.

We are now interested in studying the derivative of g_∞ in order to find singular points which correspond to maximum or minimum points. Deriving (26) it can be shown that if $w > 0$

$$g'_\infty(w) = 0 \iff 2\lambda(1-w) + 2(1-w) + 2(u-1) = 0.$$

Hence the (unique) singular point is

$$w = \frac{u+\lambda}{\lambda+1}$$

and it is positive if and only if $\lambda > -u$. Deriving (30), instead, it can be shown that if $w < 0$

$$g'_\infty(w) = 0 \iff 2\lambda(1+w) + 2(1+w) + 2(u+1) = 0$$

leading to the following singular point

$$w = \frac{u-\lambda}{\lambda+1}.$$

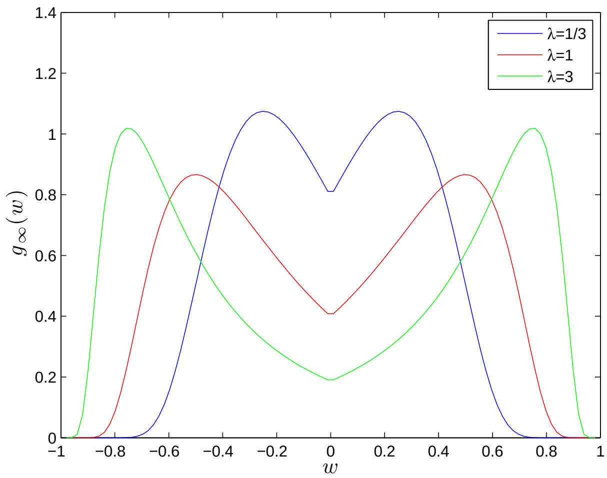

Fig. 1. Stationary profiles g_∞ for $u = 0$ and $\lambda = 1/3$ (blue line), $\lambda = 1$ (red line), and $\lambda = 3$ (green line)

Observe that this value is negative if and only if $\lambda > u$. Finally, the following cases can be considered

- if $u = 0$ then $g'_\infty(w) = 0$ in two points that are symmetric with respect to 0, namely $w = \pm\frac{\lambda}{\lambda+1}$
- if $u > 0$
  - if $0 < \lambda < u$ then $g'_\infty(w) = 0$ in a unique point, namely $w = \frac{u+\lambda}{\lambda+1}$
  - if $\lambda > u$ then $g'_\infty(w) = 0$ in two points, namely $w = \frac{u\pm\lambda}{\lambda+1}$
- if $u < 0$
  - if $0 < \lambda < -u$ then $g'_\infty(w) = 0$ in a unique point, namely $w = \frac{u-\lambda}{\lambda+1}$
  - if $\lambda > -u$ then $g'_\infty(w) = 0$ in two points, namely $w = \frac{u\pm\lambda}{\lambda+1}$

Observe that simple manipulations shows that

$$\begin{aligned} &\lim_{w\to 0^+} g'_\infty(w) > 0 \\ &\lim_{w\to 0^-} g'_\infty(w) < 0 \end{aligned} \tag{38}$$

so that $w = 0$, which is a non-differentiable point, can be considered as a point of minimum.

IV. Numerical Simulations

In this section, relevant numerical results are shown for stationary profiles for different values of u and λ. We focus on values of $u \geq 0$ as the stationary profiles relative to negative values of u can be obtained by symmetry, according to (34). The constant $c_{u,\lambda}$, which appears in g_∞, is evaluated numerically, using Newton-Cotes formulas [12].

First, we assume that $u = 0$ so that the average opinion corresponds to the middle point of I. As already observed in the previous section, if $u = 0$ then g_∞ is symmetric with respect to 0 and it has two maxima at

$$w = \pm\frac{\lambda}{\lambda+1}.$$

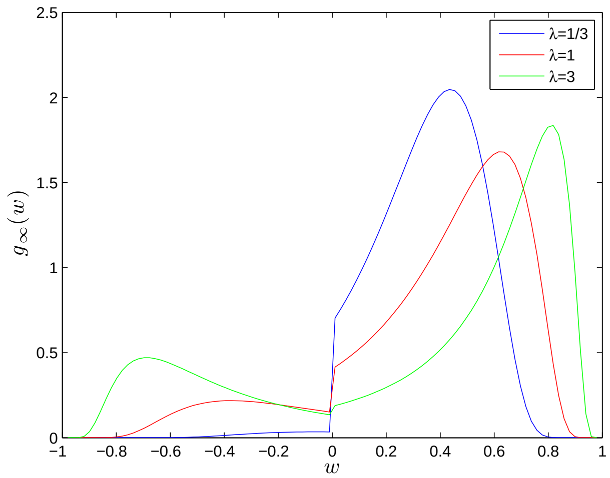

Fig. 2. Stationary profiles g_∞ for $u = 1/4$ and $\lambda = 1/3$ (blue line), $\lambda = 1$ (red line), and $\lambda = 3$ (green line)

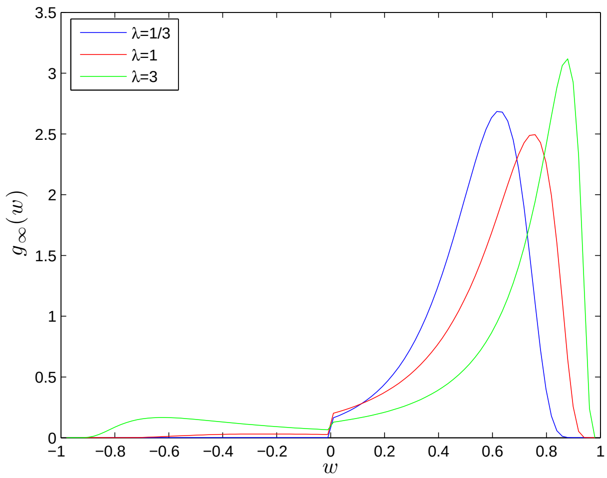

Fig. 3. Stationary profiles g_∞ for $u = 1/2$ and $\lambda = 1/3$ (blue line), $\lambda = 1$ (red line), and $\lambda = 3$ (green line)

As λ increases, such points get nearer to the extremal values of I, namely to extremal opinions. Observe that increasing the value of λ corresponds to incrementing the impact of diffusion with respect to compromise.

Fig. 1 shows the stationary profiles $g_\infty(w)$ when $u = 0$ for different values of λ. As expected from (34), the function $g_\infty(w)$ is symmetric with respect to 0 and it has a minimum in correspondence of $w = 0$ and two maxima whose values depend on λ. The stationary profiles $g_\infty(w)$ in Fig. 1 correspond to $\lambda = 1/3$ (blue line), $\lambda = 1$ (red line), and $\lambda = 3$ (green line). If $\lambda = 1/3$ the two maxima are in correspondence of $w = \pm 1/4$. Observe that in this case extremal distributions are associated with a very low probability. If $\lambda = 1$, instead, the maxima correspond to $w = \pm 1/2$; while in $\lambda = 3$ they correspond to $w = \pm 3/4$.

Therefore, it can be concluded that as λ increases the points of maximum move towards the extremes of the considered interval I. Observe that, according to its definition, any increase of λ corresponds to assuming that the contribution of diffusion is more relevant than that of compromise. Moreover, according to the results in Fig. 1, any increase of λ leads to stationary profiles with small values in correspondence of opinions in the middle of the interval I.

In Fig. 2 the stationary profiles $g_\infty(w)$ are shown when considering as average opinion the value $u = 1/4$. In this case, the function $g_\infty(w)$ is not symmetric and, as expected, it has a local minimum in $w = 0$. We consider the same values of λ as in the previous case. For each of these values, the number of maxima is two, since the condition $\lambda > u$ is always satisfied. If $\lambda = 1/3$ the positive maximum point is $w = 7/16$ and the negative one is $w = -1/16$. While the negative maximum is near the middle of the interval I, the positive one is farther. In Fig. 2 the stationary profile $g_\infty(w)$ obtained with $\lambda = 1/3$ is shown (blue line) and it can be observed that the value of the maximum in $w = 7/16$ is far more significant than that corresponding to $w = -1/16$, namely the positive opinions are far more likely than the negative ones. This is in agreement with the fact that the average opinion u is positive. If $\lambda = 1$ the maxima correspond to $w = \pm 5/8$ and $w = \mp 3/8$, as shown in Fig. 2 (red line). Finally, Fig. 2 also shows the results obtained with $\lambda = 3$ (green line). In this case the maximum points are $w = \pm 13/16$ and $w = \mp 11/16$. As in the previous case, the points of maximum get nearer to ± 1, namely to the extremes of I, as λ increases. Moreover, observe that largest values of λ correspond to increasing the likelihood of negative opinions.

Let us now increase the value of the average opinion to $u = 1/2$. The stationary profiles $g_\infty(w)$ are shown in Fig. 3 for $\lambda = 1/3$ (blue line), $\lambda = 1$ (red line), and $\lambda = 3$ (green line). Observe that if $\lambda = 1/3$ the function $g_\infty(w)$ has only one maximum, namely the positive one. As a matter of fact, according the results in the previous section, the negative one only exists if $\lambda > u$. The maximum point is $w = 5/8$. This value is greater than the one obtained for the same λ in the case $u = 1/4$, accordingly with the fact that, in this case, we consider a higher average opinion u. When considering $\lambda = 1$, the function $g_\infty(w)$ has two maxima since the condition $\lambda > u$ is satisfied. Such points are $w = 3/4$ and $w = -1/4$. As in Fig. 2, the value of the maximum corresponding to the negative value of w is less significant with respect to that relative to the positive value of w. If $\lambda = 3$ the two maxima correspond to $w = 5/8$ and $w = -7/8$ and they are nearer to the extremes of I with respect to those obtained with lower λ. A comparison of the results in Fig. 2 with those in Fig. 3 shows that in the latter the values of the positive maxima are greater while those of the negative maxima are smaller.

Finally, we consider a greater value of the average opinion, namely $u = 3/4$. This corresponds to considering an extremist society. Fig. 4 shows the stationary profiles for $\lambda = 1/3$ (blue line), $\lambda = 1$ (red line), and $\lambda = 3$ (green line). As in the previous case, since $\lambda < u$, the profile $g_\infty(w)$ has only one maximum if $\lambda = 1/3$. The point of maximum is $w = 13/16$ and it is closer to 1 than the other points of maximum obtained with the same λ for lower values of the average opinion u. From Fig. 4 it can be shown that, once again, the positive maximum point moves towards the extreme 1 as λ increases, since it corresponds to $w = 7/8$ if $\lambda = 1$ and to $w = 15/16$

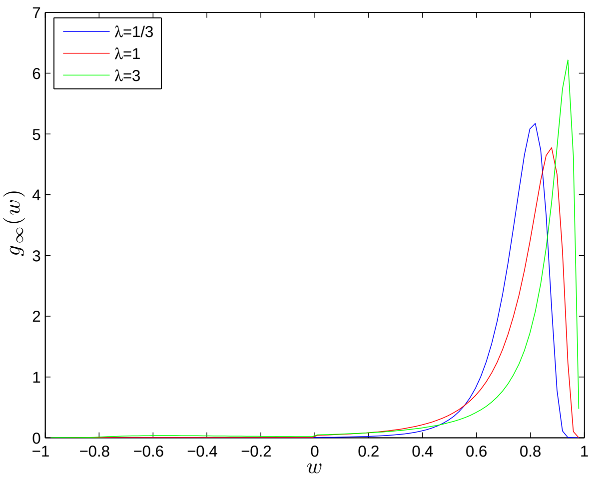

Fig. 4. Stationary profiles g_∞ for $u = 3/4$ and $\lambda = 1/3$ (blue line), $\lambda = 1$ (red line), and $\lambda = 3$ (green line)

if $\lambda = 3$. Moreover, Fig. 4 shows that an increase of the value of the average opinion significantly reduces the value of the negative maximum, which in this case is negligible. This result is not surprising since if the average opinion is near the positive extreme, then the number of agents with negative opinion has to be small.

V. Conclusions

In this paper we discussed the opinion dynamics in a multi-agent system through a kinetic approach. We considered an opinion evolution model inspired from the interactions of molecules in a gas and we studied, from an analytic point of view, the asymptotic behaviour of the opinion distribution. Assuming that the opinion of an agent can change each time it interacts with any other agent because of compromise and diffusion, we showed that the average opinion of the system is conserved. The stationary profiles can have different characteristics, depending on the parameters of the model and on the explicit expressions of the function which represents the diffusion. For a particular diffusion function, we showed that the asymptotic distribution is characterized by one or two maxima, depending on the parameters of the model. The stationary profiles are shown for different values of the average opinion and for different parameters of the model.

We recognize in kinetic models the possibility of describing complex and decentralized systems that can exhibit interesting emergent behaviours. In particular, we are mainly interested in using kinetic models as a conceptual framework that captures essential characteristics of opinion formation in multi-agent systems, and to adopt it in the design of mobile scenarios that would eventually use general-purpose industrial strength technology (see, e.g., [13], [14]). Moreover, we recognize that kinetic models can be effectively used to model agent-based cooperation (like the ones discussed in, e.g., [15]), and that they can be used to study large scale systems (like the ones discussed in, e.g., [16]). Finally, we are interested in modeling the emergent behaviors of wireless sensor networks used to support accurate localization (see, e.g., [17], [18]).

Further investigation on this subject is currently under development. In particular, we are interested in deriving the explicit expressions of the stationary profiles with a different choice of the diffusion function. We aim at studying the properties of such stationary profiles for different parameters of the model. At the same time, we are studying the application of kinetic models to multi-agents systems also from a simulative point of view. More precisely, we are interested in comparing analytic results with simulation experiments and in studying the number of iterations necessary to approximate to a certain degree an analytic stationary profile.

References

[1] S. Monica and F. Bergenti, "A stochastic model of self-stabilizing cellular automata for consensus formation," in *Proceedings of* 15^{th} *Workshop "Dagli Oggetti agli Agenti"* (WOA 2014), Catania, Italy, September 2014.

[2] L. Pareschi and G. Toscani, *Interacting Multiagent Systems: Kinetic Equations and Montecarlo Methods*. Oxford: Oxford University Press, 2013.

[3] M. Groppi, S. Monica, and G. Spiga, "A kinetic ellipsoidal BGK model for a binary gas mixture," *EPL: Europhysics Letter*, vol. 96, December 2011.

[4] W. Weidlich, *Sociodynamics: a systematic approach to mathematical modelling in the social sciences*. Amsterdam: Harwood Academic Publisher, 2000.

[5] B. K. Chakraborti, A. Chakrabarti, and A. Chatterjee, *Econophysics and sociophysics: Trends and perspectives*. Berlin: Wiley, 2006.

[6] F. Slanina, "Inelastically scattering particles and wealth distribution in an open economy," *Physical Review E*, vol. 69, pp. 46–102, 2004.

[7] S. Cordier, L. Pareschi, and G. Toscani, "On a kinetic model for simple market economy," *Journal of Statistical Physics*, vol. 120, pp. 253–277, 2005.

[8] K. Sznajd-Weron and J. Sznajd, "Opinion evolution in closed community," *International Journal of Modern Physics C*, vol. 11, pp. 1157–1166, 2000.

[9] G. Toscani, "Kinetic models of opinion formation," *Communications in Mathematical Sciences*, vol. 4, pp. 481–496, 2006.

[10] E. Ben-Naim, "Opinion dynamics: Rise and fall of political parties," *Europhysics Letters*, vol. 69, pp. 671–677, 2005.

[11] G. Toscani, "One-dimensional kinetic models of granular flows," *ESAIM: Mathematical Modelling and Numerical Analysis*, vol. 34, pp. 1277–1291, 2000.

[12] G. Naldi, L. Pareschi, and G. Russo, *Introduzione al Calcolo Scientifico*. The Mcgraw-Hill Company, 2001.

[13] F. Bergenti, G. Caire, and D. Gotta, "Agents on the move: JADE for Android devices," in *Procs. Workshop From Objects to Agents*, 2014.

[14] F. Bergenti, G. Caire, and D. Gotta, "Agent-based social gaming with AMUSE," in *Procs.* 5^{th} *Int'l Conf. Ambient Systems, Networks and Technologies (ANT 2014) and* 4^{th} *Int'l Conf. Sustainable Energy Information Technology (SEIT 2014)*, ser. Procedia Computer Science. Elsevier, 2014, pp. 914–919.

[15] F. Bergenti, A. Poggi, and M. Somacher, "A collaborative platform for fixed and mobile networks," *Communications of the ACM*, vol. 45, no. 11, pp. 39–44, 2002.

[16] F. Bergenti, G. Caire, and D. Gotta, "Large-scale network and service management with WANTS," in *Industrial Agents: Emerging Applications of Software Agents in Industry*. Elsevier, 2015, pp. 231–246.

[17] S. Monica and G. Ferrari, "Accurate indoor localization with UWB wireless sensor networks," in *Procs.* 23^{rd} *IEEE International Conference on Enabling Technologies: Infrastructure for Collaborative Enterprises* (WETICE 2014), Parma, Italy, June 2014, pp. 287–289.

[18] S. Monica and G. Ferrari, "Swarm intelligent approaches to auto-localization of nodes in static UWB networks," *Applied Soft Computing*, vol. 25, pp. 426–434, December 2014.