A Kinetic Model of the Dynamics of Compromise in Large Multi-Agent Systems

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***Abstract*—Compromise is one of the primary phenomena that govern the dynamics of the opinion in multi-agent systems. In this paper, compromise is isolated from other phenomena, and it is studied using a statistical framework designed to investigate collective properties of large multi-agent systems. The proposed framework is completed with the details needed to model compromise, and differential problems which describe the dynamics of the opinion under suitable hypotheses are presented. Long-time asymptotic solutions of obtained differential problems are discussed to confirm that compromise makes multi-agent systems tend to reach consensus. It is proved that compromise makes all agents tend to share the same opinion, and that the value of the asymptotic opinion can be expressed in terms of the characteristics of the multi-agent system and of the initial distribution of the opinion. Obtained analytic results are confirmed by independent simulations in an illustrative case.**

I. Introduction

The study of the multiple aspects of *opinion formation* in multi-agent systems is an important research topic that finds applications in various fields, e.g., control theory, robotics, biology, sociology, and artificial intelligence. Usually, the study of opinion formation in multi-agent systems assumes that each agent has an opinion on a given topic, and that the opinion can be expressed in terms of a value in a suitable range. Agents interact by exchanging messages on a discussed topic, and interactions make agents change their opinions. Interactions are typically described by suitable *interaction rules* [1] that model how each interaction changes the opinions of involved agents. Interaction rules take into account the sociological phenomena that describe how agents form their opinions, and they model the dynamics of the opinion of each agent. When the number of agents in the considered multi-agent system is large, the study of the dynamics of the opinion of each agent is not feasible, and the analysis of collective properties of the opinion of the multi-agent system as a whole is preferred. Such collective properties are typically investigated using statistical approaches that account for the dynamics of the opinion in terms of the long-time asymptotic dynamics of aggregate values, like the average opinion and the variance of the opinion. The literature already proposes various approaches to study the dynamics of collective properties of the opinion, which include those based on thermodynamics [2], on Bayesian networks [3], on gossip protocols [3], on flocking models [4], on graph Laplacians [5], and on cellular programming [6], [7].

In this paper, the long-time asymptotic behaviour of the collective dynamics of the opinion in multi-agent systems is studied using the very general approach proposed by *mathematical kinetic theories* (e.g., [8] and referenced literature), which are intended to investigate the collective properties of groups of interacting peers. The prototypical example of a mathematical kinetic theory is the classic kinetic theory of gases, which studies collective properties of gases, like temperature and pressure, starting from the details of interactions among molecules (or atoms, for noble gases). When studied gases are made of different types of molecules, classic kinetic theory of gases is normally generalized to the kinetic theory of gas mixtures, which is another mathematical kinetic theory that accounts for gases with molecules with different properties. A rather obvious parallelism between the molecules of a gas and the agents of a multi-agent system can be drawn to adopt generalisations of the kinetic theory of gases to study collective properties of multi-agent systems. This has already been done, e.g., in [9], where the similarity between the distribution of wealth in a simple economy and the density of molecules in a gas is studied, or in [10], where simple models of opinion dynamics are studied. Note that besides the general framework of mathematical kinetic theories, few results from the kinetic theory of gases can be adapted to other contexts because the details of the interaction rules which model collisions among molecules in gases are significantly different from those of the interaction rules that model cooperation and competition among agents in multi-agent systems.

The general framework of mathematical kinetic theories is used in this paper to study the characteristics of *compromise* [10], which is one of the most important phenomena that govern the dynamics of the opinion in multi-agent systems. The major contribution of this study is to generalise results from the literature (e.g., [1], [11]) by allowing each agent to have a specific propensity to change its opinion because of interactions. Such a generalisation is important because it is meant to model agents that act with some degree of autonomy.

This paper is organized as follows. Section II completes the generic framework of mathematical kinetic theories with the details needed to model compromise, and it provides results on the dynamics of compromise. Section III reports results of illustrative simulations which confirm the behaviors predicted by the proposed model of compromise. Finally, Section IV concludes the paper and outlines future developments.

II. A Kinectic Model of Compromise

The study of the dynamics of the opinion normally considers a number of sociological phenomena that can be used to model the behaviours of agents (e.g., [11] and referenced literature). Among considered phenomena, some of the most extensively studied are:

- *Compromise:* the tendency of agents to move their opinions towards those of agents they interact with, trying to reach consensus [10];
- *Diffusion:* the phenomenon according to which the opinion of each agent can be influenced by the social context [12];
- *Homophily:* the process according to which agents interact only with those with similar opinions [13];
- *Negative influence:* the idea according to which agents evaluate their peers, and they only interact with those with positive scores [14];
- *Opinion noise:* the process according to which a random additive variable may lead to arbitrary opinion changes with small probability [15]; and
- *Striving for uniqueness:* the phenomenon based on the idea that agents want to distinguish themselves from others and, hence, they decide to change their opinions if too many agents share the same opinion [16].

Models based on mathematical kinetic theories have already been proposed in the literature to study all mentioned phenomena analytically (e.g., [1], [11], [17]–[28], and referenced literature). The major contribution of this paper with respect to existing literature regards the possibility to consider agents with different propensity to change their opinions because of interactions with other agents. The paper studies a kinetic framework that associates agents with some level of autonomy by allowing each agent to have a specific propensity to change its opinion because of interactions. Compromise is isolated from all other phenomena and its dynamics is studied quantitatively using the proposed framework. Note that the analytic model described in this section can be enriched to incorporate all mentioned phenomena by adding specific contributions to adopted interaction rules, but such a generalization is not discussed here.

Starting from the pioneering work of De Groot [29], a key ingredient of opinion models has been the idea of updating the opinions of agents after an interaction by properly weighting the *pre-interaction* opinions of all interacting agents. Here, only binary interactions are considered, and an agent s with opinion $v \in I$ is supposed to interact with another agent r with opinion $w \in I$, where $I = [-1, 1]$ without loss of generality. For binary interactions, the *post-interaction* opinions of the two interacting agents are assumed to depend on the respective pre-interaction opinions as described by the following interaction rules

$$\begin{cases} v^* = v - \gamma_{s,r}(v - w) \\ w^* = w - \gamma_{r,s}(w - v), \end{cases} \tag{1}$$

where v^* and w^* are the opinions of agents s and r, respectively, after the interaction. Observe that if n is the number of agents in the multi-agent system, the considered model involves n^2 parameters $\{\gamma_{s,r}\}_{s,r=1}^n$, where $\gamma_{s,r}$ measures the propensity of a generic agent s to change its opinion in favor of the opinion of another agent r. Note that, according to (1), if $\gamma_{s,r}$ is nearly 0, agent s is not inclined to change its opinions towards that of agent r. For this reason, values of $\gamma_{s,r}$ close to 0 characterise skeptical agents. At the opposite, if $\gamma_{s,r} \simeq 1/2$, then $v^* \simeq 1/2(v + w)$, and such a $\gamma_{s,r}$ can be used to characterize agents that get easily convinced.

The choice of parameters $\{\gamma_{s,r}\}_{s,r=1}^n$ is crucial to determine the characteristics of the model, and it deserves further discussions. First, note that post-interaction opinions v^* and w^* still belong to interval I where opinions are defined because

$$|v^*| \le (1 - \gamma_{s,r} + \gamma_{s,r}) \max\{|v|, |w|\}$$
$$|w^*| \le (1 - \gamma_{r,s} + \gamma_{r,s}) \max\{|v|, |w|\},$$

and, since $\max\{|v|, |w|\} \le 1$, it can be readily concluded that $|v^*| \le 1$ and $|w^*| \le 1$. Then, from (1) it can be derived that the difference of the opinions of two interacting agents after an interaction is

$$v^* - w^* = \varepsilon_{rs}(v - w), \tag{2}$$

where

$$\varepsilon_{rs} = 1 - (\gamma_{r,s} + \gamma_{s,r}). \tag{3}$$

Observe that the model aims at describing compromise, which is the idea that the opinions of two interacting agents get closer after an interaction, and from (2), such a requirement corresponds to

$$|v^* - w^*| = |\varepsilon_{rs}||v - w| < |v - w| \tag{4}$$

for values of ε_{rs} such that $|\varepsilon_{rs}| \le 1$. This condition can be written in terms of the parameters of the model as

$$0 < \gamma_{s,r} + \gamma_{r,s} < 2, \tag{5}$$

and it is certainly satisfied under the assumption that

$$0 < \gamma_{s,r} < 1. \tag{6}$$

Under this assumption, it is reasonable to expect that, after a sufficiently large number of interactions, all agents would end up with the same opinion. In addition, a complete model of compromise requires that if two agents with different opinions interact, they tend to preserve their opinions. This can be modelled by introducing the assumption that the post-interaction opinion of an agent is normally closer to its pre-interaction opinion than to the pre-interaction opinion of the agent it interacts with, which corresponds to the following conditions

$$|v^* - v| < |v^* - w| \quad \text{and} \quad |w^* - w| < |w^* - v|. \tag{7}$$

Observe that conditions (6) are not sufficient to guarantee that inequalities (7) are satisfied. However, by imposing the additional condition

$$0 < \gamma_{s,r} < \frac{1}{2} \tag{8}$$

it can be easily proved that the post-interaction opinion of an agent is normally closer to its pre-interaction opinion than to the pre-interaction opinion of the agent it interacts with

$$\begin{aligned} |v^* - v| &= \gamma_{s,r}|v - w| < |v^* - w| \\ |w^* - w| &= \gamma_{r,s}|w - v| < |w^* - v|. \end{aligned} \tag{9}$$

In summary, in order to properly model compromise, from now on all parameters $\{\gamma_{s,r}\}_{s,r=1}^{n}$ are assumed to be defined in the interval $(0, 1/2)$. Such a choice of parameters can be used to study relevant collective properties of multi-agent systems where only compromise is relevant to opinion formation, and where adopted interaction rules are (1).

The interest now is on investigating the temporal evolution of the opinion of a generic agent s. This can be done using the general results of mathematical kinetic theory, as described, e.g., in [1]. In particular, using the adopted interaction rules, the *weak form of the Boltzmann equation* [1] relative to agent s and test function $\phi(v) = v$ can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}u_s(t) = \sum_{\substack{r=1 \\ r\neq s}}^{n}\gamma_{s,r}\beta_{s,r}u_r(t) - u_s(t)\sum_{\substack{r=1 \\ r\neq s}}^{n}\gamma_{s,r}\beta_{s,r}, \tag{10}$$

where $u_s(t)$ is the opinion of agent s at time $t \geq 0$, $u_r(t)$ is the opinion of another agent r at the same time t, and $\beta_{s,r}$ is the probability that agent s interacts with agent r per unit time, with $\beta_{s,s} = 0$ by definition.

Equation (10) is for a generic agent s, and when considering n agents, it represents a single equation of a homogeneous system of first-order linear differential equations. Such a kind of system can be solved in closed form, and its solution is generally expressed using matrix notation. Let $\underline{u}$ be the vector of size n whose $s-$th component is u_s, then the system of equations whose $s-$th equation is (10), can be written in matrix notation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{u}(t) = \underline{\underline{C}}\,\underline{u}(t) \tag{11}$$

where $\underline{\underline{C}}$ is the $n \times n$ matrix of the coefficients of the system, and it has the following explicit expression

$$\underline{\underline{C}} = \begin{pmatrix} -\sum\limits_{\substack{r=1 \\ r\neq 1}}^{n}\gamma_{1,r}\beta_{1,r} & \gamma_{1,2}\beta_{1,2} & \cdots & \gamma_{1,n}\beta_{1,n} \\ \gamma_{2,1}\beta_{2,1} & -\sum\limits_{\substack{r=1 \\ r\neq 2}}^{n}\gamma_{2,r}\beta_{2,r} & \cdots & \gamma_{2,n}\beta_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n,1}\beta_{n,1}f & \gamma_{n,2}\beta_{n,2} & \cdots & -\sum\limits_{\substack{r=1 \\ r\neq n}}^{n}\gamma_{n,r}\beta_{n,r} \end{pmatrix}.$$

The rest of this section is devoted to the study of the properties of $\underline{\underline{C}}$ which are needed to study the long-time asymptotic dynamics of the solutions of (11). Actually, presented properties of $\underline{\underline{C}}$ are sufficient to prove that the opinions of all agents tend to the same value, and that consensus is reached asymptotically. Note that some of the major properties of matrix $\underline{\underline{C}}$ are similar to those derived in [4] using a related, but significantly different, model.

In order to properly analyse relevant properties of $\underline{\underline{C}}$, some classic results valid for complex matrices are needed. First, recall that a generic $m \times m$ complex matrix $\underline{\underline{B}}$ is said to be *diagonally dominant* if the following inequality holds for all its elements $\{b_{s,r}\}_{s,r=1}^{m}$

$$|b_{s,s}| \geq \sum_{\substack{r=1 \\ r\neq s}}^{m}|b_{s,r}|. \tag{12}$$

Then, also recall that a generic $m \times m$ complex matrix $\underline{\underline{B}}$ is said to be *strictly diagonally dominant* if the following strict inequality holds for all its elements $\{b_{s,r}\}_{s,r=1}^{m}$

$$|b_{s,s}| > \sum_{\substack{r=1 \\ r\neq s}}^{m}|b_{s,r}|. \tag{13}$$

Two classic results on complex matrices, i.e., the Gershgorin circle theorem (e.g., [30]) and the Levy-Desplanques theorem (e.g., [30]), are sufficient to prove the following propositions, which characterise matrix $\underline{\underline{C}}$.

Proposition 1. *Matrix $\underline{\underline{C}}$ is singular.*

Proof. The singularity of matrix $\underline{\underline{C}}$ follows from the fact that the sum of the elements in each row is zero. Each diagonal element of $\underline{\underline{C}}$ is defined as the opposite of the sum of the remaining elements on the same row. Actually, denoting as $\underline{d}_s$ the $s-$th column of matrix $\underline{\underline{C}}$, the following equality holds

$$\sum_{s=1}^{n}\underline{d}_s = \underline{0} \tag{14}$$

where $\underline{0}$ is a vector of length n with all elements equal to 0. Therefore, it can then be concluded that $\det(\underline{\underline{C}}) = 0$. □

Proposition 2. *Matrix $\underline{\underline{C}}$ is diagonally dominant, but not strictly diagonally dominant, and the following hold*

$$|c_{s,s}| = \sum_{\substack{r=1 \\ r\neq s}}^{n}|c_{s,r}| \tag{15}$$

Proof. Observe that the non-diagonal elements of $\underline{\underline{C}}$ are positive. Adopted assumptions ensures that the parameters $\{\gamma_{s,r}\}_{s,r=1}^{n}$ are positive, and therefore

$$|c_{s,r}| = c_{s,r} \quad \text{for} \quad s \neq r. \tag{16}$$

At the opposite, it is easy to show that diagonal elements of $\underline{\underline{C}}$ are negative. Therefore, the following, which corresponds to (15), hold

$$|c_{s,s}| = -c_{s,s} = \sum_{\substack{r=1 \\ r\neq s}}^{n}\gamma_{s,r}\beta_{s,r} = \sum_{\substack{r=1 \\ r\neq s}}^{n}|c_{s,r}| \tag{17}$$

where the last equality follows from the definition of $\underline{\underline{C}}$ and from (16). □

Proposition 3. *The rank of matrix $\underline{\underline{C}}$ is $n-1$.*

Proof. Since, according to Proposition 1, matrix $\underline{\underline{C}}$ is singular, it is not full rank, and the following inequality holds

$$\mathrm{rank}(\underline{\underline{C}}) < n. \tag{18}$$

In order to show that $\mathrm{rank}(\underline{\underline{C}}) = n-1$, the principal minor of order $n-1$ is proved to be full rank. First, let us show that the principal minor of order $n-1$ is strictly diagonally dominant. This is easily proved by the following inequality

$$|c_{s,s}| = \sum_{\substack{r=1\\r\neq s}}^{n} |c_{s,r}| > \sum_{\substack{r=1\\r\neq s}}^{n-1} \gamma_{s,r}\beta_{s,r} = \sum_{\substack{r=1\\r\neq s}}^{n-1} |c_{s,r}|, \tag{19}$$

where the first equality follows from Proposition 2, the inequality follows from the fact that in the second sum the (positive) term relative to $r = n$ is omitted, and the last equality follows from (16). Note that the Levy-Desplanques theorem states that a matrix which is strictly diagonally dominant is also full rank, which proves the proposition because it can be concluded that the principal minor of order $n-1$ is full rank and, therefore, that the rank of $\underline{\underline{C}}$ is $n-1$. □

Proposition 4. *One of the eigenvalues of matrix $\underline{\underline{C}}$ is 0, and it has multiplicity 1.*

Proof. The fact that 0 is an eigenvalue of $\underline{\underline{C}}$ follows from the fact that, as observed in Proposition 1, $\underline{\underline{C}}$ is singular. The fact that the multiplicity of eigenvalue 0 is 1 follows from the fact that, as shown in Proposition 3, the rank of $\underline{\underline{C}}$ is $n-1$. □

Proposition 5. *The real parts of the eigenvalues of matrix $\underline{\underline{C}}$ are not positive, and they are 0 only for eigenvalue $\lambda_0 = 0$.*

Proof. Consider the Gershgorin disks relative to matrix $\underline{\underline{C}}$, which are defined as disks in the complex plane with the following structure

$$\mathcal{K}_s = \left\{ z \in \mathbb{C} : |z + \sum_{\substack{r=1\\r\neq s}}^{n} \gamma_{s,r}\beta_{s,r}| \leq \sum_{\substack{r=1\\r\neq s}}^{n} \gamma_{s,r}\beta_{s,r} \right\}. \tag{20}$$

Since the elements of $\underline{\underline{C}}$ are real, each disk $\mathcal{K}_s$ is centered on the real axis at

$$-\rho_s = \sum_{\substack{r=1\\r\neq s}}^{n} \gamma_{s,r}\beta_{s,r}, \tag{21}$$

and its radius is exactly ρ_s, so that all disks intersect the immaginary axis only at the origin. Therefore, if $\{\lambda_s\}_{s=1}^{n}$ are the eigenvalues of matrix $\underline{\underline{C}}$, from (20), the following inequalities hold

$$-2\rho_s \leq \mathrm{Re}(\lambda_s) \leq 0, \tag{22}$$

which proves the proposition and it also provides a lower bound on the real parts of eigenvalues. □

Proposition 6. *The long-time asymptotic opinions of all agents are equal to a real value which depends on initial opinions, on values $\{\gamma_{r,s}\}_{r,s=1}^{n}$, and on values $\{\beta_{s,r}\}_{r,s=1}^{n}$.*

Proof. In order to prove the proposition, the solution of the system of first-order linear differential equations (11) must be studied. Observe that the entire system depends on the $n \times n$ matrix $\underline{\underline{C}}$. Denoting as $\lambda_0 = 0$ and $\{\lambda_h\}_{h=1}^{k}$ the $(k+1)$ eigenvalues of $\underline{\underline{C}}$, and as $\{\eta_h\}_{h=0}^{k}$ their corresponding multiplicities, with $\eta_0 = 1$ for Proposition 4, the solutions of (11) can be written as

$$u_s(t) = \sum_{h=0}^{k} \mathrm{e}^{\lambda_h t} P_s^{(h)}(t) \tag{23}$$

where $P_s^{(h)}(t)$ are polynomials of degree $\eta_h - 1$

$$P_s^{(h)}(t) = \sum_{j=0}^{\eta_h - 1} a_{j,s}^{(h)} t^j. \tag{24}$$

Since, according to Proposition 4, λ_0 is an eigenvalue with multiplicity $\eta_0 = 1$, the degree of the n polynomials $\{P_s^{(0)}(t)\}_{s=1}^{n}$ is $\eta_0 - 1 = 0$, so that for all s

$$P_s^{(0)}(t) = a_{0,s}. \tag{25}$$

Moreover, according to classic results on systems of linear differential equations, $\{a_{0,s}\}_{s=1}^{n}$ are proportional to the components of an eigenvector relative to the eigenvalue $\lambda_0 = 0$. In other words

$$\begin{pmatrix} a_{0,1} & a_{0,2} & \dots & a_{0,n} \end{pmatrix}^{\mathsf{T}} = l_0 \underline{e_0}, \tag{26}$$

where $\underline{e_0}$ is a vector of length n which satisfies

$$\underline{\underline{C}}\,\underline{e_0} = \underline{0}. \tag{27}$$

Recalling the explicit expression of matrix $\underline{\underline{C}}$, it is evident that vector $\underline{1}$, whose n components are all equal to 1, satisfies (27) and it is therefore an eigenvector of $\underline{\underline{C}}$ relative to λ_0

$$\underline{e_0} = \underline{1}. \tag{28}$$

From (26) it can then be concluded that $\{a_{0,s}\}_{s=1}^{n}$ are all equal

$$a_{0,s} = l_0. \tag{29}$$

According to this results, the solutions $u_s(t)$ computed in (23) can be written as

$$u_s(t) = l_0 + \sum_{h=1}^{k} \mathrm{e}^{\lambda_h t} P_s^{(h)}(t). \tag{30}$$

From (30) it can be concluded that for all solutions

$$\lim_{t\to\infty} u_s(t) = l_0, \tag{31}$$

since all the addends in the sum in (23) converge to zero because all eigenvalues $\{\lambda_h\}_{h=1}^{k}$ have negative real part for Proposition 5. Therefore, all agents would eventually end up with the same opinion, whose value l_0 depends on initial opinions, on values $\{\gamma_{r,s}\}_{r,s=1}^{n}$, and on values $\{\beta_{s,r}\}_{r,s=1}^{n}$. □

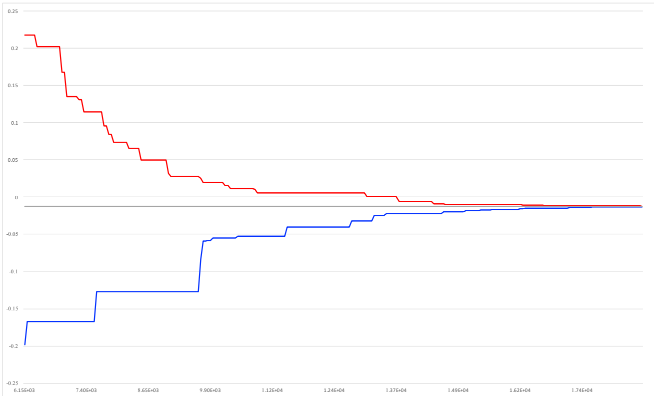

Fig. 1. Plots of minimum (blue), maximum (red), and expected (gray) values of the opinion for the studied scenario as a function of time.

III. Verification by Simulation

This section shows illustrative simulations concerning the dynamics of the opinion in a specific scenario, which is used to verify the expected long-time asymptotic behaviour modelled in previous section. A multi-agent system made of $n = 10^3$ agents is considered, and simulations implementing studied interaction rules are performed by iteratively selecting a random agent and by making selected agent interact with another randomly chosen agent. The opinions of agents are initialised to random values uniformly distributed in interval $I = [-1, 1]$, and their specific propensity to change opinion because of interactions, i.e., parameters $\{\gamma_{s,r}\}_{s,r=1}^{n}$, are fixed to random values uniformly distributed in interval $(\frac{1}{4}, \frac{1}{2})$. The distribution of the opinion after the execution of simulations is compared with the expected long-time asymptotic opinion l_0, which is computed as discussed in previous section. The coherence of the results of simulations with expected value l_0 is evaluated in terms of

$$\check{u}(t) = \min_{1 \le s \le n} u_s(t) \quad \text{and} \quad \hat{u}(t) = \max_{1 \le s \le n} u_s(t), \tag{32}$$

and all simulations are performed until $(\hat{u}(t) - \check{u}(t)) \le 10^{-3}$. Note that presented simulations do not use analytic results from previous section, rather they are direct implementation of considered interaction rules and they are intended only to validate analytic results.

Figure 1 shows the dynamics of $\check{u}(t)$ and $\hat{u}(t)$, and it also shows the expected long-time asymptotic value of the opinion l_0. Note that the opinions of agent converge to the expected value after 18.6×10^3 interactions, with each agent involved in less than 60 interactions.

IV. Conclusions

This paper presented analytic results that characterise compromise, which is one of the major phenomena used to describe opinion formation in multi-agent systems. The paper used the general framework of mathematical kinetic theories to model compromise and to derive results on the long-time asymptotic behaviour of multi-agent systems where only compromise is considered relevant. Finally, the paper showed simulations ran in an illustrative case by directly implementing chosen interaction rules to confirm analytic results.

The proposed analytic framework can be used to study all sociological phenomena that are normally associated with opinion formation, e.g., compromise, diffusion, and homophily. This paper considered only compromise in order to derive analytic results intended to characterise the dynamics of compromise independently from other phenomena. Actually, the fact that compromise makes the opinion of all agents tend to a single value is not surprising, but the dynamics of such an asymptotic behaviour was not so obvious. Compromise makes a multi-agent system reach consensus exponentially fast, and the proposed model allows estimating the rate at which the consensus value is approached on the basis of the propensity of each agent to change its opinion because of interactions, and on other relevant characteristics of the multi-agent system. Obtained analytic results are confirmed by independent simulations performed by directly implementing adopted interaction rules.

Methodologically, the major advantage that is expected from the adoption of a kinetic approach to the study of the dynamics of the opinion is that mathematical kinetic theories are inherently analytic and they can provide analytic descriptions of the collective properties of the opinion. Such a characteristic of mathematical kinetic theories ensures that obtained results can be used not only as *descriptive* tools capable of explaining observations, but that they can also be used as *prescriptive* tools to govern the dynamics of studied multi-agent systems. As a prescriptive tool, the proposed approach can support the design of multi-agent systems with desired properties because analytic results can be used to identify the actual values of specific parameters to have the multi-agent system behave as intended. In addition, as a descriptive tool, the analytic approach that is developed in this paper can be used as an alternative to simulation. The validity of results of simulations depends on how much selected simulations are representative of studied multi-agent systems. On the contrary, the validity of analytic results is clearly identified by the assumptions adopted to derive them, and such assumptions can also be studied in order to be possibly generalised.

Planned developments of the presented work involve four generalizations. First, the deterministic parameters $\{\gamma_{s,r}\}_{s,r=1}^{n}$ which characterize the propensity of agents to change their opinions because of interactions could be replaced by random variables with suitable distributions. Second, the topology of the multi-agent system could be taken into account and the hypothesis that each agent can freely interact with any other agent could be dropped. Third, more complex interaction rules could be considered to take into account how various phenomena that contribute to opinion formation interact in real situations, and how they jointly contribute to the dynamics of the opinion. Fourth, note that the ideas behind the discussed framework are not limited to the study of opinion dynamics, and the proposed approach could be applied to describe other collective properties of large multi-agent systems.

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