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# The Distribution Semantics Is Well-Defined for All Normal Programs

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**Abstract.** The distribution semantics is an approach for integrating logic programming and probability theory that underlies many languages and has been successfully applied in many domains. When the program has function symbols, the semantics was defined for special cases: either the program has to be definite or the queries must have a finite number of finite explanations. In this paper we show that it is possible to define the semantics for all programs.

**Keywords:** Distribution Semantics, Function Symbols, ProbLog, Probabilistic Logic Programming

## 1 Introduction

The distribution semantics [1, 2] was successfully applied in many domains and underlies many languages that combine logic programming with probability theory such as Probabilistic Horn Abduction, Independent Choice Logic, PRISM, Logic Programs with Annotated Disjunctions and ProbLog.

The definition of the distribution semantics can be given quite simply in the case of no function symbols in the program: a probabilistic logic program under the distribution semantics defines a probability distribution over normal logic programs called *worlds* and the probability of a ground query can be obtained by marginalizing the joint distribution of the worlds and the query. In the case the program has function symbols, however, this simple definition does not work as the probability of individual worlds is zero.

A definition of the distribution semantics for programs with function symbols was proposed in [1, 3] but restricted to definite programs. The case of normal programs was taken into account in [4] where the semantics required that the programs are acyclic. A looser condition was proposed in [5] but still required each goal to have a finite set of finite explanations.

In this paper we show that the distribution semantics can be defined for all programs, thus also for programs that have goals with an infinite number of possibly infinite explanations. We do so by adapting the definition of the well-founded semantics in terms of iterated fixpoints of [6] to the case of ProbLog, similarly to the way in which the  $T_P$  operator has been adapted in [7] to the

case of stratified ProbLog programs using parameterized interpretations. In the case of infinite number of infinite explanations, we show that the probability of queries is defined in the limit and the limit always exists.

We consider the case of ProbLog but the results are equally applicable to all other languages under the distribution semantics, as there are linear transformations from one language to another that preserve the semantics.

The paper is organized as follows. Section 2 presents preliminary material on fixpoints and the well-founded semantics. Section 3 introduces the distribution semantics for programs without function symbols. Section 4 discusses the definition of the distribution semantics with function symbols in the case of finite set of finite explanations. Section 5 represents the main contribution of this paper and discusses the case of infinite set of infinite explanations. Finally, Section 6 concludes the paper. The proofs of the main results are reported in the Appendix.

## 2 Preliminaries

A relation on a set  $S$  is a *partial order* if it is reflexive, antisymmetric and transitive. In the following, let  $S$  be a set with a partial order  $\leq$ .  $a \in S$  is an *upper bound* of a subset  $X$  of  $S$  if  $x \leq a$  for all  $x \in X$ . Similarly,  $b \in S$  is a *lower bound* of  $X$  if  $b \leq x$  for all  $x \in X$ .

$a \in S$  is the *least upper bound* of a subset  $X$  of  $S$  if  $a$  is an upper bound of  $X$  and, for all upper bounds  $a'$  of  $X$ , we have  $a \leq a'$ . Similarly,  $b \in S$  is the *greatest lower bound* of a subset  $X$  of  $S$  if  $b$  is a lower bound of  $X$  and, for all lower bounds  $b'$  of  $X$ , we have  $b' \leq b$ . The least upper bound of  $X$  is unique, if it exists, and is denoted by  $\text{lub}(X)$ . Similarly, the greatest lower bound of  $X$  is unique, if it exists, and is denoted by  $\text{glb}(X)$ .

A partially ordered set  $L$  is a *complete lattice* if  $\text{lub}(X)$  and  $\text{glb}(X)$  exist for every subset  $X$  of  $L$ . We let  $\top$  denote the top element  $\text{lub}(L)$  and  $\perp$  denote the bottom element  $\text{glb}(L)$  of the complete lattice  $L$ .

Let  $L$  be a complete lattice and  $T : L \rightarrow L$  be a mapping. We say  $T$  is *monotonic* if  $T(x) \leq T(y)$ , whenever  $x \leq y$ . We say  $a \in L$  is the *least fixpoint* of  $T$  if  $a$  is a fixpoint (that is,  $T(a) = a$ ) and for all fixpoints  $b$  of  $T$  we have  $a \leq b$ . Similarly, we define *greatest fixpoint*.

Let  $L$  be a complete lattice and  $T : L \rightarrow L$  be monotonic. Then we define  $T \uparrow 0 = \perp$ ;  $T \uparrow \alpha = T(T \uparrow (\alpha - 1))$ , if  $\alpha$  is a successor ordinal;  $T \uparrow \alpha = \text{lub}(\{T \uparrow \beta \mid \beta < \alpha\})$ , if  $\alpha$  is a limit ordinal;  $T \downarrow 0 = \top$ ;  $T \downarrow \alpha = T(T \downarrow (\alpha - 1))$ , if  $\alpha$  is a successor ordinal;  $T \downarrow \alpha = \text{glb}(\{T \downarrow \beta \mid \beta < \alpha\})$ , if  $\alpha$  is a limit ordinal.

**Proposition 1.** *Let  $L$  be a complete lattice and  $T : L \rightarrow L$  be monotonic. Then  $T$  has a least fixpoint,  $\text{lfp}(T)$  and a greatest fixpoint  $\text{gfp}(T)$ .*

A normal program  $P$  is a set of normal rules. A normal rule has the form

$$r = h \leftarrow b_1, \dots, b_n, \text{not } c_1, \dots, \text{not } c_m \quad (1)$$

where  $h, b_1, \dots, b_n, c_1, \dots, c_m$  are atoms.

The set of ground atoms that can be built with the symbols of a program  $P$  is called the Herbrand base and is denoted as  $\mathcal{B}_P$ .

A *two-valued interpretation*  $I$  is a subset of  $\mathcal{B}_P$ .  $I$  is the set of true atoms. The set  $Int2$  of two-valued interpretations for a program  $P$  forms a complete lattice where the partial order  $\leq$  is given by the subset relation  $\subseteq$ . The least upper bound and greatest lower bound are defined as  $lub(X) = \bigcup_{I \in X} I$  and  $glb(X) = \bigcap_{I \in X} I$ . The bottom and top element are respectively  $\emptyset$  and  $\mathcal{B}_P$ .

A *three-valued interpretation*  $\mathcal{I}$  is a pair  $\langle I_T; I_F \rangle$  where  $I_T$  and  $I_F$  are subsets of  $\mathcal{B}_P$  and represent respectively the set of true and false atoms. The union of two three-valued interpretations  $\langle I_T, I_F \rangle$  and  $\langle J_T, J_F \rangle$  is defined as  $\langle I_T, I_F \rangle \cup \langle J_T, J_F \rangle = \langle I_T \cup J_T, I_F \cup J_F \rangle$ . The intersection of two three-valued interpretations  $\langle I_T, I_F \rangle$  and  $\langle J_T, J_F \rangle$  is defined as  $\langle I_T, I_F \rangle \cap \langle J_T, J_F \rangle = \langle I_T \cap J_T, I_F \cap J_F \rangle$ .

The set  $Int3$  of three-valued interpretations for a program  $P$  forms a complete lattice where the partial order  $\leq$  is defined as  $\langle I_T, I_F \rangle \leq \langle J_T, J_F \rangle$  if  $I_T \subseteq J_T$  and  $I_F \subseteq J_F$ . The least upper bound and greatest lower bound are defined as  $lub(X) = \bigcup_{I \in X} I$  and  $glb(X) = \bigcap_{I \in X} I$ . The bottom and top element are respectively  $\langle \emptyset, \emptyset \rangle$  and  $\langle \mathcal{B}_P, \mathcal{B}_P \rangle$ .

The well-founded semantics (WFS) assigns a three-valued model to a program, i.e., it identifies a three-valued interpretation as the meaning of the program. The WFS was given in [8] in terms of the least fixpoint of an operator that is composed by two sub-operators, one computing consequences and the other computing unfounded sets. We give here the alternative definition of the WFS of [6] that is based on a different iterated fixpoint.

**Definition 1.** For a normal program  $P$ , sets  $Tr$  and  $Fa$  of ground atoms, and a 3-valued interpretation  $\mathcal{I}$  we define

$$\begin{aligned} OpTrue_{\mathcal{I}}^P(Tr) &= \{a \mid a \text{ is not true in } \mathcal{I}; \text{ and there is a clause } b \leftarrow l_1, \dots, l_n \text{ in } P, \\ &\quad \text{a grounding substitution } \theta \text{ such that } a = b\theta \text{ and for every } 1 \leq i \leq n \text{ either} \\ &\quad l_i\theta \text{ is true in } \mathcal{I}, \text{ or } l_i\theta \in Tr\}; \\ OpFalse_{\mathcal{I}}^P(Fa) &= \{a \mid a \text{ is not false in } \mathcal{I}; \text{ and for every clause } b \leftarrow l_1, \dots, l_n \text{ in } P \\ &\quad \text{and grounding substitution } \theta \text{ such that } a = b\theta \text{ there is some } i \text{ (} 1 \leq i \leq n \text{)} \\ &\quad \text{such that } l_i\theta \text{ is false in } \mathcal{I} \text{ or } l_i\theta \in Fa\}. \end{aligned}$$

In words, the operator  $OpTrue_{\mathcal{I}}^P$  extends the interpretation  $\mathcal{I}$  to add the new atomic facts that can be derived from  $P$  knowing  $\mathcal{I}$ , while  $OpFalse_{\mathcal{I}}^P$  adds the new negations of atomic facts that can be shown false in  $P$  by knowing  $\mathcal{I}$ .  $OpTrue_{\mathcal{I}}^P$  and  $OpFalse_{\mathcal{I}}^P$  are both monotonic [6], so they both have a least and greatest fixpoints. An iterated fixpoint operator builds up *dynamic strata* by constructing successive three-valued interpretations as follows.

**Definition 2 (Iterated Fixed Point).** For a normal program  $P$ , let  $IFP^P : Int3 \rightarrow Int3$  be defined as  $IFP^P(\mathcal{I}) = \mathcal{I} \cup \langle lfp(OpTrue_{\mathcal{I}}^P), gfp(OpFalse_{\mathcal{I}}^P) \rangle$ .

$IFP^P$  is monotonic [6] and thus as a least fixed point  $lfp(IFP^P)$ . Moreover, the well-founded model  $WFM(P)$  of  $P$  is in fact  $lfp(IFP^P)$ . Let  $\delta$  be the smallest ordinal such that  $WFM(P) = IFP^P \uparrow \delta$ . We refer to  $\delta$  as the *depth* of  $P$ . The *stratum* of atom  $a$  is the least ordinal  $\beta$  such that  $a \in IFP^P \uparrow \beta$  (where  $a$  may

be either in the true or false component of  $IFP^P \uparrow \beta$ ). Undefined atoms of the well-founded model do not belong to any stratum – i.e. they are not added to  $IFP^P \uparrow \delta$  for any ordinal  $\delta$ .

### 3 The Distribution Semantics for Programs without Function Symbols

We present the distribution semantics for the case of ProbLog [9] as it is the language with the simplest syntax. A ProbLog program  $\mathcal{P}$  is composed by a set of normal rules  $\mathcal{R}$  and a set  $\mathcal{F}$  of probabilistic facts. Each *probabilistic fact* is of the form  $p_i :: a_i$  where  $p_i \in [0, 1]$  and  $a_i$  is an atom, meaning that each ground instantiation  $a_i\theta$  of  $a_i$  is true with probability  $p_i$  and false with probability  $1 - p_i$ . Each world is obtained by selecting or rejecting each grounding of all probabilistic facts.

An *atomic choice* is the selection or not of grounding  $F\theta$  of a probabilistic fact  $F$ . It is represented with the triple  $(F, \theta, i)$  where  $i \in \{0, 1\}$ . A set  $\kappa$  of atomic choices is *consistent* if it does not contain two atomic choices  $(F, \theta, i)$  and  $(F, \theta, j)$  with  $i \neq j$  (only one alternative is selected for a ground probabilistic fact). The function *consistent*( $\kappa$ ) returns true if  $\kappa$  is consistent. A *composite choice*  $\kappa$  is a consistent set of atomic choices. The probability of composite choice  $\kappa$  is  $P(\kappa) = \prod_{(F_i, \theta, 1) \in \kappa} p_i \prod_{(F_i, \theta, 0) \in \kappa} 1 - p_i$  where  $p_i$  is the probability of the  $i$ -th probabilistic fact  $F_i$ . A *selection*  $\sigma$  is a total composite choice, i.e., it contains one atomic choice for every grounding of each probabilistic fact. A *world*  $w_\sigma$  is a logic program that is identified by a selection  $\sigma$ . The world  $w_\sigma$  is formed by including the atom corresponding to each atomic choice  $(F, \theta, 1)$  of  $\sigma$ .

The probability of a world  $w_\sigma$  is  $P(w_\sigma) = P(\sigma)$ . Since in this section we are assuming programs without function symbols, the set of groundings of each probabilistic fact is finite, and so is the set of worlds  $W_{\mathcal{P}}$ . Accordingly, for a ProbLog program  $\mathcal{P}$ ,  $W_{\mathcal{P}} = \{w_1, \dots, w_m\}$ . Moreover,  $P(w)$  is a distribution over worlds:  $\sum_{w \in W_{\mathcal{P}}} P(w) = 1$ . We call *sound* a program for which every world has a two-valued well-founded model. We consider only sound programs, as the uncertainty should be handled by the choices rather than by the semantics of negation.

Let  $q$  be a query in the form of a ground atom. We define the conditional probability of  $q$  given a world  $w$  as:  $P(q|w) = 1$  if  $q$  is true in  $w$  and 0 otherwise. Since the program is sound,  $q$  can be only true or false in a world. The probability of  $q$  can thus be computed by summing out the worlds from the joint distribution of the query and the worlds:  $P(q) = \sum_w P(q, w) = \sum_w P(q|w)P(w) = \sum_{w \models q} P(w)$ .

### 4 The Distribution Semantics for Programs with Function Symbols

When a program contains functions symbols there is the possibility that its grounding may be infinite. If so, the number of atomic choices in a selection that

defines a world is countably infinite and there is an uncountably infinite number of worlds. In this case, the probability of each individual world is zero since it is the product of infinite numbers all smaller than one. So the semantics of Section 3 is not well-defined.

*Example 1.* Consider the program

$$\begin{array}{lll} p(0) \leftarrow u(0). & t \leftarrow \neg s. & F_1 = a :: u(X). \\ p(s(X)) \leftarrow p(X), u(X). & s \leftarrow r, \neg q. & F_2 = b :: r. \\ & q \leftarrow u(X). & \end{array}$$

The set of worlds is infinite and uncountable. In fact, each world can be put in a one to one relation with a selection and a selection can be represented as a countable sequence of atomic choices of which the first involves fact  $F_2$ , the second  $F_1/\{X/0\}$ , the third  $F_1/\{X/s(0)\}$  and so on. The set of selections can be shown uncountable by Cantor's diagonal argument. Suppose the set of selections is countable. Then the selections could be listed in order, suppose from top to bottom. Suppose the atomic choices of each selection are listed from left to right. We can pick a composite choice that differs from the first selection in the first atomic choice (if  $(F_2, \emptyset, k)$  is the first atomic choice of the first selection, pick  $(F_2, \emptyset, 1 - k)$ ), from the second selection in the second atomic choice (similar to the case of the first atomic choice) and so on. In this way we have obtained a selection that is not present in the list because it differs from each selection in the list for at least an atomic choice. So it is not possible to list the selections in order.

We now present the definition of the distribution semantics for programs with function symbols following [4]. The semantics for a probabilistic logic program  $\mathcal{P}$  with function symbols is given by defining a *probability measure*  $\mu$  over the set of worlds  $W_{\mathcal{P}}$ . Informally,  $\mu$  assigns a probability to a set of *subsets* of  $W_{\mathcal{P}}$ , rather than to every element of (the infinite set)  $W_{\mathcal{P}}$ . The approach dates back to [10] who defined a probability measure  $\mu$  as a real-valued function whose domain is a  $\sigma$ -algebra  $\Omega$  on a set  $\mathcal{W}$  called the *sample space*. Together  $\langle \mathcal{W}, \Omega, \mu \rangle$  is called a *probability space*.

**Definition 3.** [11, Section 3.1] *The set  $\Omega$  of subsets of  $\mathcal{W}$  is a  $\sigma$ -algebra on the set  $\mathcal{W}$  iff ( $\sigma$ -1)  $\mathcal{W} \in \Omega$ ; ( $\sigma$ -2)  $\Omega$  is closed under complementation, i.e.,  $\omega \in \Omega \rightarrow (\mathcal{W} \setminus \omega) \in \Omega$ ; and ( $\sigma$ -3)  $\Omega$  is closed under countable union, i.e., if  $\omega_i \in \Omega$  for  $i = 1, 2, \dots$  then  $\bigcup_i \omega_i \in \Omega$ .*

The elements of  $\Omega$  are called *measurable sets*. Importantly, for defining the distribution semantics for programs with function symbols, not every subset of  $\mathcal{W}$  need be present in  $\Omega$ .

**Definition 4.** [10] *Given a sample space  $\mathcal{W}$  and a  $\sigma$ -algebra  $\Omega$  of subsets of  $\mathcal{W}$ , a probability measure is a function  $\mu : \Omega \rightarrow \mathbb{R}$  that satisfies the following axioms: ( $\mu$ -1)  $\mu(\omega) \geq 0$  for all  $\omega \in \Omega$ ; ( $\mu$ -2)  $\mu(\mathcal{W}) = 1$ ; ( $\mu$ -3)  $\mu$  is countably additive, i.e., if  $O = \{\omega_1, \omega_2, \dots\} \subseteq \Omega$  is a countable collection of pairwise disjoint sets, then  $\mu(\bigcup_{\omega \in O} \omega) = \sum_i \mu(\omega_i)$ .*

We first consider the finite additivity version of probability spaces. In this stronger version, the  $\sigma$ -algebra is replaced by an algebra.

**Definition 5.** [11, Section 3.1] *The set  $\Omega$  of subsets of  $\mathcal{W}$  is an algebra on the set  $\mathcal{W}$  iff it respects conditions ( $\sigma$ -1), ( $\sigma$ -2) and condition (a-3):  $\Omega$  is closed under finite union, i.e.,  $\omega_1 \in \Omega, \omega_2 \in \Omega \rightarrow (\omega_1 \cup \omega_2) \in \Omega$*

The probability measure is replaced by a finitely additive probability measure.

**Definition 6.** *Given a sample space  $\mathcal{W}$  and an algebra  $\Omega$  of subsets of  $\mathcal{W}$ , a finitely additive probability measure is a function  $\mu : \Omega \rightarrow \mathbb{R}$  that satisfies axioms ( $\mu$ -1) and ( $\mu$ -2) of Definition 4 and axiom (m-3):  $\mu$  is finitely additive, i.e.,  $\omega_1 \cap \omega_2 = \emptyset \rightarrow \mu(\omega_1 \cup \omega_2) = \mu(\omega_1) + \mu(\omega_2)$  for all  $\omega_1, \omega_2 \in \Omega$ .*

Towards defining a suitable algebra given a probabilistic logic program  $\mathcal{P}$ , we define the *set of worlds  $\omega_\kappa$  compatible with a composite choice  $\kappa$*  as  $\omega_\kappa = \{w_\sigma \in W_{\mathcal{P}} | \kappa \subseteq \sigma\}$ . Thus a composite choice identifies a set of worlds. For programs without function symbols  $P(\kappa) = \sum_{w \in \omega_\kappa} P(w)$ .

Given a set of composite choices  $K$ , the *set of worlds  $\omega_K$  compatible with  $K$*  is  $\omega_K = \bigcup_{\kappa \in K} \omega_\kappa$ . Two composite choices  $\kappa_1$  and  $\kappa_2$  are *incompatible* if their union is not consistent. A set  $K$  of composite choices is *pairwise incompatible* if for all  $\kappa_1 \in K, \kappa_2 \in K, \kappa_1 \neq \kappa_2$  implies that  $\kappa_1$  and  $\kappa_2$  are incompatible.

Regardless of whether a probabilistic logic program has a finite number of worlds or not, obtaining pairwise incompatible sets of composite choices is an important problem. This is because the *probability of a pairwise incompatible set  $K$  of composite choices* is defined as  $P(K) = \sum_{\kappa \in K} P(\kappa)$  which is easily computed. Two sets  $K_1$  and  $K_2$  of finite composite choices are *equivalent* if they correspond to the same set of worlds:  $\omega_{K_1} = \omega_{K_2}$ .

One way to assign probabilities to a set  $K$  of composite choices is to construct an equivalent set that is pairwise incompatible; such a set can be constructed through the technique of *splitting*. More specifically, if  $F\theta$  is an instantiated fact and  $\kappa$  is a composite choice that does not contain an atomic choice  $(F, \theta, k)$  for any  $k$ , the *split* of  $\kappa$  on  $F\theta$  is the set of composite choices  $S_{\kappa, F\theta} = \{\kappa \cup \{(F, \theta, 0)\}, \kappa \cup \{(F, \theta, 1)\}\}$ . It is easy to see that  $\kappa$  and  $S_{\kappa, F\theta}$  identify the same set of possible worlds, i.e., that  $\omega_\kappa = \omega_{S_{\kappa, F\theta}}$ , and that  $S_{\kappa, F\theta}$  is pairwise incompatible. The technique of splitting composite choices on formulas is used for the following result [12].

**Theorem 1** (*Existence of a pairwise incompatible set of composite choices [12]*) *Given a finite set  $K$  of composite choices, there exists a finite set  $K'$  of pairwise incompatible composite choices such that  $K$  and  $K'$  are equivalent.*

**Proof:** Given a finite set of composite choices  $K$ , there are two possibilities to form a new set  $K'$  of composite choices so that  $K$  and  $K'$  are equivalent:

1. **removing dominated elements:** if  $\kappa_1, \kappa_2 \in K$  and  $\kappa_1 \subset \kappa_2$ , let  $K' = K \setminus \{\kappa_2\}$ .
2. **splitting elements:** if  $\kappa_1, \kappa_2 \in K$  are compatible (and neither is a superset of the other), there is a  $(F, \theta, k) \in \kappa_1 \setminus \kappa_2$ . We replace  $\kappa_2$  by the split of  $\kappa_2$  on  $F\theta$ . Let  $K' = K \setminus \{\kappa_2\} \cup S_{\kappa_2, F\theta}$ .

In both cases  $\omega_K = \omega_{K'}$ . If we repeat this two operations until neither is applicable we obtain a splitting algorithm that terminates because  $K$  is a finite set of composite choices. The resulting set  $K'$  is pairwise incompatible and is equivalent to the original set.  $\diamond$

**Theorem 2** (*Equivalence of the probability of two equivalent pairwise incompatible finite set of finite composite choices [13]*) *If  $K_1$  and  $K_2$  are both pairwise incompatible finite sets of finite composite choices such that they are equivalent then  $P(K_1) = P(K_2)$ .*

For a probabilistic logic program  $\mathcal{P}$ , we can thus define a unique probability measure  $\mu : \Omega_{\mathcal{P}} \rightarrow [0, 1]$  where  $\Omega_{\mathcal{P}}$  is defined as the set of sets of worlds identified by finite sets of finite composite choices:  $\Omega_{\mathcal{P}} = \{\omega_K | K \text{ is a finite set of finite composite choices}\}$ .  $\Omega_{\mathcal{P}}$  is an algebra over  $W_{\mathcal{P}}$  since  $W_{\mathcal{P}} = \omega_K$  with  $K = \{\emptyset\}$ . Moreover, the complement  $\omega_K^c$  of  $\omega_K$  where  $K$  is a finite set of finite composite choice is  $\omega_{\bar{K}}$  where  $\bar{K}$  is a finite set of finite composite choices. In fact,  $\bar{K}$  can be obtained with the function  $duals(K)$  of [12] that performs Reiter's hitting set algorithm over  $K$ , generating an element  $\kappa$  of  $\bar{K}$  by picking an atomic choice  $(F, \theta, k)$  from each element of  $K$  and inserting in  $\kappa$  the atomic choice  $(F, \theta, 1 - k)$ . After this process is performed in all possible ways, inconsistent sets of atom choices are removed obtaining  $\bar{K}$ . Since the possible choices of the atomic choices are finite, so is  $\bar{K}$ . Finally, condition (a-3) holds since the union of  $\omega_{K_1}$  with  $\omega_{K_2}$  is equal to  $\omega_{K_1 \cup K_2}$  for the definition of  $\omega_K$ .

The corresponding measure  $\mu$  is defined by  $\mu(\omega_K) = P(K')$  where  $K'$  is a pairwise incompatible set of composite choices equivalent to  $K$ .  $\langle W_{\mathcal{P}}, \Omega_{\mathcal{P}}, \mu \rangle$  is a finitely additive probability space according to Definition 6 because  $\mu(\omega_{\{\emptyset\}}) = 1$ ,  $\mu(\omega_K) \geq 0$  for all  $K$  and if  $\omega_{K_1} \cap \omega_{K_2} = \emptyset$  and  $K'_1$  ( $K'_2$ ) is pairwise incompatible and equivalent to  $K_1$  ( $K_2$ ), then  $K'_1 \cup K'_2$  is pairwise incompatible and

$$\mu(\omega_{K_1 \cup K_2}) = \sum_{\kappa \in K'_1 \cup K'_2} P(\kappa) = \sum_{\kappa_1 \in K'_1} P(\kappa_1) + \sum_{\kappa_2 \in K'_2} P(\kappa_2) = \mu(\omega_{K_1}) + \mu(\omega_{K_2}).$$

Given a query  $q$ , a composite choice  $\kappa$  is an *explanation* for  $q$  if  $\forall w \in \omega_{\kappa} : w \models q$ . A set  $K$  of composite choices is *covering* wrt  $q$  if every world in which  $q$  is true belongs to  $\omega_K$

**Definition 7.** *For a probabilistic logic program  $\mathcal{P}$ , the probability of a ground atom  $q$  is given by  $P(q) = \mu(\{w | w \in W_{\mathcal{P}}, w \models q\})$ .*

If  $q$  has a finite set  $K$  of finite explanations such that  $K$  is covering then  $\{w | w \in W_{\mathcal{P}} \wedge w \models q\} = \omega_K \in \Omega_{\mathcal{P}}$  and we say that  $P(q)$  is *finitely well-defined* for the distribution semantics. A program  $\mathcal{P}$  is *finitely well-defined* if the probability of all ground atoms in the grounding of  $\mathcal{P}$  is finitely well-defined.

*Example 2.* Consider the program of Example 1. The set  $K = \{\kappa\}$  with  $\kappa = \{(F_1, \{X/0\}, 1), (F_1, \{X/s(0)\}, 1)\}$  is a pairwise incompatible finite set of finite explanations that are covering for the query  $p(s(0))$ . Definition 7 therefore applies, and  $P(p(s(0))) = P(\kappa) = a^2$

## 5 Infinite Covering Set of Explanations

In this section we go beyond [4] and we remove the requirement of the finiteness of the covering set of explanations and of each explanation for a query  $q$ .

*Example 3.* In Example 1, the query  $s$  has the pairwise incompatible covering set of explanations  $K^s = \{\kappa_0^s, \kappa_1^s, \dots\}$  with

$$\kappa_i^s = \{(F_2, \emptyset, 1), (F_1, \{X/0\}, 1), \dots, (F_1, \{X/s^{i-1}(0)\}, 1), (F_1, \{X/s^i(0)\}, 0)\}$$

where  $s^i(0)$  is the term where the functor  $s$  is applied  $i$  times to 0. So  $K^s$  is countable and infinite. A covering set of explanation for  $t$  is  $K^t = \{(F_2, \emptyset, 0)\}, \kappa^t\}$  where  $\kappa^t$  is the infinite composite choice

$$\kappa^t = \{(F_2, \emptyset, 1), (F_1, \{X/0\}, 1), (F_1, \{X/s(0)\}, 1), \dots\}$$

For a probabilistic logic program  $\mathcal{P}$ , we can define the probability measure  $\mu : \Omega_{\mathcal{P}} \rightarrow [0, 1]$  where  $\Omega_{\mathcal{P}}$  is defined as the set of sets of worlds identified by countable sets of countable composite choices:  $\Omega_{\mathcal{P}} = \{\omega_K | K \text{ is a countable set of countable composite choices}\}$ .

**Lemma 3**  $\Omega_{\mathcal{P}}$  is a  $\sigma$ -algebra over  $W_{\mathcal{P}}$ .

**Proof:** ( $\sigma$ -1) is true as in the algebra case. To see that the complement  $\omega_K^c$  of  $\omega_K$  is in  $\Omega_{\mathcal{P}}$ , let us prove by induction that the dual  $\overline{K}$  of  $K$  is a countable set of countable composite choices and then that  $\omega_K^c = \omega_{\overline{K}}$ . In the base case, if  $K_1 = \{\kappa_1\}$ , then we can obtain  $\overline{K}_1$  by picking each atomic choice  $(F, \theta, k)$  of  $\kappa_1$  and inserting in  $\overline{K}_1$  the composite choice  $\{(F, \theta, 1 - k)\}$ . As there is a finite or countable number of atomic choices in  $\kappa_1$ ,  $\overline{K}_1$  is a finite or countable set of composite choices each with one atomic choice.

In the inductive case, assume that  $K_{n-1} = \{\kappa_1, \dots, \kappa_{n-1}\}$  and that  $\overline{K_{n-1}}$  is a finite or countable set of composite choices. Let  $K_n = K_{n-1} \cup \{\kappa_n\}$  and  $\overline{K_{n-1}} = \{\kappa'_1, \kappa'_2, \dots\}$ . We can obtain  $\overline{K_n}$  by picking each  $\kappa'_i$  and each atomic choice  $(F, \theta, k)$  of  $\kappa_n$ . If  $(F, \theta, k) \in \kappa'_i$ , then discard  $\kappa'_i$ , else if  $(F, \theta, k') \in \kappa'_i$  with  $k' \neq k$ , insert  $\kappa'_i$  in  $\overline{K_n}$ . Otherwise generate the composite choice  $\kappa''_i$  where  $\kappa''_i = \kappa'_i \cup \{(F, \theta, 1 - k)\}$  and insert it in  $\overline{K_n}$ . Doing this for all atomic choices  $(F, \theta, k)$  in  $\kappa_n$  generates a finite set of composite choices if  $\kappa_n$  is finite and a countable number if  $\kappa_n$  is countable. Doing this for all  $\kappa'_i$  we obtain that  $\overline{K_n}$  is a countable union of countable sets which is a countable set [14, page 3].  $\omega_K^c = \omega_{\overline{K}}$  because all composite choices of  $\overline{K}$  are incompatible with each world of  $\omega_K$ , as they are incompatible with each composite choice of  $K$ . So  $\omega_K^c \in \Omega_{\mathcal{P}}$ . ( $\sigma$ -3) is true as in the algebra case.  $\diamond$

We can see  $K$  as  $\lim_{n \rightarrow \infty} K_n$  where  $K_n = \{\kappa_1, \dots, \kappa_n\}$ . Each  $K_n$  is a finite set of composite choices and we can compute an equivalent finite pairwise incompatible set of composite choices  $K'_n$ . For each  $K'_n$  we can compute the probability  $P(K'_n)$ , noting that the probability of infinite composite choices is 0.

Now consider  $\lim_{n \rightarrow \infty} P(K'_n)$ . We can see the  $P(K'_n)$ s as the partial sums of a series. Moreover, it can be shown that  $P(K'_{n-1}) \leq P(K'_n)$  so the series



has non-negative terms. Such a series converges if the sequence of partial sums is bounded from above [15, page 92]. Since  $P(K'_n)$  is bounded by 1, the limit  $\lim_{n \rightarrow \infty} P(K'_n)$  exists. So we can define measure  $\mu$  as  $\mu(\omega_K) = \lim_{n \rightarrow \infty} P(K'_n)$ .

**Theorem 4**  $\langle W_{\mathcal{P}}, \Omega_{\mathcal{P}}, \mu \rangle$  is a probability space according to Definition 4.

**Proof:**  $(\mu-1)$  and  $(\mu-2)$  hold as for the finite case and for  $(\mu-3)$  let  $O = \{\omega_{L_1}, \omega_{L_2}, \dots\}$  be a countable set of subsets of  $\Omega_{\mathcal{P}}$  such that the  $\omega_{L_i}$ s are pairwise disjoint. Let  $L'_i$  be the pairwise incompatible set equivalent to  $L_i$  and let  $\mathcal{L}$  be  $\bigcup_{i=1}^{\infty} L'_i$ . Since the  $\omega_{L_i}$ s are pairwise disjoint, then  $\mathcal{L}$  is pairwise incompatible.  $\Omega_{\mathcal{P}}$  is a  $\sigma$ -algebra, so  $\mathcal{L}$  is countable. Let  $\mathcal{K}$  be  $\{\kappa_1, \kappa_2, \dots\}$  and let  $K'_n$  be  $\{\kappa_1, \dots, \kappa_n\}$ . Then  $\mu(O) = \lim_{n \rightarrow \infty} P(K'_n) = \lim_{n \rightarrow \infty} \sum_{\kappa \in K'_n} P(\kappa) = \sum_{\kappa \in \mathcal{L}} P(\kappa)$ . Since  $\mathcal{L} = \bigcup_{i=1}^{\infty} L'_i$ , by rearranging the terms in the last summation we get  $\mu(O) = \sum_{\kappa \in \mathcal{L}} P(\kappa) = \sum_{n=1}^{\infty} P(L'_n) = \sum_{n=1}^{\infty} \mu(\omega_{L_n})$ .  $\diamond$

For a probabilistic logic program  $\mathcal{P}$ , the probability of a ground atom  $q$  is again given by  $P(q) = \mu(\{w | w \in W_{\mathcal{P}}, w \models q\})$ . If  $q$  has a countable set  $K$  of explanations such that  $K$  is covering then  $\{w | w \in W_{\mathcal{P}} \wedge w \models q\} = \omega_K \in \Omega_{\mathcal{P}}$  and we say that  $P(q)$  is *well-defined* for the distribution semantics. A program  $\mathcal{P}$  is well-defined if the probability of all ground atoms in the grounding of  $\mathcal{P}$  is well-defined.

*Example 4.* Consider Example 3. Since the explanations in  $K^s$  are pairwise incompatible the probability of  $s$  can be computed as

$$P(s) = b(1-a) + ba(1-a) + ba^2(1-a) + \dots = \frac{b(1-a)}{1-a} = b.$$

since the sum is a geometric series.  $K^t$  is also pairwise incompatible and  $P(\kappa^t) = 0$  so  $P(t) = 1 - b + 0 = 1 - b$  which is what we intuitively expect.

We now want to show that every program has countable set of countable explanations that is covering for each query. In the following, we consider only ground programs that however may be countably infinite, thus they can be the result of grounding a program with function symbols.

Given two sets of composite choices  $K_1$  and  $K_2$ , define the conjunction  $K_1 \otimes K_2$  of  $K_1$  and  $K_2$  as  $K_1 \otimes K_2 = \{\kappa_1 \cup \kappa_2 | \kappa_1 \in K_1, \kappa_2 \in K_2, \text{consistent}(\kappa_1 \cup \kappa_2)\}$

Similarly to [7], we define parametrized interpretations and a *IFPC* $_{\mathcal{P}}$  operator. Differently from [7], here parametrized interpretations associate a set of composite choices instead of a Boolean formula to each atom.

**Definition 8 (Parameterized two-valued interpretations).** A *parameterized positive two-valued interpretation*  $Tr$  of a ground probabilistic logic program  $\mathcal{P}$  with atoms  $\mathcal{B}_{\mathcal{P}}$  is a set of pairs  $(a, K_a)$  with  $a \in \text{atoms}$  and  $K_a$  a set of composite choices. A *parameterized negative two-valued interpretation*  $Fa$  of a ground probabilistic logic program  $\mathcal{P}$  with atoms  $\mathcal{B}_{\mathcal{P}}$  is a set of pairs  $(a, K_{-a})$  with  $a \in \mathcal{B}_{\mathcal{P}}$  and  $K_{-a}$  a set of composite choices.

Parametrized two-valued interpretations form a complete lattice where the partial order is defined as  $I \leq J$  if  $\forall (a, K_a) \in I, (a, L_a) \in J : \omega_{K_a} \subseteq \omega_{L_a}$ . The

least upper bound and greatest lower bound always exist and are  $\text{lub}(X) = \{(a, \bigcup_{(a, K_a) \in I, I \in X} K_a) | a \in \mathcal{B}_{\mathcal{P}}\}$  and  $\text{glb}(X) = \{(a, \bigotimes_{(a, K_a) \in I, I \in X} K_a) | a \in \mathcal{B}_{\mathcal{P}}\}$ . The top element  $\top$  is  $\{(a, \{\emptyset\}) | a \in \mathcal{B}_{\mathcal{P}}\}$  and the bottom element  $\perp$  is  $\{(a, \emptyset) | a \in \mathcal{B}_{\mathcal{P}}\}$ .

**Definition 9 (Parameterized three-valued interpretation).** A parameterized three-valued interpretation  $\mathcal{I}$  of a ground probabilistic logic program  $\mathcal{P}$  with atoms  $\mathcal{B}_{\mathcal{P}}$  is a set of triples  $(a, K_a, K_{\neg a})$  with  $a \in \mathcal{B}_{\mathcal{P}}$  and  $K_a$  and  $K_{\neg a}$  sets of composite choices.

Parameterized three-valued interpretations form a complete lattice where the partial order is defined as  $I \leq J$  if  $\forall (a, K_a, K_{\neg a}) \in I, (a, L_a, L_{\neg a}) \in J : \omega_{K_a} \subseteq \omega_{L_a}$  and  $\omega_{K_{\neg a}} \subseteq \omega_{L_{\neg a}}$ . The least upper bound and greatest lower bound always exist and are  $\text{lub}(X) = \{(a, \bigcup_{(a, K_a, K_{\neg a}) \in I, I \in X} K_a, \bigcup_{(a, K_a, K_{\neg a}) \in I, I \in X} K_{\neg a}) | a \in \mathcal{B}_{\mathcal{P}}\}$  and  $\text{glb}(X) = \{(a, \bigotimes_{(a, K_a, K_{\neg a}) \in I, I \in X} K_a, \bigotimes_{(a, K_a, K_{\neg a}) \in I, I \in X} K_{\neg a}) | a \in \mathcal{B}_{\mathcal{P}}\}$ . The top element  $\top$  is  $\{(a, \{\emptyset\}, \{\emptyset\}) | a \in \mathcal{B}_{\mathcal{P}}\}$ , the bottom element  $\perp$  is  $\{(a, \emptyset, \emptyset) | a \in \mathcal{B}_{\mathcal{P}}\}$ .

**Definition 10.** For a ground program  $\mathcal{P}$ , a two-valued parametrized positive interpretation  $\text{Tr}$  with pairs  $(a, L_a)$ , a two-valued parametrized negative interpretation  $\text{Fa}$  with pairs  $(a, M_{\neg a})$  and a three-valued parametrized interpretation  $\mathcal{I}$  with triples  $(a, K_a, K_{\neg a})$ , we define  $\text{OpTrueC}_{\mathcal{I}}^{\mathcal{P}}(\text{Tr}) = \{(a, L'_a) | a \in \mathcal{B}_{\mathcal{P}}\}$  where

$$L'_a = \begin{cases} \{(a, \emptyset, 1)\} & \text{if } a \in \mathcal{F} \\ \bigcup_{a \leftarrow b_1, \dots, b_n, \neg c_1, \dots, c_m \in \mathcal{R}} ((L_{b_1} \cup K_{b_1}) \otimes \dots \otimes (L_{b_n} \cup K_{b_n}) \otimes K_{\neg c_1} \otimes \dots \otimes K_{\neg c_m}) & \text{if } a \in \mathcal{B}_{\mathcal{P}} \setminus \mathcal{F} \end{cases}$$

$\text{OpFalseC}_{\mathcal{I}}^{\mathcal{P}}(\text{Fa}) = \{(a, M'_a) | a \in \mathcal{B}_{\mathcal{P}}\}$  where

$$M'_{\neg a} = \begin{cases} \{(a, \emptyset, 0)\} & \text{if } a \in \mathcal{F} \\ \bigotimes_{a \leftarrow b_1, \dots, b_n, \neg c_1, \dots, c_m \in \mathcal{R}} ((M_{\neg b_1} \otimes K_{\neg b_1}) \cup \dots \cup (M_{\neg b_n} \otimes K_{\neg b_n}) \cup K_{c_1} \cup \dots \cup K_{c_m}) & \text{if } a \in \mathcal{B}_{\mathcal{P}} \setminus \mathcal{F} \end{cases}$$

**Proposition 5**  $\text{OpTrueC}_{\mathcal{I}}^{\mathcal{P}}$  and  $\text{OpFalseC}_{\mathcal{I}}^{\mathcal{P}}$  are monotonic.

Since  $\text{OpTrueC}_{\mathcal{I}}^{\mathcal{P}}$  and  $\text{OpFalseC}_{\mathcal{I}}^{\mathcal{P}}$  are monotonic, they have a least fixpoint and a greatest fixpoint.

**Definition 11 (Iterated Fixed Point).** For a ground program  $\mathcal{P}$ , let  $\text{IFPC}^{\mathcal{P}}$  be defined as  $\text{IFPC}^{\mathcal{P}}(\mathcal{I}) = \{(a, K_a, K_{\neg a}) | (a, K_a) \in \text{lfp}(\text{OpTrueC}_{\mathcal{I}}^{\mathcal{P}}), (a, K_{\neg a}) \in \text{lfp}(\text{OpFalseC}_{\mathcal{I}}^{\mathcal{P}})\}$ .

**Proposition 6**  $\text{IFPC}^{\mathcal{P}}$  is monotonic.

So  $\text{IFPC}^{\mathcal{P}}$  has a least fixpoint. Let  $\text{WFMC}(\mathcal{P})$  denote  $\text{lfp}(\text{IFPC}^{\mathcal{P}})$ , and let  $\delta$  the smallest ordinal such that  $\text{IFPC}^{\mathcal{P}} \uparrow \delta = \text{WFMC}(\mathcal{P})$ . We refer to  $\delta$  as the depth of  $\mathcal{P}$ .

**Theorem 7** For a ground probabilistic logic program  $\mathcal{P}$  with atoms  $\mathcal{B}_{\mathcal{P}}$ , let  $K_a^\alpha$  and  $K_{\neg a}^\alpha$  be the formulas associated with atom  $a$  in  $IFPC^{\mathcal{P}} \uparrow \alpha$ . For every atom  $a$  and total choice  $\sigma$ , there is an iteration  $\alpha_0$  such that for all  $\alpha > \alpha_0$  we have:

$$w_\sigma \in \omega_{K_a^\alpha} \leftrightarrow WFM(w_\sigma) \models a \quad w_\sigma \in \omega_{K_{\neg a}^\alpha} \leftrightarrow WFM(w_\sigma) \models \neg a$$

**Theorem 8** For a ground probabilistic logic program  $\mathcal{P}$ , let  $K_a^\alpha$  and  $K_{\neg a}^\alpha$  be the formulas associated with atom  $a$  in  $IFPC^{\mathcal{P}} \uparrow \alpha$ . For every atom  $a$  and every iteration  $\alpha$ ,  $K_a^\alpha$  and  $K_{\neg a}^\alpha$  are countable sets of countable composite choices.

So every query for every program has a countable set of countable explanations that is covering and the probability measure is well defined. Moreover, since the program is sound, for all atoms  $a$ ,  $\omega_{K_a^\delta} = \omega_{K_{\neg a}^\delta}^c$  where  $\delta$  is the depth of the program, as in each world  $a$  is either true or false.

### 5.1 Comparison with Sato and Kameya's Definition

Sato and Kameya [3] define the distribution semantics for definite programs. They build a probability measure on the sample space  $W_{\mathcal{P}}$  from a collection of finite distributions. Let  $\mathcal{F}$  be  $\{F_1, F_2, \dots\}$  and let  $X_i$  be a random variable associated to  $F_i$  whose domain is  $\{0, 1\}$ .

The finite distributions  $P_{\mathcal{P}}^{(n)}(X_1 = k_1, \dots, X_n = k_n)$  for  $n \geq 1$  must be such that

$$\begin{cases} 0 \leq P_{\mathcal{P}}^{(n)}(X_1 = k_1, \dots, X_n = k_n) \leq 1 \\ \sum_{k_1, \dots, k_n} P_{\mathcal{P}}^{(n)}(X_1 = k_1, \dots, X_n = k_n) = 1 \\ \sum_{k_{n+1}} P_{\mathcal{P}}^{(n+1)}(X_1 = k_1, \dots, X_{n+1} = k_{n+1}) = P_{\mathcal{P}}^{(n)}(X_1 = k_1, \dots, X_n = k_n) \end{cases} \quad (2)$$

The last equation is called the compatibility condition. It can be proved [16] from the compatibility condition that there exists a probability space  $(W_{\mathcal{P}}, \Psi_{\mathcal{P}}, \eta)$  where  $\eta$  is a probability measure on  $\Psi_{\mathcal{P}}$ , the minimal  $\sigma$ -algebra containing open sets of  $W_{\mathcal{P}}$  such that for any  $n$ ,

$$\eta(X_1 = k_1, \dots, X_n = k_n) = P_T^{(n)}(X_1 = k_1, \dots, X_n = k_n). \quad (3)$$

$P_{\mathcal{P}}^{(n)}(X_1 = k_1, \dots, X_n = k_n)$  is defined as  $P_{\mathcal{P}}^{(n)}(X_1 = k_1, \dots, X_n = k_n) = p_1 \dots p_n$  where  $p_i$  is the annotation of alternative  $k_i$  in fact  $F_i$ . This definition clearly satisfies the properties in (2).  $P_{\mathcal{P}}^{(n)}(X_1 = k_1, \dots, X_n = k_n)$  is then extended to a probability measure over  $\mathcal{B}_{\mathcal{P}}$ .

We conjecture that this definition of the distribution semantics with function symbols coincides for definite programs with the one given above.

To show that the two definition coincide, we conjecture that  $\Psi_{\mathcal{P}} = \Omega_T$ . Moreover,  $X_1 = k_1, \dots, X_n = k_n$  is equivalent to the set of composite choices  $K = \{(F_1, \emptyset, k_1), \dots, (F_n, \emptyset, k_n)\}$  and  $\mu(\omega_K)$  gives  $p_1 \dots p_n$  which satisfies equation (3).

## 6 Conclusions

We have presented a definition of the distribution semantics in terms of an iterated fixpoint operator that allowed us to prove that the semantics is well defined for all programs. The operator we have presented is also interesting from an inference point of view, as it can be used for forward inference similarly to [7].

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## A Proofs of Theorems

**Proposition 5**  $OpTrueC_{\mathcal{I}}^{\mathcal{P}}$  and  $OpFalseC_{\mathcal{I}}^{\mathcal{P}}$  are monotonic.

**Proof:** Let us consider  $OpTrueC_{\mathcal{I}}^{\mathcal{P}}$ . We have to prove that if  $Tr_1 \leq Tr_2$  then  $OpTrueC_{\mathcal{I}}^{\mathcal{P}}(Tr_1) \leq OpTrueC_{\mathcal{I}}^{\mathcal{P}}(Tr_2)$ .  $Tr_1 \leq Tr_2$  means that  $\forall (a, L_a) \in Tr_1, (a, M_a) \in Tr_2 : L_a \subseteq M_a$ . Let  $(a, L'_a)$  be the elements of  $OpTrueC_{\mathcal{I}}^{\mathcal{P}}(Tr_1)$  and  $(a, M'_a)$  the elements of  $OpTrueC_{\mathcal{I}}^{\mathcal{P}}(Tr_2)$ . We have to prove that  $L'_a \subseteq M'_a$ .

If  $a \in \mathcal{F}$  then  $L'_a = M'_a = \{\{(a, \theta, 1)\}\}$ . If  $a \in \mathcal{B}_{\mathcal{P}} \setminus \mathcal{F}$ , then  $L'_a$  and  $M'_a$  have the same structure. Since  $\forall b \in \mathcal{B}_{\mathcal{P}} : L_b \subseteq M_b$ , then  $L'_a \subseteq M'_a$ .

We can prove similarly that  $OpFalseC_{\mathcal{I}}^{\mathcal{P}}$  is monotonic.  $\diamond$

**Proposition 6**  $IFPC^{\mathcal{P}}$  is monotonic.

**Proof:** We have to prove that if  $\mathcal{I}_1 \leq \mathcal{I}_2$  then  $IFPC^{\mathcal{P}}(\mathcal{I}_1) \leq IFPC^{\mathcal{P}}(\mathcal{I}_2)$ .  $\mathcal{I}_1 \leq \mathcal{I}_2$  means that  $\forall (a, L_a, L_{-a}) \in \mathcal{I}_1, (a, M_a, M_{-a}) \in \mathcal{I}_2 : L_a \subseteq M_a, L_{-a} \subseteq M_{-a}$ . Let  $(a, L'_a, L'_{-a})$  be the elements of  $IFPC^{\mathcal{P}}(\mathcal{I}_1)$  and  $(a, M'_a, M'_{-a})$  the elements of  $IFPC^{\mathcal{P}}(\mathcal{I}_2)$ . We have to prove that  $L'_a \subseteq M'_a$  and  $L'_{-a} \subseteq M'_{-a}$ . This follows from the monotonicity of  $OpTrueC_{\mathcal{I}_1}^{\mathcal{P}}$  and  $OpFalseC_{\mathcal{I}_2}^{\mathcal{P}}$  in  $\mathcal{I}_1$  and  $\mathcal{I}_2$  respectively, which can be proved as in Proposition 5.  $\diamond$

**Lemma 9** For a ground probabilistic logic program  $\mathcal{P}$  with probabilistic facts  $\mathcal{F}$ , rules  $\mathcal{R}$  and atoms  $\mathcal{B}_{\mathcal{P}}$ , let  $L_a^\alpha$  be the formula associated with atom  $a$  in  $OpTrueC_{\mathcal{I}}^{\mathcal{P}} \uparrow \alpha$ . For every atom  $a$ , total choice  $\sigma$  and iteration  $\alpha$ , we have:

$$w_\sigma \in \omega_{L_a^\alpha} \rightarrow WFM(w_\sigma | \mathcal{I}) \models a$$

where  $w_\sigma | \mathcal{I}$  is obtained by adding to  $w_\sigma$  the atoms  $a$  for which  $(a, K_a, K_{-a}) \in \mathcal{I}$  and  $w_\sigma \in K_a$  as facts and by removing all the rules with  $a$  in the head for which  $(a, K_a, K_{-a}) \in \mathcal{I}$  and  $w_\sigma \in K_{-a}$ .

**Proof:** Let us prove the lemma by transfinite induction: let us assume the thesis for all  $\beta < \alpha$  and let us prove it for  $\alpha$ . If  $\alpha$  is a successor ordinal, then it is easily verified for  $a \in \mathcal{F}$ . Otherwise assume  $w_\sigma \in \omega_{L_a^\alpha}$  where

$$L_a^\alpha = \bigcup_{a \leftarrow b_1, \dots, b_n, \neg c_1, \dots, c_m \in \mathcal{R}} ((L_{b_1}^{\alpha-1} \cup K_{b_1}) \otimes \dots \otimes (L_{b_n}^{\alpha-1} \cup K_{b_n}) \otimes K_{\neg c_1} \otimes \dots \otimes K_{\neg c_m})$$

This means that there is rule  $a \leftarrow b_1, \dots, b_n, \neg c_1, \dots, c_m \in \mathcal{R}$  such that  $w_\sigma \in \omega_{L_{b_i}^{\alpha-1} \cup K_{b_i}}$  for  $i = 1, \dots, n$  and  $w_\sigma \in \omega_{K_{\neg c_j}}$  for  $j = 1, \dots, m$ . By the inductive assumption and because of how  $w_\sigma | \mathcal{I}$  is built then  $WFM(w_\sigma | \mathcal{I}) \models b_i$  and  $WFM(w_\sigma | \mathcal{I}) \models \neg c_j$  so  $WFM(w_\sigma | \mathcal{I}) \models a$ .

If  $\alpha$  is a limit ordinal, then

$$L_a^\alpha = lub(\{L_a^\beta | \beta < \alpha\}) = \bigcup_{\beta < \alpha} L_a^\beta$$

If  $w_\sigma \in \omega_{L_a^\alpha}$  then there must exist a  $\beta < \alpha$  such that  $w_\sigma \in \omega_{L_a^\beta}$ . By the inductive assumption the hypothesis holds.  $\diamond$

**Lemma 10** For a ground probabilistic logic program  $\mathcal{P}$  with probabilistic facts  $\mathcal{F}$ , rules  $\mathcal{R}$  and atoms  $\mathcal{B}_{\mathcal{P}}$ , let  $M_{\neg a}^{\alpha}$  be the set of composite choices associated with atom  $a$  in  $OpFalseC_{\mathcal{I}}^{\mathcal{P}} \downarrow \alpha$ . For every atom  $a$ , total choice  $\sigma$  and iteration  $\alpha$ , we have:

$$w_{\sigma} \in \omega_{M_{\neg a}^{\alpha}} \rightarrow WFM(w_{\sigma}|\mathcal{I}) \models \neg a$$

where  $w_{\sigma}|\mathcal{I}$  is built as in Lemma 9.

**Proof:** Similar to the proof of Theorem Lemma 9.  $\diamond$

**Lemma 11** For a ground probabilistic logic program  $\mathcal{P}$  with probabilistic facts  $\mathcal{F}$ , rules  $\mathcal{R}$  and atoms  $\mathcal{B}_{\mathcal{P}}$ , let  $K_a^{\alpha}$  and  $K_{\neg a}^{\alpha}$  be the formulas associated with atom  $a$  in  $IFPC^{\mathcal{P}} \uparrow \alpha$ . For every atom  $a$ , total choice  $\sigma$  and iteration  $\alpha$ , we have:

$$w_{\sigma} \in \omega_{K_a^{\alpha}} \rightarrow WFM(w_{\sigma}) \models a \quad (4)$$

$$w_{\sigma} \in \omega_{K_{\neg a}^{\alpha}} \rightarrow WFM(w_{\sigma}) \models \neg a \quad (5)$$

**Proof:** Let us first prove that for all  $\alpha$ ,  $WFM(w_{\sigma}) = WFM(w_{\sigma}|IFPC^{\mathcal{P}} \uparrow \alpha)$ . We can prove it by transfinite induction. Consider the case of  $\alpha$  a successor ordinal. Consider an atom  $b$ . If  $w_{\sigma} \notin \omega_{K_b^{\alpha}}$  and  $w_{\sigma} \notin \omega_{K_{\neg b}^{\alpha}}$  then the rules for  $b$  in  $w_{\sigma}$  and  $w_{\sigma}|IFPC^{\mathcal{P}} \uparrow \alpha$  are the same. If  $w_{\sigma} \in \omega_{K_b^{\alpha}}$  then  $b$  is a fact in  $w_{\sigma}|IFPC^{\mathcal{P}} \uparrow \alpha$  but, according to Lemma 9,  $WFM(w_{\sigma}|IFPC^{\mathcal{P}} \uparrow (\alpha-1)) \models b$ . For the inductive hypothesis  $WFM(w_{\sigma}) \models b$  so  $b$  has the same truth value in  $WFM(w_{\sigma})$  and  $WFM(w_{\sigma}|IFPC^{\mathcal{P}} \uparrow \alpha)$ . Similarly, if  $w_{\sigma} \in \omega_{K_{\neg b}^{\alpha}}$ , then  $WFM(w_{\sigma}) \models \neg b$  and  $b$  has the same truth value in  $WFM(w_{\sigma})$  and  $WFM(w_{\sigma}|IFPC^{\mathcal{P}} \uparrow \alpha)$ . So overall  $WFM(w_{\sigma}) = WFM(w_{\sigma}|IFPC^{\mathcal{P}} \uparrow \alpha)$ .

If  $\alpha$  is a limit ordinal, then  $K_b^{\alpha} = \bigcup_{\beta < \alpha} K_b^{\beta}$  and  $K_{\neg b}^{\alpha} = \bigcup_{\beta < \alpha} K_{\neg b}^{\beta}$ . So if  $w_{\sigma} \in \omega_{K_b^{\alpha}}$  there is a  $\beta$  such  $w_{\sigma} \in \omega_{K_b^{\beta}}$  and for the inductive hypothesis  $WFM(w_{\sigma}) \models b$  so  $b$  has the same truth value in  $WFM(w_{\sigma})$  and  $WFM(w_{\sigma}|IFPC^{\mathcal{P}} \uparrow \alpha)$ . Similarly if  $w_{\sigma} \in \omega_{K_{\neg b}^{\alpha}}$ .

We can now prove the lemma by transfinite induction. Consider the case of  $\alpha$  a successor ordinal. Since  $(a, K_a^{\alpha}) \in lfp(OpTrueC_{IFPC \uparrow (\alpha-1)}^{\mathcal{P}})$ , by Lemma 9

$$w_{\sigma} \in \omega_{K_a^{\alpha}} \rightarrow WFM(w_{\sigma}|IFPC^{\mathcal{P}} \uparrow (\alpha-1)) \models a$$

Since  $WFM(w_{\sigma}|IFPC^{\mathcal{P}} \uparrow (\alpha-1)) = WFM(w_{\sigma})$ , (4) is proved.

Since  $(a, K_{\neg a}^{\alpha}) \in gfp(OpFalseC_{IFPC \uparrow (\alpha-1)}^{\mathcal{P}})$ , by Lemma 10

$$w_{\sigma} \in \omega_{K_{\neg a}^{\alpha}} \rightarrow WFM(w_{\sigma}|IFPC^{\mathcal{P}} \uparrow (\alpha-1)) \models \neg a$$

Since  $WFM(w_{\sigma}|IFPC^{\mathcal{P}} \uparrow (\alpha-1)) = WFM(w_{\sigma})$ , (5) is proved.

If  $\alpha$  is a limit ordinal,  $K_a^{\alpha} = \bigcup_{\beta < \alpha} K_a^{\beta}$  and  $K_{\neg a}^{\alpha} = \bigcup_{\beta < \alpha} K_{\neg a}^{\beta}$ . If  $w_{\sigma} \in \omega_{K_a^{\alpha}}$  there is a  $\beta$  such that  $w_{\sigma} \in \omega_{K_a^{\beta}}$  and by the inductive hypothesis (4) is proved. Similarly for (5).  $\diamond$

**Lemma 12** For a ground probabilistic logic program  $\mathcal{P}$  with probabilistic facts  $\mathcal{F}$ , rules  $\mathcal{R}$  and atoms  $\mathcal{B}_{\mathcal{P}}$ , let  $K_a^\alpha$  and  $K_{\neg a}^\alpha$  be the formulas associated with atom  $a$  in  $IFPC^{\mathcal{P}} \uparrow \alpha$ . For every atom  $a$ , total choice  $\sigma$  and iteration  $\alpha$ , we have:

$$\begin{aligned} a \in IFP^{w_\sigma} \uparrow \alpha &\rightarrow w_\sigma \in K_a^\alpha \\ \neg a \in IFP^{w_\sigma} \uparrow \alpha &\rightarrow w_\sigma \in K_{\neg a}^\alpha \end{aligned}$$

**Proof:** Let us prove it by double transfinite induction. If  $\alpha$  is a successor ordinal, assume that

$$\begin{aligned} a \in IFP^{w_\sigma} \uparrow (\alpha - 1) &\rightarrow w_\sigma \in K_a^{\alpha-1} \\ \neg a \in IFP^{w_\sigma} \uparrow (\alpha - 1) &\rightarrow w_\sigma \in K_{\neg a}^{\alpha-1} \end{aligned}$$

Let us perform transfinite induction on the iterations of  $OpTrue_{IFPC^{\mathcal{P}} \uparrow (\alpha-1)}^{\mathcal{P}}$ . Let us consider a successor ordinal  $\delta$ : assume that

$$\begin{aligned} a \in OpTrue_{IFP^{w_\sigma} \uparrow (\alpha-1)}^{w_\sigma} \uparrow (\delta - 1) &\rightarrow w_\sigma \in L_a^{\delta-1} \\ \neg a \in OpFalse_{IFP^{w_\sigma} \uparrow (\alpha-1)}^{w_\sigma} \downarrow (\delta - 1) &\rightarrow w_\sigma \in M_{\neg a}^{\delta-1} \end{aligned}$$

and prove that

$$\begin{aligned} a \in OpTrue_{IFP^{w_\sigma} \uparrow (\alpha-1)}^{w_\sigma} \uparrow \delta &\rightarrow w_\sigma \in L_a^\delta \\ \neg a \in OpFalse_{IFP^{w_\sigma} \uparrow (\alpha-1)}^{w_\sigma} \downarrow \delta &\rightarrow w_\sigma \in M_{\neg a}^\delta \end{aligned}$$

Consider  $a$ . If  $a \in \mathcal{F}$  then it is easily proved.

For other atoms  $a$ ,  $a \in OpTrue_{IFP^{w_\sigma} \uparrow (\alpha-1)}^{w_\sigma} \uparrow \delta$  means that there is a rule  $a \leftarrow b_1, \dots, b_n, \neg c_1, \dots, \neg c_m$  such that for all  $i = 1, \dots, n$   $b_i \in OpTrue_{IFP^{w_\sigma} \uparrow (\alpha-1)}^{w_\sigma} \uparrow (\delta - 1)$  and for all  $j = 1, \dots, m$   $\neg c_j \in IFP^{w_\sigma} \uparrow (\alpha - 1)$ . For the inductive hypothesis  $\forall i : w_\sigma \in L_{b_i}^{\delta-1} \vee w_\sigma \in K_{b_i}^{\alpha-1}$  and  $\forall j : w_\sigma \in K_{\neg c_j}^{\alpha-1}$  so, for the definition of  $OpTrue_{IFP^{w_\sigma} \uparrow (\alpha-1)}^{w_\sigma}$ ,  $w_\sigma \in L_a^\delta$ . Analogously for  $\neg a$ .

If  $\delta$  is a limit ordinal, then  $L_a^\delta = \bigcup_{\mu < \delta} L_a^\mu$  and  $M_{\neg a}^\delta = \bigotimes_{\mu < \delta} M_{\neg a}^\mu$ . For the inductive hypothesis for all  $\mu < \delta$

$$\begin{aligned} a \in OpTrue_{IFP^{w_\sigma} \uparrow (\alpha-1)}^{w_\sigma} \uparrow \mu &\rightarrow w_\sigma \in L_a^\mu \\ \neg a \in OpFalse_{IFP^{w_\sigma} \uparrow (\alpha-1)}^{w_\sigma} \downarrow \mu &\rightarrow w_\sigma \in M_{\neg a}^\mu \end{aligned}$$

If  $a \in OpTrue_{IFP^{w_\sigma} \uparrow (\alpha-1)}^{w_\sigma} \uparrow \delta$ , then there exists a  $\mu < \delta$  such that  $a \in OpTrue_{IFP^{w_\sigma} \uparrow (\alpha-1)}^{w_\sigma} \uparrow \mu$ . For the inductive hypothesis,  $w_\sigma \in L_a^\delta$ .

If  $\neg a \in OpFalse_{IFP^{w_\sigma} \uparrow (\alpha-1)}^{w_\sigma} \downarrow \delta$ , then, for all  $\mu < \delta$ ,  $\neg a \in OpFalse_{IFP^{w_\sigma} \uparrow (\alpha-1)}^{w_\sigma} \downarrow \mu$ . For the inductive hypothesis,  $w_\sigma \in M_{\neg a}^\delta$ .

Consider a limit  $\alpha$ . Then  $K_a^\alpha = \bigcup_{\beta < \alpha} K_a^\beta$  and  $K_{\neg a}^\alpha = \bigcup_{\beta < \alpha} K_{\neg a}^\beta$ . The inductive hypothesis is

$$\begin{aligned} a \in IFP^{w_\sigma} \uparrow \beta &\rightarrow w_\sigma \in K_a^\beta \\ \neg a \in IFP^{w_\sigma} \uparrow \beta &\rightarrow w_\sigma \in K_{\neg a}^\beta \end{aligned}$$

If  $a \in IFP^{w_\sigma} \uparrow \alpha$ , then there exists a  $\beta < \alpha$  such that  $a \in IFP^{w_\sigma} \uparrow \beta$ . For the inductive hypothesis  $w_\sigma \in K_a^\beta$  so  $w_\sigma \in K_a^\alpha$ . Similarly for  $\neg a$ .  $\diamond$

**Theorem 7** For a ground probabilistic logic program  $\mathcal{P}$  with atoms  $\mathcal{B}_{\mathcal{P}}$ , let  $K_a^\alpha$  and  $K_{\neg a}^\alpha$  be the formulas associated with atom  $a$  in  $IFPC^{\mathcal{P}} \uparrow \alpha$ . For every atom  $a$  and total choice  $\sigma$ , there is an iteration  $\alpha_0$  such that for all  $\alpha > \alpha_0$  we have:

$$w_\sigma \in \omega_{K_a^\alpha} \leftrightarrow WFM(w_\sigma) \models a \quad w_\sigma \in \omega_{K_{\neg a}^\alpha} \leftrightarrow WFM(w_\sigma) \models \neg a$$

**Proof:** The  $\rightarrow$  direction is Lemma 11. In the other direction,  $WFM(w_\sigma) \models a$  implies  $\exists \alpha_0 \forall \alpha \geq \alpha_0 : IFP_{w_\sigma}^{\mathcal{P}} \uparrow \alpha \models a$ . For Lemma 12,  $w_\sigma \in \omega_{K_a^\alpha}$ .  $WFM(w_\sigma) \models \neg a$  implies  $\exists \alpha_0 \forall \alpha \geq \alpha_0 : IFP^{w_\sigma} \uparrow \alpha \models \neg a$ . For Lemma 12,  $w_\sigma \in \omega_{K_{\neg a}^\alpha}$ .  $\diamond$

**Theorem 8** For a ground probabilistic logic program  $\mathcal{P}$ , let  $K_a^\alpha$  and  $K_{\neg a}^\alpha$  be the formulas associated with atom  $a$  in  $IFPC^{\mathcal{P}} \uparrow \alpha$ . For every atom  $a$  and every iteration  $\alpha$ ,  $K_a^\alpha$  and  $K_{\neg a}^\alpha$  are countable sets of countable composite choices.

**Proof:** It can be proved by observing that each iteration of  $OpTrueC_{IFPC^{\mathcal{P}} \uparrow \beta}^{\mathcal{P}}$  and  $OpFalseC_{IFPC^{\mathcal{P}} \uparrow \beta}^{\mathcal{P}}$  generates countable sets of countable explanations since the set of rules is countable.  $\diamond$