

Reducing the Size of the Optimization Problems in Fuzzy Ontology Reasoning

Fernando Bobillo¹ and Umberto Straccia²

¹ Dpt. of Computer Science & Systems Engineering, University of Zaragoza, Spain

² Istituto di Scienza e Tecnologie dell'Informazione (ISTI - CNR), Pisa, Italy
Email: fbobillo@unizar.es, straccia@isti.cnr.it

Abstract. Fuzzy ontologies allow the representation of imprecise structured knowledge, typical in many real-world application domains. A key factor in the practical success of fuzzy ontologies is the availability of highly optimized reasoners. This short paper discusses a novel optimization technique: a reduction of the size of the optimization problems obtained during the inference by the fuzzy ontology reasoner *fuzzyDL*.

1 Introduction

In recent years, we have noticed an increase in the number of applications for mobile devices that could benefit from the use of semantic reasoning services [1]. Because of the limited capabilities of mobile devices, it is especially important to develop reasoning algorithms performing efficiently in practice. In order to deal with imprecise knowledge, such applications could use fuzzy ontologies [8]. In fuzzy ontologies, concepts and relations are fuzzy. Consequently, the axioms are not in general either true or false, but they may hold to some degree of truth.

However, little effort has been paid so far to the study and implementation of optimization techniques for fuzzy ontology reasoning, which is essential to reason with real-world scenarios in practice (some exceptions are [3,4,5,6]). This short paper discusses some optimization techniques to improve the performance of the reasoning algorithm by reducing the size of optimization problems obtained during the inference. In particular, we will provide optimized MILP encodings of the restrictions involving n -ary operators and fuzzy membership functions. Such optimizations have been implemented in *fuzzyDL*, arguably the most popular and advanced fuzzy ontology reasoner [2], and proved their usefulness.

2 Background on fuzzyDL reasoning

We assume the reader to be familiar with the syntax and semantics of fuzzy Description Logics (DLs) [8]. The reasoning algorithm implemented in *fuzzyDL* combines tableau rules with an optimization problem. After some preprocessing, *fuzzyDL* applies tableau rules decomposing complex concept expressions into simpler ones, as usual in tableau algorithms, but also generating a system of inequation constraints. These inequations have to hold in order to respect

the semantics of the DL constructors. After all rules have been applied, an optimization problem must be solved before obtaining the final solution. The tableau rules are deterministic and the optimization problem is unique.

This optimization problem has a solution iff the fuzzy KB is consistent. In fuzzyDL, we obtain a bounded Mixed Integer Linear Programming [7] (MILP) problem, that is, minimising a linear function with respect to a set of constraints that are linear inequations in which rational and integer variables can occur. The problem is bounded, with rational variables ranging over $[0, 1]$ and some integer variables ranging over $\{0, 1\}$. For example, in Łukasiewicz fuzzy DLs, the restriction $x_1 \otimes_{\mathbf{L}} x_2 = z$ can be encoded using the set of constraints $\{x_1 + x_2 - 1 \leq z, x_1 + x_2 - 1 \geq z - y, z \leq 1 - y, y \in \{0, 1\}\}$. Observe that the MILP encoding of the restriction has introduced a new variable y : the two possibilities $y = 0$ and $y = 1$ encode the non-deterministic choice implicit in the interpretation of the conjunction under Łukasiewicz fuzzy logic. The complexity of solving a MILP problem is NP-complete and it depends on the number of variables, so it is convenient to reduce the number of new variables.

Let x, z be $[0, 1]$ -variables, and x_u be a rational unbounded variable. fuzzyDL has to solve some restrictions involving fuzzy connectives, such as $x_1 = \ominus x_2$, $x_1 \otimes x_2 = z$, $x_1 \oplus x_2 = z$, or $x_1 \Rightarrow x_2 = z$. Furthermore, it also needs to solve some restrictions $\mathbf{d}(x_u) \geq z$ involving fuzzy membership functions \mathbf{d} such as the *trapezoidal*($k1, k2, q1, q2, q3, q4$) (see Table 1 (a)), the *triangular*($k1, k2, q1, q2, q3$), *left*($k1, k2, q1, q2$), or *right*($k1, k2, q1, q2$) [8].

3 Optimizing Łukasiewicz N-ary Operators

Let us start with the case of conjunction concepts in Łukasiewicz fuzzy DLs. An n -ary concept of the form $(C_1 \sqcap C_2 \sqcap \dots \sqcap C_n)$ can be represented, using associativity, only using binary conjunctions $(C_1 \sqcap (C_2 \sqcap (\dots \sqcap C_n))) \dots$. A binary conjunction concept introduces a restriction of the form $x_1 \otimes_{\mathbf{L}} x_2 = z$ which, as shown in Section 2, can be encoded adding a new binary variable y . Hence, in order to represent the n -ary conjunction, $n - 1$ new variables y_i would be needed. However, it is possible to give a more efficient representation by considering the conjunction as an n -ary operator. Indeed, a restriction of the form $x_1 \otimes_{\mathbf{L}} x_2 \otimes \dots \otimes x_n = z$ can be encoded using only *one* new binary variable and, thus, saves 2^{n-2} possible alternative assignments to the variables y_i .

$$\begin{aligned} \sum_{i=1}^n x_i - (n-1) &\leq z, \\ y &\leq 1 - z, \\ \sum_{i=1}^n x_i - (n-1) &\geq z - (n-1)y, \\ y &\in \{0, 1\}. \end{aligned}$$

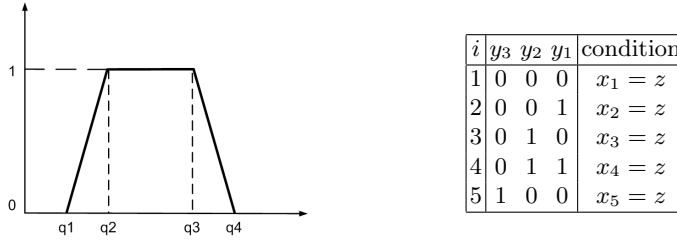


Table 1. (a) Trapezoidal membership function; (b) Encoding of 5 states.

$y = 0$ encodes the case $z = \sum_{i=1}^n x_i - (n - 1) \geq 0$, and $y = 1$ encodes the case $z = 0$ and $\sum_{i=1}^n x_i - (n - 1) < 0$. Let us consider now disjunction concepts in Lukasiewicz fuzzy DLs. A binary disjunction can be represented adding a new binary variable y as $\{x_1 + x_2 \leq z + y, y \leq z, x_1 + x_2 \geq z, y \in \{0, 1\}\}$. Again, $n - 1$ new binary variables would be needed but, similarly as before, considering the disjunction as an n -ary operator we would need only *one* new binary variable:

$$\begin{aligned} \sum_{i=1}^n x_i &\leq z + (n - 1)y, \\ y &\leq z, \\ \sum_{i=1}^n x_i &\geq z, \\ y &\in \{0, 1\}. \end{aligned}$$

4 Optimizing Gödel N-ary Operators

An n -ary conjunction can be represented using binary conjunctions adding restrictions of the form $x_1 \otimes_G x_2 = z$, which can be encoded as follows:

$$\begin{aligned} z &\leq x_1, \\ z &\leq x_2, \\ x_1 &\leq z + y, \\ x_2 &\leq z + (1 - y), \\ y &\in \{0, 1\}. \end{aligned}$$

The idea is that if $y = 0$, $x_1 = z$ is the minimum; whereas if $y = 1$, $x_2 = z$ is the minimum. This adds a new variable y , so in the case of n -ary conjunctions there would be $n - 1$ new variables. Treating the conjunction as an n -ary operator, a more efficient representation is possible. An n -ary conjunction introduces a restriction of the form $x_1 \otimes_G x_2 \otimes \cdots \otimes x_n = z$. To represent that the minimum of n variables x_i is equal to z , we can use n binary variables y_i such that if y_i

takes the value 0 then x_i (representing the minimum) is equal to z , and such that the sum of the y_i is 1, so z takes the value of some x_i . Note that the minimum may not be unique. Such a representation is as follows:

$$\begin{aligned} z &\leq x_i, \text{ for } i \in \{1, \dots, n\}, \\ x_i &\leq z + y_i, \text{ for } i \in \{1, \dots, n\}, \\ \sum_{i=1}^n y_i &= 1, \\ y_i &\in \{0, 1\}, \text{ for } i \in \{1, \dots, n\}. \end{aligned}$$

Now, we will show that it is possible to give a more efficient representation. Essentially, we need to encode n possible states. However, n possible states can be encoded using $m = \lceil \log_2 n \rceil$ new binary variables only. For instance, for $n = 5$, only $\lceil \log_2 5 \rceil = 3$ binary variables are necessary, where we use the encoding of the $n = 5$ states in Table 1 (b).

The main point is now to correctly encode the condition $x_i \leq z + y_i$ of the old encoding. We proceed as follows. Let b_i be a string of length m , representing the value $i - 1$ in base 2 ($1 \leq i \leq n$). For instance, for $i = 4$, $b = 011$, as illustrated in the table above. Let us define the expression e_{ij} ($1 \leq i \leq n$, $1 \leq j \leq m$) as:

$$e_{ij} = \begin{cases} y_j & \text{if the } j\text{th bit of } b_i \text{ is 0} \\ 1 - y_j & \text{otherwise.} \end{cases}$$

For $i = 4$, we have $b = 011$ and, thus, $e_{41} = 1 - y_1$, $e_{42} = 1 - y_2$, and $e_{43} = y_3$. Now we are ready to provide the whole encoding:

$$\begin{aligned} z &\leq x_i, \text{ for } i = 1, \dots, n \\ x_i &\leq z + \sum_{j=1}^m e_{ij}, \text{ for } i = 1, \dots, n \\ \sum_{j=1}^m 2^{j-1} y_j &\leq n - 1, \\ y_j &\in \{0, 1\}, \text{ for } j = 1, \dots, m. \end{aligned}$$

The first condition is the same as before. The second condition guarantees that $x_i \leq z$ in the state b_i . Finally, the third condition ensures that we are not addressing more than n states. For instance, for $n = 5$ we have:

$$\begin{aligned} z &\leq x_1, \\ z &\leq x_2, \\ z &\leq x_3, \\ z &\leq x_4, \\ z &\leq x_5, \\ x_1 &\leq z + y_1 + y_2 + y_3, \end{aligned}$$

$$\begin{aligned}
x_2 &\leq z + (1 - y_1) + y_2 + y_3, \\
x_3 &\leq z + y_1 + (1 - y_2) + y_3, \\
x_4 &\leq z + (1 - y_1) + (1 - y_2) + y_3, \\
x_5 &\leq z + y_1 + y_2 + (1 - y_3), \\
y_1 + 2y_2 + 4y_3 &\leq 4, \\
y_1 &\in \{0, 1\}, \\
y_2 &\in \{0, 1\}, \\
y_3 &\in \{0, 1\}.
\end{aligned}$$

The case of the disjunction in Gödel fuzzy DLs is dual. If an n -ary concept of the form $(C_1 \sqcup C_2 \sqcup \dots \sqcup C_n)$ is represented using binary disjunctions, $n - 1$ new binary variables are needed. However, if we consider it as an n -ary concept, it is possible to use $\lceil \log_2 n \rceil$ new binary variables only:

$$\begin{aligned}
z &\geq x_i, \text{ for } i = 1, \dots, n \\
x_i + \sum_{j=1}^m e_{ij} &\geq z \text{ for } i = 1, \dots, n \\
\sum_{j=1}^m 2^{j-1} y_j &\leq n - 1, \\
y_j &\in \{0, 1\} \text{ for } j = 1, \dots, m.
\end{aligned}$$

5 Optimizing Fuzzy Membership Functions

Let us start with the case of trapezoidal functions, which introduce a restriction of the form $\text{trapezoidal}(k_1, k_2, q_1, q_2, q_3, q_4)(x_u) \geq z$. A restriction of that form can be represented by adding 5 new binary variables y_i as follows:

$$\begin{aligned}
x_u + (k_1 - q_1)y_2 &\geq k_1, \\
x_u + (k_1 - q_2)y_3 &\geq k_1, \\
x_u + (k_1 - q_3)y_4 &\geq k_1, \\
x_u + (k_1 - q_4)y_5 &\geq k_1, \\
x_u + (k_2 - q_1)y_1 &\leq k_2, \\
x_u + (k_2 - q_2)y_2 &\leq k_2, \\
x_u + (k_2 - q_3)y_3 &\leq k_2, \\
x_u + (k_2 - q_4)y_4 &\leq k_2, \\
x_u &\leq 1 - y_1 - y_5, \\
x_u &\geq y_3, \\
x_u + (q_1 - q_2)x_u + (k_2 - q_1)y_2 &\leq k_2, \\
x_u + (q_1 - q_2)x_u + (k_1 - q_2)y_2 &\geq k_1 + q_1 - q_2,
\end{aligned}$$

$$\begin{aligned}
x_u + (q4 - q3)x_u + (k_2 - q3)y4 &\leq k_2 + q4 - q3, \\
x_u + (q4 - q3)x_u + (k_1 - q4)y4 &\geq k_1, \\
y_1 + y_2 + y_3 + y_4 + y_5 &= 1, \\
y_i &\in \{0, 1\}, \text{ for } i = 1, \dots, 5.
\end{aligned}$$

To reduce now the number of binary variables, the idea is to have 5 binary variables encoding the 5 possible states: $x_u \leq q1$ ($y_1 = 1$), $x_u \in [q1, q2]$ ($y_2 = 1$), $x_u \in [q2, q3]$ ($y_3 = 1$), $x_u \in [q3, q4]$ ($y_4 = 1$), and $x_u \geq q4$ ($y_5 = 1$). However, as shown in Table 1 (b), it is possible to represent 5 states using only 3 variables.

The case of other fuzzy membership functions is similar. In triangular functions, a naïve encoding introduces 4 new variables to represent the 4 possible states, but it is possible to consider only 2. Finally, in left and right shoulder functions, it is necessary to consider 3 states, which can be achieved by adding 2 new binary variables, instead of the 3 ones needed in the non-optimal encoding.

By considering the fact that even for moderate sized ontologies we may easily generate thousands of such constraints, it is evident that the number of saved binary variables n , and hence the number of saved assignments 2^n , is non-negligible.

Acknowledgement This research work has been partially supported by the CICYT project TIN2013-46238-C4-4-R and DGA-FSE.

References

1. C. Bobed, R. Yus, F. Bobillo, and E. Mena. Semantic reasoning on mobile devices: Do androids dream of efficient reasoners? *Journal of Web Semantics*, In press.
2. F. Bobillo and U. Straccia. fuzzyDL: An expressive fuzzy description logic reasoner. In *Proceedings of the 17th IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2008)*, pages 923–930, 2008.
3. F. Bobillo and U. Straccia. On partitioning-based optimisations in expressive fuzzy description logics. In *Proceedings of the 24th IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2015)*, 2015.
4. F. Bobillo and U. Straccia. Optimising fuzzy description logic reasoners with general concept inclusions absorption. *Fuzzy Sets and Systems*, In press.
5. V. Haarslev, H.-I. Pai, and N. Shiri. Optimizing tableau reasoning in \mathcal{ALC} extended with uncertainty. In *Proceedings of the 20th International Workshop on Description Logics (DL 2007)*, volume 250, pages 307–314. CEUR Workshop Proceedings, 2007.
6. G. S. N. Simou, T. Mailis and G. Stamou. Optimization techniques for fuzzy description logics. In *Proceedings of the 23rd International Workshop on Description Logics (DL 2010)*, volume 573. CEUR Workshop Proceedings, 2010.
7. H. M. Salkin and K. Mathur. *Foundations of Integer Programming*. North-Holland, 1989.
8. U. Straccia. *Foundations of Fuzzy Logic and Semantic Web Languages*. CRC Studies in Informatics Series. Chapman & Hall, 2013.