Malmquist productivity index for network production systems

Chiang Kao

Department of Industrial and Information Management
National Cheng Kung University, Tainan 701, Taiwan
e-mail: ckao@mail.ncku.edu.tw

Abstract. The conventional Malmquist productivity index (MPI) measures the performance improvement of a production system between two periods, where the system is treated as a black box, ignoring the internal operations of the component processes. Based on a relational model of the data envelopment analysis (DEA) for measuring the system and process efficiencies, this paper develops a methodology for calculating the system and process MPIs in one model. Moreover, relationships between the system and process MPIs are derived. By defining the ratio of the inefficiencies of a unit in two periods as complementary MPIs, this paper finds that the system complementary MPI is a linear combination of the process complementary MPIs, and the former is also close to a weighted average of the latter. Knowing the relationship between the system and process MPIs helps identify the processes that deter the improvement of the system; amendments to them will improve the system performance in the future.

Keywords: data envelopment analysis, Malmquist productivity index, network system.

1 Introduction

Efficiency measurement is important for organizations to identify unsatisfactory operations so that making improvements to them will produce more outputs with the same amount of inputs. If an organization is relatively efficient as compared to other similar ones at a point of time, yet it is actually declining as compared to its past performance, certain indexes for alerting the decision maker are necessary. One such index is the Malmquist productivity index (MPI), which measures efficiency changes between two periods for an organization, or any decision making unit (DMU).

The MPI has been widely applied to measuring performance changes between two periods, especially due to an act or policy (Banker et al., 2005; Chang et al., 2009; Kao, 2000). Different forms of the MPI have been proposed in the literature. Suppose the efficiency change of a DMU between periods t and t+1 is to be measured. The early work of Caves et al. (1982a, 1982b) calculated the relative efficiencies of the two periods based on the production technology of period *t*. Since the production

In: A. Kononov et al. (eds.): DOOR 2016, Vladivostok, Russia, published at http://ceur-ws.org

technology of period t+1 can also be used for calculating the relative efficiencies, and the results are probably different from those calculated from the technology of period t, Färe et al. (1994) suggested using the geometric mean of the two measures as the MPI to solve the problem of disparity.

A system is usually composed of several processes connected in a network structure. The conventional MPI measures the performance improvement of a system considered as a whole unit, neglecting the operations of its component processes. However, for a network system, it is possible that some processes are worsened while the system is improved. It is also possible that the system is worsened, while some processes are improving. Merely looking at the system MPI cannot identify the processes that cause the deterioration of the aggregate performance. The objective of this paper is to develop a methodology for measuring the system and process MPIs at the same time, and explore the relationship between them so that unsatisfactory processes can be identified. The MPI used for discussion is the global MPI.

The basic component of MPI is relative efficiency, and the data envelopment analysis (DEA) technique (Charnes et al., 1978) has been widely used for its calculation. To calculate the MPI of a network system, a network DEA model is needed. Various network DEA models have been developed in the literature (see, for example, the review of Kao (2014)), and they can be classified into three types: independent, connected (or descriptive), and relational. For independent models, the system and individual process efficiencies are calculated independently and separately, without considering the relations among them.

For descriptive models, the operations of all component processes are described in the model in calculating the system efficiency. The results obtained are more reasonable; nevertheless, the process efficiencies still need to be calculated separately. The system and process efficiencies do not have any relationship, either. The third type of models, relational, on the other hand, takes the relations between the system and processes into consideration in developing the model. The system and process efficiencies can be calculated at the same time. Moreover, there exist some mathematical relationships between the system and process efficiencies (Chen et al., 2009a; 2009b; Kao and Hwang, 2008). Due to this property, this paper uses the relational model to calculate efficiencies. Based on the mathematical relationship between the system and process efficiencies, certain relationships between the system and process MPIs will be derived.

In the next section, relational models for different network structures are firstly reviewed. Then, in Section 3, models for calculating the global MPI for network systems are developed. Relationships between the system and process MPIs are investigated. After that, Section 4 uses three examples to explain the methodology proposed in this paper. Finally, in Section 5, some conclusions are drawn.

2 The Relational Model

Consider a set of n DMUs, each uses the same m inputs to produce the same s outputs. Denote X_{ij} and Y_{rj} as the ith input, i=1,…, m, and rth output, r=1,…, s, respec-

tively, of the jth DMU, j=1,…, n. The CCR model of DEA for calculating the efficiency of DMU k under the assumption of constant returns-to-scale can be formulated as (Charnes et al., 1978):

$$\begin{aligned} E_k = \max.\ & \sum_{r=1}^{s} u_r Y_{rk} \\ \text{s.t.}\ & \sum_{i=1}^{m} v_i X_{ik} = 1 \\ & \sum_{r=1}^{s} u_r Y_{rj} - \sum_{i=1}^{m} v_i X_{ij} \le 0, \qquad j = 1,\ldots,n \\ & u_r, v_i \ge \varepsilon, \quad r = 1,\ldots,s, \qquad i = 1,\ldots,m \end{aligned} \tag{1}$$

where u_r and v_i are virtual multipliers and ε is a small non-Archimedean number (Charnes and Cooper, 1984) imposed to prevent any input/output factor from being ignored in calculating the efficiency.

Usually a system is composed of several processes connected as a network. Model (1) treats the system as a black-box, neglecting the operations of the component processes. Consequently, it is possible that all processes are not efficient while the system, as a whole, is. The network DEA takes the operations of the processes into consideration in calculating the system efficiency so that unreasonable results can be excluded. Network systems have various structures. The two fundamental ones are series and parallel.

The series structure is a basic network structure where a number of processes are connected in series. The characteristic of this type of structure is that the inputs used by all processes, except the first, are produced by their preceding one, and the outputs produced by all processes, except the last, are utilized by their succeeding one. The series structure is the most widely discussed network structure in the DEA literature.

Let $Z_{fj}^{(p)}$ denote the fth intermediate product, f=1,…, g, produced by process p, p=1,…, q–1. Note that the intermediate products produced by the last process, q, are the outputs of the system, Y_{rj}. Kao and Hwang (2008) showed that the system efficiency is the product of the q process efficiencies:

$$\prod_{p=1}^{q} E_k^{(p)} = \sum_{r=1}^{s} u_r^* Y_{rk} = E_k^S \tag{2}$$

The parallel structure is another basic network structure which is composed of a number of q processes, and each applies inputs $X_{ij}^{(p)}$ to produce outputs $Y_{rj}^{(p)}$. The total inputs consumed by the system are $X_{ij} = \sum_{p=1}^{q} X_{ij}^{(p)}$, and the total outputs produced are $Y_{rj} = \sum_{p=1}^{q} Y_{rj}^{(p)}$.

The relational model assigns the same multiplier to the same factor, regardless of which process it is associated with. Kao (2009) showed that the system efficiency is a weighted average of the q process efficiencies:

$$\sum_{p=1}^{q} \omega^{(p)} E_k^{(p)} = \sum_{p=1}^{q} \left(\sum_{i=1}^{m} v_i^* X_{ik}^{(p)} \right) \left(\frac{\sum_{r=1}^{s} u_r^* Y_{rk}^{(p)}}{\sum_{i=1}^{m} v_i^* X_{ik}^{(p)}} \right) = \sum_{p=1}^{q} \left(\sum_{r=1}^{s} u_r^* Y_{rk}^{(p)} \right) = E_k^S \tag{3}$$

where $\omega^{(p)} = \sum_{i=1}^{m} v_i^* X_{ik}^{(p)} / \sum_{i=1}^{m} v_i^* X_{ik} = \sum_{i=1}^{m} v_i^* X_{ik}^{(p)}$.

Most network systems are a mixture of the series and parallel structures. Theoretically, they can have numerous forms of structure, although the most complicated structure that appears in the literature only has five processes (Lewis and Sexton, 2004). Denote $I_j^{(p)}$, $O_j^{(p)} \subseteq \{1, 2,\ldots, g\}$ as the index sets of the input and output intermediate products, respectively, of process p for DMU j. To be generic, we consider the very general network structure shown in Figure 1, where each process p consumes exogenous inputs $X_{ij}^{(p)}$ and intermediate products $Z_{fj}^{(p)}$, $j \in I_j^{(p)}$ that are produced by other processes to produce exogenous outputs $Y_{rj}^{(p)}$ and intermediate products $Z_{fj}^{(p)}$, $j \in O_j^{(p)}$ for other processes to use. The total inputs consumed and the total outputs produced by the system are $X_{ij} = \sum_{p=1}^{q} X_{ij}^{(p)}$ and $Y_{rj} = \sum_{p=1}^{q} Y_{rj}^{(p)}$, respectively. Let the same factor have the same multiplier; the relational model for calculating the system efficiency is:

$$
\begin{aligned}
E_k^S = \max.\ & \sum_{r=1}^{s} u_r Y_{rk} \\
\text{s.t.}\ & \sum_{i=1}^{m} v_i X_{ik} = 1 \\
& \sum_{r=1}^{s} u_r Y_{rj} - \sum_{i=1}^{m} v_i X_{ij} + s_j = 0, \quad j = 1,\ldots,n \\
& (\sum_{r=1}^{s} u_r Y_{rj}^{(p)} + \sum_{f \in O_j^{(p)}} w_f Z_{fj}^{(p)}) - (\sum_{i=1}^{m} v_i X_{ij}^{(p)} + \sum_{f \in I_j^{(p)}} w_f Z_{fj}^{(p)}) + s_j^{(p)} = 0, \\
& \qquad\qquad j = 1,\ldots,n, \quad p = 1,\ldots,q \\
& u_r, v_i, w_f \geq \varepsilon, \quad r = 1,\ldots,s, \quad i = 1,\ldots,m, \quad f = 1,\ldots,g \\
& s_j, s_j^{(p)} \geq 0, \quad j = 1,\ldots,n, \quad p = 1,\ldots,q
\end{aligned}
\tag{4}
$$

Since all the intermediate products are produced and consumed in the system, the sum of the constraints corresponding to the q processes is equal to the constraint corresponding to the system for each DMU.

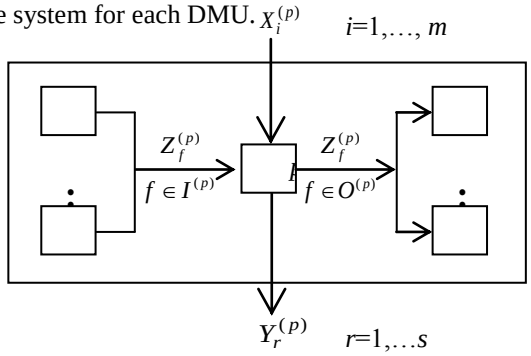

Fig. 1. The general network structure.

Based on Model (4), the system and process efficiencies of DMU k can be calculated as:

$$
E_k^S = \sum_{r=1}^{s} u_r^* Y_{rk} \,/ \sum_{i=1}^{m} v_i^* X_{ik} = \sum_{r=1}^{s} u_r^* Y_{rk} \tag{5}
$$

$$E_k^{(p)} = [\sum_{r=1}^{s} u_r^* Y_{rk}^{(p)} + \sum_{f \in O_k^{(p)}} w_f^* Z_{fk}^{(p)}] / [\sum_{i=1}^{m} v_i^* X_{ik}^{(p)} + \sum_{f \in I_k^{(p)}} w_f^* Z_{fk}^{(p)}], \quad p = 1, \ldots, q$$

where (u_r^*, v_i^*, w_f^*) is a set of optimal solutions. Since the sum of the q process constraints is equal to the system constraint for each DMU, that is, $s\,s_k = \sum_{p=1}^{q} s_k^{(p)}$, we have: $\sum_{r=1}^{s} u_r^* Y_{rk} - \sum_{i=1}^{m} v_i^* X_{ik} = \sum_{p=1}^{q} [\sum_{r=1}^{s} u_r^* Y_{rk}^{(p)} + \sum_{f \in O_k^{(p)}} w_f^* Z_{fk}^{(p)}] - (\sum_{i=1}^{m} v_i^* X_{ik}^{(p)} + \sum_{f \in I_k^{(p)}} w_f^* Z_{fk}^{(p)})]$, or, from the expression of E_k^S and $E_k^{(p)}$ in Equation (9):

$$1 - E_k^S = \sum_{p=1}^{q} (1 - E_k^{(p)}) (\sum_{i=1}^{m} v_i^* X_{ik}^{(p)} + \sum_{f \in I_k^{(p)}} w_f^* Z_{fk}^{(p)}) = \sum_{p=1}^{q} (1 - E_k^{(p)}) \omega^{(p)} \tag{6}$$

where $\omega^{(p)} = \sum_{i=1}^{m} v_i^* X_{ik}^{(p)} + \sum_{f \in I_k^{(p)}} w_f^* Z_{fk}^{(p)}$. That is, the system inefficiency, $1 - E_k^S$, is a linear combination of the q process inefficiencies, $1 - E_k^{(p)}$. Nevertheless, the former is not a weighted average of the latter because the sum of the weights, $\sum_{p=1}^{q} \omega^{(p)} = \sum_{p=1}^{q} [\sum_{i=1}^{m} v_i^* X_{ik}^{(p)} + \sum_{f \in I_k^{(p)}} w_f^* Z_{fk}^{(p)}] = 1 + \sum_{p=1}^{q} \sum_{f \in I_k^{(p)}} w_f^* Z_{fk}^{(p)}$, is clearly greater than 1.

3 Malmquist Productivity Index

The MPI is an index for representing the efficiency improvement between two periods. Various forms of MPI have been developed. The global MPI proposed by Pastor and Lovell (2005) has several attractive properties and is used in this paper. The basic idea of the global MPI is to use the observations of all periods to construct the production frontier. Based on which, the relative efficiencies of a DMU in two periods are calculated, and their ratio is the MPI. Since all observations have been included in constructing the frontier, the resulting efficiencies for both periods will not exceed one.

Let the superscript h, t, and t+1, denote period. The relational model for calculating the relative efficiency of DMU k in period t+1 for the series structure based on the technology constructed from the observations of periods t and t+1 is:

$$\begin{aligned}
(E_k^S)^{t+1} = \max.\ & \sum_{r=1}^{s} u_r (Y_{rk})^{t+1} \\
\text{s.t.}\ & \sum_{i=1}^{m} v_i (X_{ik})^{t+1} = 1 \\
& \sum_{f=1}^{g} w_f (Z_{fj}^{(1)})^h - \sum_{i=1}^{m} v_i (X_{ij})^h \le 0, \qquad h = t, \ldots, t+1, \quad j = 1, \ldots, n \\
& \sum_{f=1}^{g} w_f Z_{fj}^{(p)})^h - \sum_{f=1}^{g} w_f (Z_{fj}^{(p-1)})^h \le 0, \ h = t, \ldots, t+1, \\
& \qquad\qquad\qquad\qquad\qquad\qquad j = 1, \ldots, n, \qquad p = 2, \ldots, q-1 \\
& \sum_{r=1}^{s} u_r (Y_{rj})^h - \sum_{f=1}^{g} w_f (Z_{fj}^{(q-1)})^h \le 0, \qquad h = t, \ldots, t+1, \quad j = 1, \ldots, n
\end{aligned} \tag{7}$$

$$u_r, v_i, w_f \geq \varepsilon, \quad r=1,\ldots,s, \quad i=1,\ldots,m, \quad f=1,\ldots,g$$

The system MPI for DMU k, M_k^S, is the ratio of $(E_k^S)^{t+1}$ to $(E_k^S)^t$. Kao and Hwang (2014) showed that it can be expressed as:

$$M_k^S = \frac{(E_k^S)^{t+1}}{(E_k^S)^t} = \frac{\prod_{p=1}^q (E_k^{(p)})^{t+1}}{\prod_{p=1}^q (E_k^{(p)})^t} = \prod_{p=1}^q \frac{(E_k^{(p)})^{t+1}}{(E_k^{(p)})^t} = \prod_{p=1}^q M_k^{(p)} \tag{8}$$

where $M_k^{(p)} = (E_k^{(p)})^{t+1} / (E_k^{(p)})^t$ is, by definition, the MPI of process p. Hence, the system MPI is the product of the process MPIs.

The way of calculating the system MPI for the parallel structure is similar to that for the series structure. The model for calculating the system efficiency of period t+1 based on the combined technology is:

$$\begin{aligned}
(E_k^S)^{t+1} = \text{ max. } & \sum_{r=1}^s u_r (Y_{rk})^{t+1} \\
\text{s.t. } & \sum_{i=1}^m v_i (X_{ik})^{t+1} = 1 \\
& \sum_{r=1}^s u_r (Y_{rj}^{(p)})^h - \sum_{i=1}^m v_i (X_{ij}^{(p)})^h \leq 0, \quad h=t,\ldots,t+1, \\
& \qquad\qquad\qquad\qquad\qquad\qquad j=1,\ldots,n, \quad p=1,\ldots,q \\
& u_r, v_i \geq \varepsilon, \quad r=1,\ldots,s, \quad i=1,\ldots,m
\end{aligned} \tag{9}$$

According to Equation (3), the system efficiency for the parallel structure is a weighted average of the process efficiencies: $(E_k^S)^h = \sum_{p=1}^q [\sum_{i=1}^m v_i^h (X_{ik}^{(p)})^h](E_k^{(p)})^h$, $h=t,\ldots,t+1$, where $(E_k^S)^h = \sum_{r=1}^s u_r^h (Y_{rk})^h$ and $(E_k^{(p)})^h = \sum_{r=1}^s u_r^h (Y_{rk}^{(p)})^h / \sum_{i=1}^m v_i^h (X_{ik}^{(p)})^h$. The system MPI can then be expressed as (Kao 2016):

$$\begin{aligned}
M_k^S &= \frac{(E_k^S)^{t+1}}{(E_k^S)^t} = \frac{\sum_{p=1}^q [\sum_{i=1}^m v_i^{t+1} (X_{ik}^{(p)})^{t+1}](E_k^{(p)})^{t+1}}{(E_k^S)^t} \\
&= \sum_{p=1}^q \left(\sum_{i=1}^m v_i^{t+1} (X_{ik}^{(p)})^{t+1} \right) \frac{(E_k^{(p)})^{t+1}}{(E_k^{(p)})^t} \frac{(E_k^{(p)})^t}{(E_k^S)^t} \\
&= \sum_{p=1}^q \left(\sum_{i=1}^m v_i^{t+1} (X_{ik}^{(p)})^{t+1} \right) M_k^{(p)} \frac{\sum_{r=1}^s u_r^t (Y_{rk}^{(p)})^t / \sum_{i=1}^m v_i^t (X_{ik}^{(p)})^t}{\sum_{r=1}^s u_r^t (Y_{rk})^t} \\
&= \sum_{p=1}^q M_k^{(p)} \left(\frac{\sum_{r=1}^s u_r^t (Y_{rk}^{(p)})^t}{\sum_{r=1}^s u_r^t (Y_{rk})^t} \frac{\sum_{i=1}^m v_i^{t+1} (X_{ik}^{(p)})^{t+1}}{\sum_{i=1}^m v_i^t (X_{ik}^{(p)})^t} \right) = \sum_{p=1}^q M_k^{(p)} \varpi^{(p)}
\end{aligned} \tag{10}$$

A result that the system MPI is a linear combination of the process MPIs is derived. However, the former is not a weighted average of the latter because the sum of the weights, $\sum_{p=1}^q \varpi^{(p)}$, is not necessarily equal to 1.

Theoretically, if the total weight, $\sum_{p=1}^q \varpi^{(p)}$, in a linear combination is much smaller than 1, then it is possible that all process MPIs are greater than 1, yet their linear combination is less than 1. The opposite situation may also happen, if the total weight is much greater than 1. In practice, the chance of having a very large or a very small total weight is quite low. The reason is because the observations of DMU k for each process p in period t, $((X_{ik}^{(p)})^t, (Y_{rk}^{(p)})^t)$, and period t+1, $((X_{ik}^{(p)})^{t+1}, (Y_{rk}^{(p)})^{t+1})$, are usually not too different, which make $\sum_{i=1}^m v_i^{t+1} (X_{ik}^{(p)})^{t+1}$ approximately equal to

$\sum_{i=1}^{m} v_i^t (X_{ik}^{(p)})^t$, especially when they are under the same frontier facet where $v_i^t = v_i^{t+1}$. In this case, the ratio of $\sum_{i=1}^{m} v_i^{t+1} (X_{ik}^{(p)})^{t+1}$ to $\sum_{i=1}^{m} v_i^t (X_{ik}^{(p)})^t$ in $\varpi^{(p)}$ is approximately equal to 1, which simplifies $\varpi^{(p)}$ to $\hat{\varpi}^{(p)} = \sum_{r=1}^{s} u_r^t (Y_{rk}^{(p)})^t \,/\, \sum_{r=1}^{s} u_r^t (Y_{rk})^t$. Since $\sum_{p=1}^{q} \hat{\varpi}^{(p)} = 1$, a result in which the system MPI is approximately equal to a weighted average of the process MPIs is obtained, where the weight associated with process *p* is the proportion of the aggregate output of this process in all processes.

For network systems with a general structure of Figure 1, Model (4) can be adapted to suit the combined technology for calculating the efficiency of DMU *k* in period *t*+1:

$$
\begin{aligned}
(E_k^S)^{t+1} = \max. \ & \sum_{r=1}^{s} u_r (Y_{rk})^{t+1} \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i (X_{ik})^{t+1} = 1 \\
& \sum_{r=1}^{s} u_r (Y_{rj})^h - \sum_{i=1}^{m} v_i (X_{ij})^h + (s_j)^h = 0, \quad h = t, \ldots, t+1, \quad j = 1, \ldots, n \quad (11) \\
& [\sum_{r=1}^{s} u_r (Y_{rj}^{(p)})^h + \sum_{f \in O_j^{(p)}} w_f (Z_{fj}^{(p)})^h] - [\sum_{i=1}^{m} v_i (X_{ij}^{(p)})^h + \sum_{f \in I_j^{(p)}} w_f (Z_{fj}^{(p)})^h] + \\
& \qquad (s_j^{(p)})^h = 0, \ h = t, \ldots, t+1, \ j = 1, \ldots, n, \ p = 1, \ldots, q \\
& u_r, v_i, w_f \geq \varepsilon, \qquad r = 1, \ldots, s, \qquad i = 1, \ldots, m, \quad f = 1, \ldots, g \\
& (s_j)^h, \ (s_j^{(p)})^h \geq 0, \quad h = t, \ldots, t+1, \quad j = 1, \ldots, n, \ p = 1, \ldots, q
\end{aligned}
$$

From Equation (6), the system inefficiency is a linear combination of the process inefficiencies: $1-(E_k^S)^h = \sum_{p=1}^{q} (1-(E_k^{(p)})^h)(\omega^{(p)})^h$ for $h = t, \ldots, t+1$, where $(\omega^{(p)})^h = \sum_{i=1}^{m} v_i^h (X_{ik}^{(p)})^h + \sum_{f \in I_k^{(p)}} w_f^h (Z_{fk}^{(p)})^h$. The conventional MPI is defined as the ratio of the efficiencies of two periods. A value greater (or less) than 1 indicates that the efficiency has improved (or is worsened). Conceptually, the MPI can also be defined as the ratio of the inefficiencies of two periods. Under this definition, a value less (or greater) than 1 indicates that the efficiency has improved (or is worsened). Term this type of MPI the complementary MPI, and denote it as $\breve{M}$. The system complementary MPI for DMU *k* can be expressed as:

$$
\begin{aligned}
\breve{M}_k^S &= \frac{1-(E_k^S)^{t+1}}{1-(E_k^S)^t} = \frac{\sum_{p=1}^{q} [1-(E_k^{(p)})^{t+1}][\sum_{i=1}^{m} v_i^{t+1} (X_{ik}^{(p)})^{t+1} + \sum_{f \in I_k^{(p)}} w_f^{t+1} (Z_{fk}^{(p)})^{t+1}]}{1-(E_k^S)^t} \\
&= \sum_{p=1}^{q} \left(\frac{1-(E_k^{(p)})^{t+1}}{1-(E_k^{(p)})^t} \right) \left(\frac{[1-(E_k^{(p)})^t][\sum_{i=1}^{m} v_i^{t+1} (X_{ik}^{(p)})^{t+1} + \sum_{f \in I_k^{(p)}} w_f^{t+1} (Z_{fk}^{(p)})^{t+1}]}{1-(E_k^S)^t} \right)
\end{aligned}
$$

From the constraints of Model (11), we have $1-(E_k^S)^t = (s_k)^t$ (because $\sum_{i=1}^{m} v_i (X_{ik})^t = 1$) and $1-(E_k^{(p)})^t = (s_k^{(p)})^t / \sum_{i=1}^{m} v_i^t (X_{ik}^{(p)})^t + \sum_{f \in I_k^{(p)}} w_f^t (Z_{fk}^{(p)})^t]$. Denote

$$
\varpi^{(p)} = \frac{(s_k^{(p)})^t [\sum_{i=1}^{m} v_i^{t+1} (X_{ik}^{(p)})^{t+1} + \sum_{f \in I_k^{(p)}} w_f^{t+1} (Z_{fk}^{(p)})^{t+1}]}{(s_k)^t [\sum_{i=1}^{m} v_i^t (X_{ik}^{(p)})^t + \sum_{f \in I_k^{(p)}} w_f^t (Z_{fk}^{(p)})^t]}
$$

$\breve{M}_k^S$ becomes:

$$
\breve{M}_k^S = \sum_{p=1}^{q} \breve{M}_k^{(p)} \varpi^{(p)} \qquad (12)
$$

where $\breve{M}_k^{(p)} = [1-(E_k^{(p)})^{t+1}]/[1-(E_k^{(p)})^t]$ is the complementary MPI of process *p*. Here a property, that the system complementary MPI is a linear combination of the process complementary MPIs, is obtained.

For cases where $(E_k^S)^t = 1$, $\breve{M}_k^S$ is undefined. However, since $(E_k^S)^t = 1$ implies $(E_k^{(p)})^t = 1$ for all *p*, the system and process MPIs become: $M_k^S = (E_k^S)^{t+1}$ and $M_k^{(p)} = (E_k^{(p)})^{t+1}$. The relationship between the system and process MPIs then is:

$$\begin{aligned} 1-M_k^S = 1-(E_k^S)^{t+1} &= \sum_{p=1}^{q}(1-M_k^{(p)})\left(\sum_{i=1}^{m} v_i^{t+1}(X_{ik}^{(p)})^{t+1} + \sum_{f\in I_k^{(p)}} w_f^{t+1}(Z_{fk}^{(p)})^{t+1}\right) \\ &= \sum_{p=1}^{q}(1-M_k^{(p)})\varpi^{(p)} \end{aligned} \tag{13}$$

The value of M_k^S in this case is always less than or equal to 1, indicating that the performance is worsened. If, for period *t*, the system is not efficient, yet a process is, then the corresponding term $\breve{M}_k^{(p)}$ in Equation (12) is changed to $[1-M_k^{(p)}] \times [\sum_{i=1}^{m} v_i^{t+1}(X_{ik}^{(p)})^{t+1} + \sum_{f\in I_k^{(p)}} w_f^{t+1}(Z_{fk}^{(p)})^{t+1}]/\ (s_k)^t$.

4 Example

In this section we use an example to explain how the models developed in Section 3 are used to calculate the system and process MPIs, and the relationships between them.

Consider a system of three processes connected in a general structure shown in Figure 2, where process 1 applies input X_1 to produce output Y_1, process 2 applies input X_2 to produce output Y_2, and process 3 applies input X_3 and portions of Y_1 and Y_2 to produce output Y_3. There are eight DMUs to be compared, with the data shown in Table 1. Note that the output of process 1, Y_1, is split into $Y_1^{(I)}$ and $Y_1^{(O)}$, where the former is used by process 3 for producing Y_3 and the latter is an output of the system. Similarly, Y_2, is split into $Y_2^{(I)}$ and $Y_2^{(O)}$.

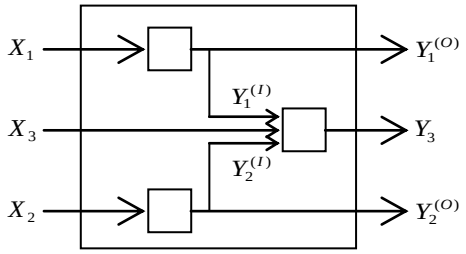

Fig. 2. Structure of the general network example.

By applying Model (11), the system and process inefficiencies are calculated as shown in the first two rows of each DMU in Table 2. As discussed in Section 2, the system inefficiency is a linear combination of the process inefficiencies, and the sum

of the weights for linear combination is greater than 1 for all DMUs (shown in the last column in parentheses). The ratio of the inefficiencies in periods t+1 and t is the complementary MPI, and is shown in the third row and denoted as cMPI. The weight associated with each process is calculated from Equation (12) shown in parentheses next to the complementary MPI. The system complementary MPI is a linear combination of the process complementary MPIs.

Table 1. Data for the general structure example.

DMU	Period	X_1	X_2	X_3	Y_1	$(Y_1^{(I)}\ Y_1^{(O)})$	Y_2	$(Y_2^{(I)}\ Y_2^{(O)})$	Y_3
1	t	2	4	3	5	(2 3)	5	(2 3)	4
	t+1	2	4	2	6	(3 3)	5	(2 3)	4
2	t	3	5	3	5	(3 2)	6	(3 3)	5
	t+1	2	5	4	5	(3 2)	7	(3 4)	6
3	t	3	6	3	6	(2 4)	6	(2 4)	5
	t+1	4	7	4	7	(3 4)	8	(4 4)	7
4	t	4	6	4	7	(3 4)	6	(3 3)	7
	t+1	3	5	3	7	(3 4)	6	(3 3)	6
5	t	5	6	4	7	(4 3)	7	(3 4)	7
	t+1	4	5	4	7	(4 3)	6	(3 3)	7
6	t	5	7	5	8	(4 4)	8	(3 5)	8
	t+1	5	7	6	9	(4 5)	9	(4 5)	9
7	t	5	8	5	9	(4 5)	9	(4 5)	9
	t+1	6	7	5	9	(4 5)	9	(5 4)	10
8	t	6	9	5	9	(5 4)	9	(4 5)	8
	t+1	5	8	4	9	(5 4)	9	(5 4)	9

A complementary MPI of greater (or less) than one indicates that the performance of the associated unit is worsened (or has improved). When the efficiency of period t, $(E_k^{(p)})^t$, is close to 1, the inefficiency, $1-(E_k^{(p)})^t$, will be close to 0, which may result in peculiar numbers in calculating ratios. The complementary MPIs of process 3 of DMUs 3 and 6 are examples where the former has a value of 391.75 and the latter has a value of 674.76. These large values are difficult for human to interpret how worse the units have performed. A contrary situation is when $(E_k^{(p)})^{t+1}$ is close to 1, which makes the complementary MPI close to 0. For example, both process 1 of DMU 1 and process 2 of DMU 2 have a complementary MPI of 0. There is no way of knowing which one has improved more.

Since the MPI and the complementary MPI have opposite trends intersecting at 1, one can rely on the MPI to help judge the extent of the performance change. The fourth row of Table 2 shows the MPI, which is the ratio of $E^{(t+1)}/E^{(t)}$, of each DMU. Comparing the complementary MPI in the third row with the MPI in the fourth, it is clear that when one has a value greater (or less) than 1, the other has a value less (or greater) than 1. For the example of 391.75 and 674.76 mentioned in the preceding paragraph, we know that both processes are worsened, yet their extents are difficult to judge. Their corresponding MPIs of 0.8753 and 0.90021 make the judgment much easier. Similarly, for process 1 of DMU 1 and process 2 of DMU 2, where both processes have a complementary MPI of 0, the former has an MPI of 1.2 and the latter an MPI of 1.1667, indicating that the former has improved more than the latter. Hence, one may use the MPI to judge the extent of performance changes. The complementary MPI is merely for showing the mathematical relationship between the performance change of the system and the processes.

Table 2. Inefficiencies, complementary MPIs, and MPIs of the general structure example.

DMU		Process 1	Process 2	Process 3	System
		Score (weight)	Score(weight)	Score(weight)	Score(weight)
1	$1-E^{(t)}$	0.1667 (0.0006)	0.1071 (0.6516)	0.2000 (0.5807)	0.1861 (1.2329)
	$1-E^{(t+1)}$	0.0000 (0.0006)	0.1071 (0.7360)	0.0003 (0.5266)	0.0790 (1.2632)
	cMPI	0.0000 (0.0005)	1.0000 (0.4238)	0.0013 (0.5660)	0.4246 (0.9904)
	MPI	1.2000	1.0000	1.2497	1.1315
2	$1-E^{(t)}$	0.4444 (0.0009)	0.1429 (0.6988)	0.1668 (0.6001)	0.2003 (1.2998)
	$1-E^{(t+1)}$	0.1667 (0.0006)	0.0000 (0.6370)	0.1430 (0.6357)	0.0910 (1.2733)
	cMPI	0.3750 (0.0013)	0.0000 (0.4544)	0.8575 (0.5292)	0.4543 (0.9849)
	MPI	1.5000	1.1667	1.0285	1.1367
3	$1-E^{(t)}$	0.3333 (0.0009)	0.2857 (0.7356)	0.0003 (0.4389)	0.2106 (1.1753)
	$1-E^{(t+1)}$	0.4167 (0.0012)	0.1837 (0.7085)	0.1250 (0.5798)	0.2031 (1.2895)
	cMPI	1.2500 (0.0019)	0.6429 (0.9612)	391.75 (0.0009)	0.9643 (0.9640)
	MPI	0.8750	1.1429	0.8753	1.0095
4	$1-E^{(t)}$	0.4167 (0.0012)	0.2857 (0.6758)	0.0003 (0.5646)	0.1938 (1.2417)
	$1-E^{(t+1)}$	0.2222 (0.0009)	0.1429 (0.6988)	0.0001 (0.6001)	0.1001 (1.2998)
	cMPI	0.5333 (0.0019)	0.5000 (1.0304)	0.3521 (0.0009)	0.5165 (1.0332)
	MPI	1.3333	1.2000	1.0002	1.1162
5	$1-E^{(t)}$	0.5333 (0.0015)	0.1667 (0.6764)	0.0002 (0.5641)	0.1137 (1.2420)
	$1-E^{(t+1)}$	0.4167 (0.0012)	0.1429 (0.6356)	0.0002 (0.6360)	0.1914 (0.9932)
	cMPI	0.7813 (0.0056)	0.8571 (0.9321)	0.8867 (0.0012)	0.8044 (0.9389)
	MPI	1.2500	1.0286	1.0000	1.0251
6	$1-E^{(t)}$	0.4667 (0.0015)	0.1837 (0.6612)	0.0001 (0.5401)	0.1222 (1.2028)
	$1-E^{(t+1)}$	0.0400 (0.0015)	0.0816 (0.6193)	0.1000 (0.6324)	0.1144 (1.2532)
	cMPI	0.8571 (0.0057)	0.4444 (0.9307)	674.76 (0.0008)	0.9360 (0.9372)
	MPI	1.1250	1.1250	0.9001	1.0089
7	$1-E^{(t)}$	0.4000 (0.0015)	0.1964 (0.6894)	0.0003 (0.5557)	0.1362 (1.2466)
	$1-E^{(t+1)}$	0.5000 (0.0018)	0.0816 (0.6600)	0.0000 (0.6753)	0.0548 (1.3372)
	cMPI	1.2500 (0.0053)	0.4156 (0.9520)	0.0000 (0.0016)	0.4022 (0.9733)
	MPI	0.8333	1.1429	1.0003	1.0942
8	$1-E^{(t)}$	0.5000 (0.0018)	0.2857 (0.7136)	0.1116 (0.5116)	0.2619 (1.2432)
	$1-E^{(t+1)}$	0.4000 (0.0015)	0.1964 (0.0011)	0.0000 (0.9984)	0.0008 (1.0010)
	cMPI	0.8000 (0.0029)	0.6875 (0.0012)	0.0000 (0.4254)	0.0031 (0.4295)
	MPI	1.2000	1.1245	1.1256	1.3537

5 Conclusion

A system is usually composed of several processes operating interdependently. The conventional MPI measures the efficiency improvement between two periods by treating the system as a black box, neglecting the operations of the component processes. Once the MPI is calculated, it is not clear which processes cause the improvement of deterioration of the system. More seriously, it is possible that the system shows an improvement while most of the component processes are actually worsened.

This paper adopts the idea of the relational DEA model to calculate the system and process efficiencies at the same time. Most importantly, there exist mathematical relationships between them. Based on the relationships between the system and process efficiencies, relationships between those of MPIs are derived. The relationships show the effects of the process MPIs on the system MPI, and the processes which

cause the improvement or deterioration of the system performance can also be identified.

By defining the complementary MPI as the ratio of the inefficiencies of two periods, this paper finds that the system complementary MPI is a linear combination of the process complementary MPIs, and the former is approximately a weighted average of the latter. Although the complementary MPI can also be used to judge whether a unit has improved or not between two periods, its magnitude is difficult to interpret as was illustrated by the general-structure example. For these cases, the conventional MPI can be calculated to help the interpretation.

The DEA model used in this study is the CCR model under the assumption of constant returns-to-scale. For series systems with only two processes, Kao and Hwang (2011) were able to use the BCC model (Banker et al., 1984) to calculate the efficiency under the assumption of variable returns-to-scale. Its extension to the calculation and decomposition of MPI should be straightforward. For parallel systems, replacing the CCR model by the BCC model is not difficult, as briefly mentioned in Kao (2009). How to generalize to cases of variable returns-to-scale for series structures with more than two processes and general network structures are not so simple, and is a direction for future research.

References

1. Banker, R. D., Chang, H. S., Natarajan, R., 2005. Productivity change, technical progress, and relative efficiency change in the public accounting industry. Management Science 51 291-304.
2. Banker, R. D., Charnes, A., Cooper, W. W., 1984. Some models for estimating technical and scale efficiencies in data envelopment analysis. Management Science 30 1078-1092.
3. Caves, D. W., Christensen, L. R., Diewert, W. E., 1982a. The economic theory of index numbers and the measurement of input, output, and productivity. Econometrica 50 1393-1414.
4. Caves, D. W., Christensen, L. R., Diewert, W. E., 1982b. Multilateral comparisons of output, input, and productivity using superlative index numbers. The Economic Journal 92 73-86.
5. Chang, H., Choy, H. L., Cooper, W. W., Ruefli, T. W., 2009. Using Malmquist indexes to measure changes in the productivity and efficiency of US accounting firms before and after the Sarbanes-Oxley Act. Omega 37 951-960.
6. Charnes, A., Cooper, W.W., 1984 The non-Archimedean CCR ratio for efficiency analysis: A rejoinder to Boyd and Färe. European J. Operational Research 15 333- 334.
7. Charnes, A., Cooper, W. W., Rhodes, E., 1978. Measuring the efficiency of decision making units. European J. Operational Research 2 429-444.
8. Chen, Y., Cook, W. D., Li, N., Zhu, J., 2009a. Additive efficiency decomposition in two-stage DEA. European J. Operational Research 196 1170-1176.
9. Chen, Y., Liang, L., Zhu, J., 2009b. Equivalence in two-stage DEA approaches. European Journal of Operational Research 193 600-604.
10. Färe, R., Grosskopf, S., Norris, M., Zhang, Z., 1994. Productivity growth, technical progress and efficiency change in industrialized countries. American Economic Review 84 66-83.
11. Kao, C., 2000. Measuring the performance improvement of Taiwan forests after reorganization. Forest Science 46 577-584.

12. Kao, C., 2009. Efficiency measurement for parallel production systems. European J. Operational Research 196 1107-1112.
13. Kao, C., 2014. Network data envelopment analysis: A review. European J. Operational Research 239, 1-16.
14. Kao, C., 2016. Measurement and decomposition of the Malmquist productivity index for parallel production systems. (under journal review)
15. Kao, C., Hwang, S.N., 2008. Efficiency decomposition in two-stage data envelopment analysis: An application to non-life insurance companies in Taiwan. European J. Operational Research 185 418-429.
16. Kao, C., Hwang, S.N., 2011. Decomposition of technical and scale efficiencies in two-stage production systems. European J. Operational Research 211, 515-519
17. Kao, C., Hwang, S. N., 2014. Multi-period efficiency and Malmquist productivity index in two-stage production systems. European J. Operational Research 232, 512-521
18. Lewis, H. F., Sexton, T. R., 2004. Network DEA: Efficiency analysis of organizations with complex internal structure, Computers & Operations Research 31 1365-1410.
19. Pastor, J. T., Lovell, C. A. K., 2005. A global Malmquist productivity index. Economics Letters 88 266-271.