Different tree approaches to the problem of counting numerical semigroups by genus

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1 Numerical semigroups

Let $\mathbb{N}_0$ denote the set of non-negative integers. A numerical semigroup is a subset of $\mathbb{N}_0$ which is closed under addition, contains 0, and its complement in $\mathbb{N}_0$ is finite.

Numerical semigroups model, for instance, the amounts of money that can be withdrawn from an ideal cash point or the number of nodes of combinatorial configurations [BS12]. They appear in algebraic geometry, as they model Weierstrass non-gaps, and in music theory as they are the inherent structure of the set of numbers of semitones of the intervals of each overtone of a given fundamental tone with respect to the fundamental tone, when the physical model of the harmonic series is discretized into an equal temperament [Bra17].

For a numerical semigroup Λ the elements in $\mathbb{N}_0 \setminus \Lambda$ are called *gaps* and the number of gaps is the *genus* of the semigroup. The largest gap is called the *Frobenius number* and the non-gap right after the Frobenius number is the *conductor*. The *multiplicity* m of a numerical semigroup is its first non-zero non-gap. A numerical semigroup different than $\mathbb{N}_0$ is said to be *ordinary* if its gaps are all in a row. The *generators* of a numerical semigroup are those non-zero non-gaps which can not be obtained as the sum of two smaller non-gaps. One general reference for numerical semigroups is [RG09].

As an example, the set $\Lambda = \{0, 4, 5, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, \dots\}$ is a numerical semigroup with Frobenius number 11, conductor 12, genus 6 and multiplicity 4. Its generators are 4 and 5.

It was conjectured in [Bra08] that the number n_g of numerical semigroups of genus g asymptotically behaves like the Fibonacci numbers. More precisely, it was conjectured that $n_g \geq n_{g-1} + n_{g-2}$, that the limit of the ratio $\frac{n_g}{n_{g-1}+n_{g-2}}$ is 1 and so that the limit of the ratio $\frac{n_g}{n_{g-1}}$ is the the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. Many other papers deal with the sequence n_g [Kom89, Kom98, Bra09, BdM07, BB09, Eli10, Zha10, BGP11, Kap12, BR12, Bra12, ODo13, BT17, FH16, BF18, Kap17] and Alex Zhai gave a proof for the asymptotic Fibonacci-like behavior of n_g [Zha13]. However, it has still not been proved that n_g is increasing.

We will see how we can approach this problem and other problems using three different constructions of trees and forests whose nodes are numerical semigroups.

2 The tree of all numerical semigroups

All numerical semigroups can be organized in an infinite tree $\mathcal{T}$ whose root is the semigroup $\mathbb{N}_0$ and in which the parent of a numerical semigroup Λ is the numerical semigroup Λ' obtained by adjoining to Λ its Frobenius number. For instance, the parent of the semigroup $\Lambda = \{0, 4, 5, 8, 9, 10, 12, \dots\}$ is the semigroup $\Lambda' = \{0, 4, 5, 8, 9, 10, \mathbf{11}, 12, \dots\}$. In turn, the descendants of a numerical semigroup are the semigroups we obtain by taking away one by one the generators that are larger or equal to the conductor of the semigroup. The parent of a numerical semigroup of genus g has genus $g-1$ and all numerical semigroups are in $\mathcal{T}$, at a

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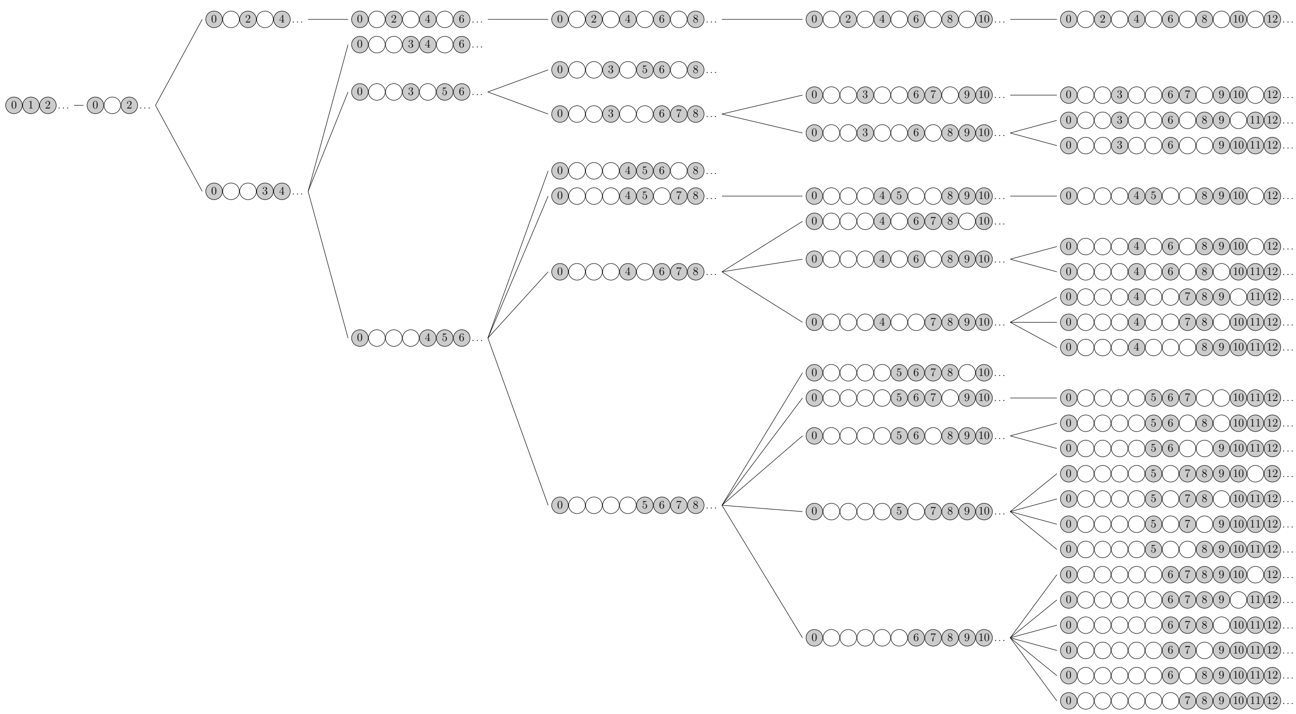

Figure 1: The tree $\mathcal{T}$ up to depth 7

depth equal to its genus. In particular, n_g is the number of nodes of $\mathcal{T}$ at depth g. This construction was already considered in [RGGJ03]. Figure 1 shows the tree up to depth 7.

3 The tree of numerical semigroups of a given genus

In [Bra12] a new tree construction is introduced as follows. The *ordinarization transform* of a non-ordinary semigroup Λ with Frobenius number F and multiplicity m is the set $\Lambda' = \Lambda \setminus \{m\} \cup \{F\}$. For instance, the ordinarization transform of the semigroup $\Lambda = \{0, \mathbf{4}, 5, 8, 9, 10, 12, \dots\}$ is the semigroup $\Lambda' = \{0, 5, 8, 9, 10, \mathbf{11}, 12, \dots\}$. As an extension, the ordinarization transform of an ordinary semigroup is itself. Note that the genus of the ordinarization transform of a semigroup is the genus of the semigroup.

The definition of the ordinarization transform of a numerical semigroup allows the construction of a tree $\mathcal{T}_g$ on the set of all numerical semigroups of a given genus rooted at the unique ordinary semigroup of this genus, where the parent of a semigroup is its ordinarization transform and the descendants of a semigroup are the semigroups obtained by taking away a generator larger than the Frobenius number and adding a new non-gap smaller than the multiplicity in a licit place. To illustrate this construction with an example in Figure 2 we depicted $\mathcal{T}_7$.

One significant difference between $\mathcal{T}_g$ and $\mathcal{T}$ is that the first one has only a finite number of nodes, indeed, it has n_g nodes, while $\mathcal{T}$ is an infinite tree. It was conjectured in [Bra12] that the number of numerical semigroups in $\mathcal{T}_g$ at a given depth is at most the number of numerical semigroups in $\mathcal{T}_{g+1}$ at the same depth. This was proved in the same reference for the lowest and largest depths. This conjecture would prove that $n_{g+1} \geq n_g$.

4 The forest of numerical semigroups of a given genus with a tree per conductor

Almost-ordinary semigroups are those semigroups for which the multiplicity equals the genus and so, there is a unique gap larger than m. The *sub-Frobenius number* of a non-ordinary semigroup Λ with conductor c is the Frobenius number of $\Lambda \cup \{c-1\}$. The subconductor and dominant of a semigroup are, respectively, the smallest and largest integers in its interval of non-gaps immediately previous to the conductor. If Λ is a non-ordinary and non almost-ordinary semigroup, with multiplicity m and genus g, and sub-Frobenius number u, then $\Lambda \cup \{u\} \setminus \{m\}$ is another numerical semigroup which we denote the *almost-ordinarization transform* of Λ. For instance, the almost-ordinarization transform of the semigroup $\Lambda = \{0, \mathbf{4}, 5, 8, 9, 10, 12, \dots\}$ is the semigroup $\Lambda' = \{0, 5, \mathbf{7}, 8, 9, 10, 12, \dots\}$.

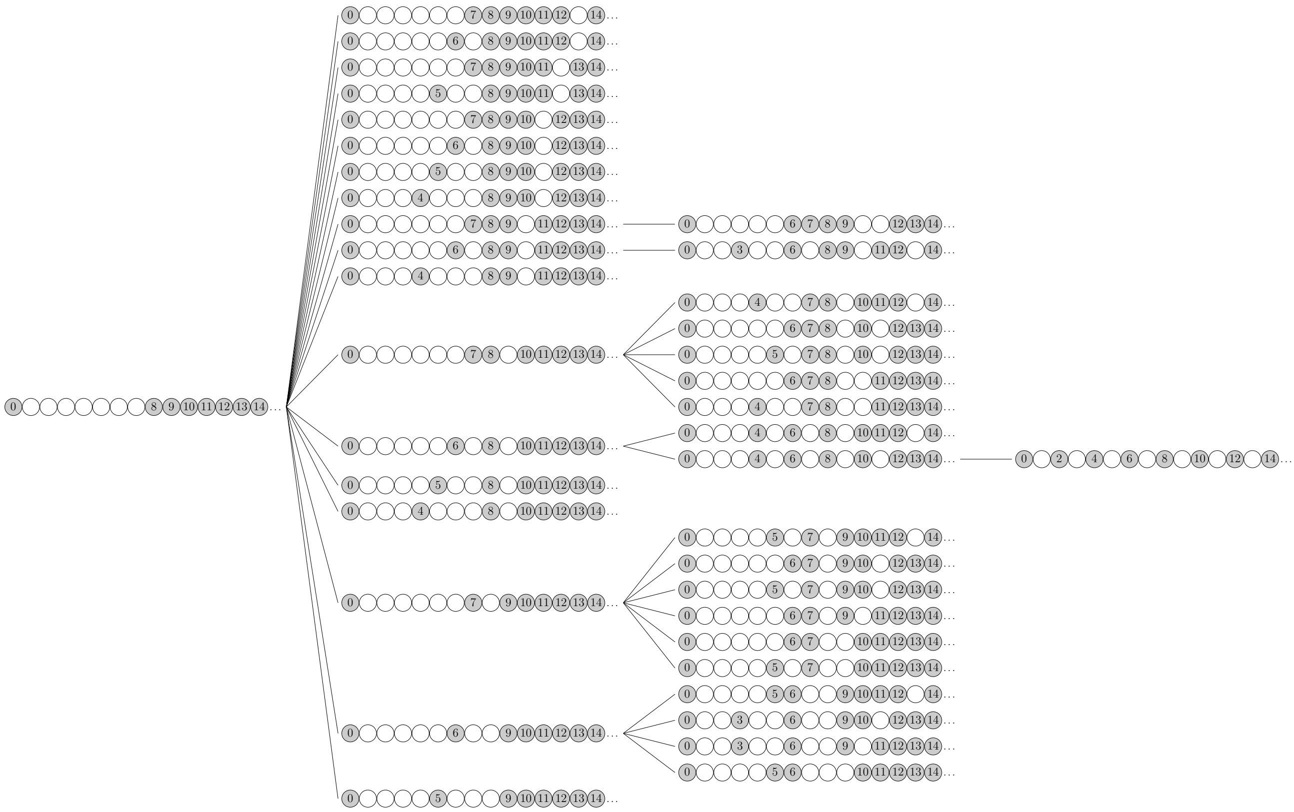

Figure 2: The tree $\mathcal{T}_7$

This transform can be applied subsequently and at some step we will attain the unique almost-ordinary semigroup of that genus and conductor, that is, the semigroup $\{0, g, g+1, \dots, c-2, c, c+1, \dots\}$. This defines, for each fixed genus and conductor, a tree rooted at this unique almost-ordinary semigroup of that genus and conductor. The parent of a semigroup is its almost-ordinarization. The descendants of a numerical semigroup are the semigroups we obtain by taking away one by one the generators between the subconductor and the dominant of the semigroup and adjoining a non-gap in a licit place between 0 and the multiplicity of the semigroup.

Figure 3 shows the forest of genus 7.

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References

[BT17] M. Bernardini and F. Torres. Counting numerical semigroups by genus and even gaps. *Discrete Mathematics*, 340(12):2853–2863, 2017.

[BR12] V. Blanco and J. C. Rosales. The set of numerical semigroups of a given genus. *Semigroup Forum*, 85(2):255–267, 2012.

[BGP11] V. Blanco, P. A. García-Sánchez and J. Puerto. Counting numerical semigroups with short generating functions. *International Journal of Algebra and Computation*, 21(7):1217–1235, 2011.

[Bra17] M. Bras-Amorós. Metric molds, fractal molds on the golden ratio, half-closed-pipe admissible molds, and the mathematical emergence of the well tempered harmonic semigroup. At *https://arxiv.org/abs/1703.01077*, 2017 (to appear on *Journal of Mathematics and Music*).

[Bra08] M. Bras-Amorós. Fibonacci-like behavior of the number of numerical semigroups of a given genus. *Semigroup Forum*, 76(2):379–384, 2008.

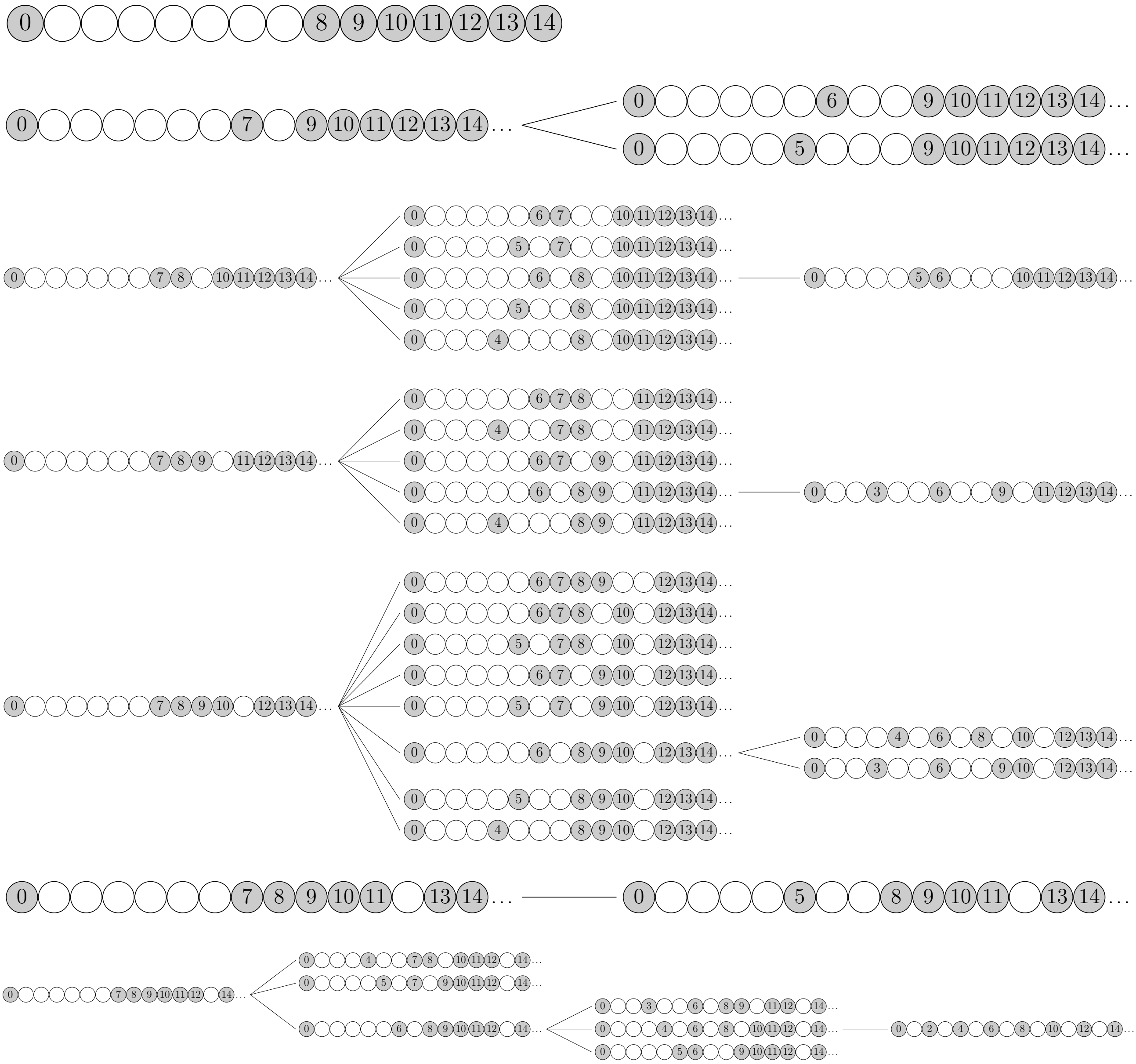

Figure 3: The forest of genus 7

[Bra09] M. Bras-Amorós. Bounds on the number of numerical semigroups of a given genus. *Journal of Pure and Applied Algebra*, 213(6):997–1001, 2009.

[Bra12] M. Bras-Amorós. The ordinarization transform of a numerical semigroup and semigroups with a large number of intervals. *Journal of Pure and Applied Algebra*, 216(11):2507–2518, 2012.

[BB09] M. Bras-Amorós and S. Bulygin. Towards a better understanding of the semigroup tree. *Semigroup Forum*, 79(3):561–574, 2009.

[BdM07] M. Bras-Amorós and A. de Mier. Representation of numerical semigroups by Dyck paths. *Semigroup Forum*, 75(3):677–682, 2007.

[BF18] M. Bras-Amorós and J. Fernández-González. Computation of numerical semigroups by means of seeds. *Mathematics of Computation*, 87(313):2539–2550, 2018.

[BS12] M. Bras-Amorós and K. Stokes. The semigroup of combinatorial configurations. *Semigroup Forum*, 84(1):91–96, 2012.

[Eli10] S. Elizalde. Improved bounds on the number of numerical semigroups of a given genus. *Journal of Pure and Applied Algebra*, 214(10):1862–1873, 2010.

[FH16] J. Fromentin and F. Hivert. Exploring the tree of numerical semigroups. *Mathematics of Computation*, 85(301):2553–2568, 2016.

[Kap12] N. Kaplan. Counting numerical semigroups by genus and some cases of a question of Wilf. *Journal of Pure and Applied Algebra*, 216(5):1016–1032, 2012.

[Kap17] N. Kaplan. Counting numerical semigroups. *American Mathematical Monthly*, 124(9):862–875, 2017.

[Kom89] J. Komeda. On non-Weierstrass gap sequqences. Technical report, Kanagawa Institute of Technology B-13, 1989.

[Kom98] J. Komeda. Non-Weierstrass numerical semigroups. *Semigroup Forum*, 57(2):157–185, 1998.

[ODo13] E. O'Dorney. Degree asymptotics of the numerical semigroup tree. *Semigroup Forum*, 87(3):601–616, 2013.

[RG09] J. C. Rosales and P. A. García-Sánchez. *Numerical semigroups.* Vol. 20 of Developments in Mathematics. Springer, New York, 2009.

[RGGJ03] J. C. Rosales, P. A. García-Sánchez, J. I. García-García, and J. A. Jiménez Madrid. The oversemigroups of a numerical semigroup. *Semigroup Forum*, 67(1):145–158, 2003.

[Zha13] A. Zhai. Fibonacci-like growth of numerical semigroups of a given genus. *Semigroup Forum*, 86(3):634–662, 2013.

[Zha10] Y. Zhao. Constructing numerical semigroups of a given genus. *Semigroup Forum*, 80(2):242–254, 2010.