Pink Panthers and Toothless Tigers: Three Problems in Classification

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Abstract. Many aspects of how humans form and combine concepts are notoriously difficult to capture formally. In this paper, we focus on the representation of three particular such aspects, namely overextension, underextension, and dominance. Inspired in part by the work of Hampton, we consider concepts as given through a prototype view, and by considering the interdependencies between the attributes that define a concept. To approach this formally, we employ a recently introduced family of operators that enrich Description Logic languages. These operators aim to characterise complex concepts by collecting those instances that apply, in a finely controlled way, to 'enough' of the concept's defining attributes. Here, the meaning of 'enough' is technically realised by accumulating weights of satisfied attributes and comparing with a given threshold that needs to be met.

1 Introduction

A *green chair* is a chair that is also green, it is an untroubled combination of attributes that gives us a conjunctive concept $G \sqcap C$. It is untroubled because the specific colour of a chair is an attribute that does not affect the 'chairness' of the chair: changing or losing the colour green will not make the chair a non-chair.

However, many combinations of concepts are not easily understood as simple conjunctions, because some defining features of the combined concepts can enter into a subtle conversation: this paper discusses and formally analyses some of those phenomena studied within psychology and cognitive science, where the combination of two (or more) concepts has a more subtle semantics than intersection of sets.

In the classical view of concepts, the extensions of concepts are precisely defined: for each object it would be possible to specify whether it falls within the definition of the concept or not. If the object fits all the necessary and sufficient conditions in the definition of the concept, it will be categorised as an instance of the concept, otherwise it will not. Moreover the combination of concepts is defined in a compositional way. This means that the meaning of a complex concept functionally depends on the meaning of the primitive concepts (or predicates) that compose it. In other words, the definition of a complex concept is exclusively determined by its basic components, and this is often seen to explain in part the ease and prolific ability of humans in creating meaningful phrases. This perspective, that can be traced back to Frege [1891], became a cornerstone of Classical Logic, and moved from there to be also a paradigm in Description Logic. In spite of this, experimental cognitive sciences have shown that this view has great limitations and have proposed alternative frameworks. Cognitive models of concepts and classification are usually grouped into the *prototype view*, the *exemplar view*, and the *knowledge view* also called *theory-theory* (see Margolis and Laurence [1999]; Murphy [2002]), but also Gärdenfors's *theory of conceptual spaces* Gärdenfors [2000] and Barsalou's *theory of frames* Barsalou [1992] enter this category.

The crisp view of concepts in the traditional logic-based paradigm excludes indeterminacy, graded membership and typicality effects from the process of categorisation, while the merely extensional compositionality of concepts does not allow for the representation of a number of cognitive phenomena linked to concept combination.

Overextension in conjunction [Hampton, 2017], situational effects [Barsalou, 1992], dominance effects [Hampton, 1988a], and attributes emergence [Hampton, 1987] are just a subset of the phenomena that hardly reconcile with compositionality of the extensions: the sole sets of entitites that fall under two concepts do not determine the set of entities that fall under the combined concept. How to integrate these phenomena in a logic-based framework is still an open challenge, and the topic of this paper.

We begin with some examples to set the scene. A *toothless tiger* is a tiger that has no teeth.[1] Analysing the literal, rather than the metaphorical, meaning of this conjunctive concept, we include all those tigers which are, although tigers, lacking teeth, thereby losing an important, although not essential property. Similar to the *green chair*, this can be understood as a simple conjunctive concept when the concept of 'tiger' is given in such a way that it allows for grades of typicality (e.g., wrt. a prototype). However, a definition of tiger in terms of classical logic will exclude toothless tigers as non-tigers.

Now consider a *pink panther* as compared to the *green chair*. Whereas a *green chair* might be quite like a *red chair*, just disagreeing in the colour, a *pink panther* is quite unlike a *black panther*. Whilst it is extremely rare for a panther to be black, it is still an optional colour quality attached to a panther.[2] A *pink panther*, on the other hand, assumes a colour which, by definition of the natural

[1] The phrase comes from the Chinese cultural sphere, and a 'toothless tiger' in English is essentially synonymous to a 'paper tiger' (a tiger made from paper, Chinese phrase 'zhilaohu'): both are metaphorical concepts refering to "something or someone that claims or appears to be powerful and/or threatening, but is actually ineffectual and unable to withstand challenge" (see `https://en.wikipedia.org/wiki/Paper_tiger`, accessed May 2019).

[2] In 2018, a black panther was spotted in Kenya, the first scientifically confirmed sighting in almost a century (1909). The location was very close to the fictional kingdom

concept of *panther*, makes it a non-*panther* (compare Gärdenfors [2000][p. 114] for the related discussion of a *pink elephant*).

This is an example for the first of our three classification problems, **overextension**, where we include **more** instances that belong to (the intersection of) the defining concepts. To see the effect in a more straightforward example, take the concepts of *domestic worker* and *home appliance*: clearly, these do not intersect? Now consider a *robotic vacuum cleaner*: It is an autonomous domestic robot with a floor cleaning system. It is not a domestic worker in the usual sense of *worker*, since it is not a person, but often it would be categorised by people as a *domestic worker which is also a home appliance*.

If we consider the category of the *Tools* that are also *Weapons* we could consider different examples: hammers, chainsaws, razors, scissors, screwdrivers, pliers and so on. But maybe we are interested in considering only the most typical ones (say: hammer, razor and scissor).

This is an example for the second of our three classification problems, **underextension**, where we include **less** instances than belong to the (intersection of the) defining concepts. One way of making 'less' precise is to collect only the (globally) most typical members of the intersection (see below).

Finally, consider a *houseboat*. A *houseboat* is both a *house* and a *boat*, however, it is a rather special house, being able to swim and lacking a solid fundament, and it is a rather special boat, being able to be inhabited. This is an example of, on the one hand, overconstraining the conjunction (as in underextension), as well as prioritising certain aspects of a concept over others. In this case, we obtain an example of conceptual blending [Fauconnier and Turner, 2003].[3] Such blends can be viewed as examples for the third of our three classification problems, **dominance**, where we include instances that are **primarily** a C (the boat is the *head* of the concept) and **secondly** a D (the house is the *modifier*, adding unusual aspects to a house).

Such complex aspects of how humans form and combine concepts, which are notoriously difficult to capture in Cognitive Science, are largely absent in logic-based approaches for the representation of concepts. We base the approach of this paper on the work of Hampton [Hampton, 1987, 1988a,b, 2017] and we develop an abstract setting to represent the three particular concept formation aspects just discussed: **overextension**, when a concept classifies more than what is constrained by its parts, **underextension**, when it classifies less, and **dominance**, when some parts of a concept dominate the classification question. The motivation for this work is to bridge the traditional techniques for doing knowledge representation with cognitively grounded views of human classification.

of Wakanda, ruled by the Black Panther (see `https://www.nationalgeographic.com/animals/2019/02/rare-black-panther-spotted-in-africa-century/`).

[3] In conceptual blending, the combination of two concepts is both a merge of common properties, together with emergent properties [Hedblom et al., 2016; Eppe et al., 2018], which is not fully covered by our present approach in this paper (consider the case of a 'boathouse' which needs large enough doors so that boats can fit through them.).

This paper contributes to the project of extending Description Logics (DLs) to model cognitively relevant features of classification [Porello et al., 2019]. We start from familiar DL formalisms (in particular from $\mathcal{ALC}$), which is an important logical language to model concepts and concept combinations in knowledge representation. We introduce a family of operators which apply to sets of concept descriptions and return a composed concept whose instances are those that satisfy 'enough' of the listed concept descriptions. To provide a meaning of 'enough', the operator takes a list of weighted concepts as argument, as well as a threshold. The combined concept applies to an instance if the sum of the weights of the concepts it satisfies meets the threshold. Using a threshold, the presentation focuses on crisp categorisations. Although the framework of weighted concepts easily adapts to a many-valued setting, we do not admit degrees of classification here.

The approach to weighted logics that we follow here takes inspiration from the use of sets of weighted proposition for representing utility functions in Uckelman et al. [2009]. Extensions of that approach to DLs have been developed in Ragone et al. [2009]. Two related articles are [Masolo and Porello, 2018] and [Masolo and Porello, 2016], where cognitive features of categorisation have been modelled by means of sets of weighted predicative formulas. The main difference in the present approach is that we study weighted combinations of concepts by explicitly introducing *syntactic operators* on concepts extending the basic concept languages, and employing their logical expressivity for cognitive modelling.

We introduce the weighted logic in Section 2, introduce in more detail the three classification problems in Section 3, discuss the formalisation of the problems in Section 4, and conclude in Section 5.

2 Weighted concept combination: The $\nabla\!\!\!\!\nabla$-operator logic

We introduce a class of m-ary operators, denoted by the symbol $\nabla\!\!\!\!\nabla$ (spoken 'tooth'), for combining concepts. Each operator works as follows: i) it takes a list of concept descriptions, ii) it associates a vector of weights to them, and iii) it returns a complex concept that applies to those instances that satisfy a certain combination of concepts, i.e., those instances for which, by summing up the weights of the satisfied concepts, a certain threshold is met.

An initial technical study of $\mathcal{ALC}_{\nabla\!\!\!\!\nabla}$ was undertaken in Porello et al. [2019], including a number of valid principles, expressivity issues, as well as counting and maximisation principles. We here briefly introduce the basic syntax and semantics, before we employ $\mathcal{ALC}_{\nabla\!\!\!\!\nabla}$ to study the above introduced classification problems formally.

Basic Syntax and Semantics of the $\nabla\!\!\!\!\nabla$-Logic. The new logic is denoted by $\mathcal{ALC}_{\nabla\!\!\!\!\nabla^{\mathbb{R}}}$, where weights and thresholds range over real numbers $r \in \mathbb{R}$. In the following we will refer to the languages for brevity just as $\mathcal{ALC}_{\nabla\!\!\!\!\nabla}$. To define the extended language of $\mathcal{ALC}_{\nabla\!\!\!\!\nabla}$, we add combination operators as follows, which behave syntactically just like m-ary modalities. We assume a vector of

m weights $\boldsymbol{w} \in \mathbb{R}^m$ and a threshold value $t \in \mathbb{R}$. Each pair $\boldsymbol{w}$, t specifies an operator: if $C_1, \ldots, C_m$ are concepts of $\mathcal{ALC}$, then $\nabla\!\!\!\!\nabla^t_{\boldsymbol{w}}(C_1, \ldots, C_m)$ is a concept of $\mathcal{ALC}_{\nabla\!\!\!\!\nabla}$. Note that in this basic definition, the possible nesting of the operator is excluded.[4]

For $C'_i \in \mathcal{ALC}$, the set of $\mathcal{ALC}_{\nabla\!\!\!\!\nabla}$ concepts is then described by the grammar:

$$C ::= A \mid \neg C \mid C \sqcap C \mid C \sqcup C \mid \forall R.C \mid \exists R.C \mid \nabla\!\!\!\!\nabla^t_{\boldsymbol{w}}(C'_1, \ldots, C'_m)$$

The semantics of the operator is obtained by extending the definition of the semantics of $\mathcal{ALC}$ as follows. Let $I = (\Delta^I, \cdot^I)$ be an interpretation of $\mathcal{ALC}$. We define the *value* of a $d \in \Delta^I$ under a $\nabla\!\!\!\!\nabla$-concept $\mathsf{C} = \nabla\!\!\!\!\nabla^t_{\boldsymbol{w}}(C_1, \ldots, C_m)$ by setting:

$$v^I_{\mathsf{C}}(d) = \sum_{i \in \{1,\ldots,m\}} \{w_i \mid d \in C^I_i\} \tag{1}$$

The interpretation (i.e., the extension) of a $\nabla\!\!\!\!\nabla$-concept in $I = (\Delta^I, \cdot^I)$ is then:

$$(\nabla\!\!\!\!\nabla^t_{\boldsymbol{w}}(C_1, \ldots, C_m))^I = \{d \in \Delta^I \mid v^I_{\mathsf{C}}(d) \geq t\} \tag{2}$$

To better visualise the weights an operator associates to the concepts, we sometimes use the notation $\nabla\!\!\!\!\nabla^t((C_1, w_1), \ldots, (C_m, w_m))$ instead of $\nabla\!\!\!\!\nabla^t_{\boldsymbol{w}}(C_1, \ldots, C_m)$.

In the following examples, we will consider the value of an object name a (aka individual constant) wrt. a $\nabla\!\!\!\!\nabla$-concept for interpretations that satisfy a certain knowledge base $\mathcal{K}$ (i.e., a set of formulas).

Definition 1 (Weights relative to a knowledge base). *Let a be an object name of $\mathcal{ALC}$ and $\mathcal{K}$ an $\mathcal{ALC}$ knowledge base. We set*

$$v^{\mathcal{K}}_{\mathsf{C}}(a) := \sum_{i \in \{1,\ldots,m\}} \{w_i \mid \mathcal{K} \models C_i(a)\}$$

I.e., $v^{\mathcal{K}}_{\mathsf{C}}(a)$ gives the accumulated weight of those C_i that are entailed by $\mathcal{K}$ to satisfy a.

Note that for positive weights, a given name a and a fixed interpretation I such that $I \models \mathcal{K}$, we always have that $v^{\mathcal{K}}_{\mathsf{C}}(a) \leq v^I_{\mathsf{C}}(a^I)$.

Example 1. Consider the set of concepts $\mathcal{C} = \{\mathsf{Red}, \mathsf{Round}, \mathsf{Coloured}\}$ and the concept C defined by means of the $\nabla\!\!\!\!\nabla$ operator

$$\mathsf{C} = \nabla\!\!\!\!\nabla^t((\mathsf{Red} \sqcup \mathsf{Round}, w_1), (\exists \mathsf{above}.\mathsf{Coloured}, w_2))$$

[4] In a more fine-grained definition $\mathcal{ALC}^i_{\nabla\!\!\!\!\nabla^{\mathbb{K}}}$, $i \geq 0$, is the logic with i levels of allowed nesting and where weights and thresholds range over K; we will comment on this further below.

The definition of C means that the relevant information to establish the categorisation under C of an object is whether (*i*) it is red or round, and (*ii*) it is above a coloured thing.

Consider the following knowledge base $\mathcal{K} = \{\mathsf{Red}(a), \exists\mathsf{above}.\mathsf{Blue}(a), \mathsf{Blue} \sqsubseteq \mathsf{Coloured}\}$, i.e., an agent knows that the object a is red and it is above a blue thing and that blue things are coloured, $\mathsf{Blue} \sqsubseteq \mathsf{Coloured}$.

The value of a returned by $v^{\mathcal{K}}_{\mathsf{C}}$ is computed as follows. Firstly, if a satisfies Red, then a satisfies $\mathsf{Red} \sqcup \mathsf{Round}$, so the weight w_1 can be obtained. Moreover, since $\mathsf{Blue} \sqsubseteq \mathsf{Coloured} \in \mathcal{K}$ and a satisfies $\exists\mathsf{above}.\mathsf{Blue}$, then a satisfies $\exists\mathsf{above}.\mathsf{Coloured}$, so also the weight w_2 can be obtained. Thus, $v^{\mathcal{K}}_{\mathsf{C}}(a)$ is $w_1 + w_2$. If $w_1 + w_2 \geq t$, then a is classified under C.

Having defined the basic $\nabla\!\!\!\!\nabla$ operator, a number of derived operations are easily obtained. First, it is possible to define the set of instances that at most reach a given threshold:

$$\nabla\!\!\!\!\nabla^{\leq t}((C_1, w_1), \ldots (C_m, w_m)) \equiv \nabla\!\!\!\!\nabla^{-t}((C_1, -w_1), \ldots, (C_m, -w_m)) \quad (3)$$

Given this, we can define the concept of instances that exactly score a certain threshold value t:

$$\nabla\!\!\!\!\nabla^{=t}((C_1, w_1), \ldots (C_m, w_m)) \equiv \nabla\!\!\!\!\nabla^{t}((C_1, w_1), \ldots, (C_m, w_m)) \sqcap \nabla\!\!\!\!\nabla^{\leq t}((C_1, w_1), \ldots (C_m, w_m)) \quad (4)$$

Finally, it is interesting to consider the set of entities that *maximally* satisfy a combination of concepts $C_1, \ldots C_m$ of $\mathcal{ALC}$. That is, we may define an operator with the following semantics:

$$\left(\nabla\!\!\!\!\nabla^{max}((C_1, w_1), \ldots (C_m, w_m))\right)^I = \{d \in \Delta \mid v^I_{\mathsf{C}}(d) \geq v^I_{\mathsf{C}}(d') \text{ for all } d' \in \Delta\} \quad (5)$$

Defining $\nabla\!\!\!\!\nabla^{max}$ in terms of $\nabla\!\!\!\!\nabla^{t}$ would require to use a *universal role*, which significantly increases the expressive power of $\mathcal{ALC}$ [Hemaspaandra, 1996], as shown in Porello et al. [2019].

3 Three Problems in Classification

3.1 Overextension

What happens to ordinary concepts when they are subjected to the logical operation of conjunction? In the 80s Hampton started a series of experiments to evaluate how people interpret conjunctive phrases such as *sports witch are also games* or *games which are also sports* ([Hampton, 1988a]). The classical semantic interpretation of these cases would interpret their meaning in terms of conjunction: the extension of the two phrases would correspond to the intersection of the set of *games* and the set of *sports*. However, experimental results showed something different. In his experiment, Hampton first provided a list of activities and asked people to evaluate which items in the list were *sports* and which were *games* (with the possibility of overlapping). After a couple of weeks, people were asked again to evaluate which items in the list could have been considered *sports which are also games* and, conversely, *games which are also sports*, and the results were interpreted to verify if people followed a rule corresponding to an intersection or conjunctive interpretation. The answer is no: a number of *sports* (*games*) that were not considered *games* (*sports*), according to the first classification, were considered as members of the conjunction in the second part of the experiments. This phenomenon is called overextension in conjunction, but the problem of non-extensional combination applies equally to disjunctions (and negation). For instance, most people do not categorise a mushroom as *fruit*, nor as *vegetable*, but then categorise it as an instance of *fruit or vegetable* [Hampton, 2017].

Hampton's proposal is to model the combination of such concepts in terms of *intensional composition* which takes into account the features that define the combined concepts. When a complex concept arises as a conjunction, the features that define it

> will be derived in a compositional fashion as a disjunction of the features of the two conjuncts [Hampton, 2017, p. 98].

Conjunctive phrases such as *sports which are also games* should then return a *composite prototype* resulting from aggregating all the features commonly associated with either of the two conjuncts (p. 98). The new set of features would then be modified according with Hampton's model for attribute inheritance [Hampton, 1987]. This model of the construction of a conjunctive concept would also predict overextension in conjunctions. Roughly, Hamptons argument runs as follows: let us imagine that, in order to be categorised under a concept, an object must show at least two out of three features of that concept. So, let us suppose that to be categorised in a conjunction, it should show four out of six features of the component concepts. If an object presents three features of the first object but just one feature of the second object it would be categorised as an instance of the conjunction, even if it is not an instance of both components. This would reflect overextension in conjunctions [Hampton, 2017].

Interestingly, Hampton's experiments also showed that this kind of relative clause form is non-commutative: the concept *games which are also sports* is not identical to the concept *sports which are also games*. The tooth operator can capture some of these insights and distinctions, as we will show in more detail in Section 4.

3.2 Underextension

Overextension's specular, or dual, phenomenon is that of underextension, that is, when considered in the context of conjunction, the phenomenon in which objects categorised as instances of two categories are not classified as belonging to the (underextending) conjunction of the categories. For instance, an item could be categorised as *sport* and as *game*, but could be not considered an instance of *sports which are also games* [Hampton, 1988a]. Actually, when related to underextension, Hampton's experiments were mostly directed to examine the disjunction of common sense concepts, and to verify the consistency of people's judgements with respect to the set union rule [Hampton, 1988b]. So, in that context, underextension would happen when an item categorised under a concept is not categorised as an item of the disjunction of that concept and another concept. For instance, an item could be categorised as a *game*, but then could be not considered as an instance of *game or sport* (maybe because it was a very atypical *game*). Here, we are dealing with another intuition. The idea is that underextension in conjunctions would mean to identify particularly typical members of the intersection of two concepts. As a particular case, we can then define a prototype concept of the regular intersection of the concepts.

3.3 Dominance

Hampton's experiments showed another interesting phenomenon, namely the dominance effect. Roughly speaking, in the simplest case of the composition of just two concepts, one of the two concepts 'weighs more' than the other in the definition of the conjunction. More precisely, the dominance effect was analysed with respect to typicality. Hampton's aim was in fact to predict the typicality of an item in a conjunction of two classes on the basis of the typicality of the item in each of the conjuncts. In his analyses, he discovered that the weights of the two constituent concepts in the determination of an item's typicality were usually not equal. On the contrary, dominance of a concept with respect to a second was frequently observed. For instance, the weight of the concept *sports* was almost twice the weight of *games* in determining the typicality of items in both of the concepts *sports which are also games* and *games which are also sports* [Hampton, 1987]. Following these insights, one can distinguish between a concept *Head* (*sports* in the example), and a concept *Modifier* (e.g. *games*).

We can naturally model such phenomenon assigning a higher weight to the 'dominant concept' (the Head), or specific features thereof, or by considering only the most prototypical instances of the dominant concept, as discussed in more detail below.

4 $\nabla\!\!\!\!\nabla$-Modelling of Over/Underextension and Dominance

We start by proposing how to represent prototype-based classifications by means of the $\nabla\!\!\!\!\nabla$ operator. We then proceed to, firstly, show how we can capture simple intersections of concepts defined in terms of prototypes. Secondly, we present a simple modelling of all three phenomena that we have previously discussed (overextension, underextension, and dominance effect).

The general picture is informally illustrated in Fig. 1, the formal modelling of which we undertake in the this section.

The main motivation is the simple elegance of the approach, which allows to compactly define specific instances of the three problems. Although not strictly

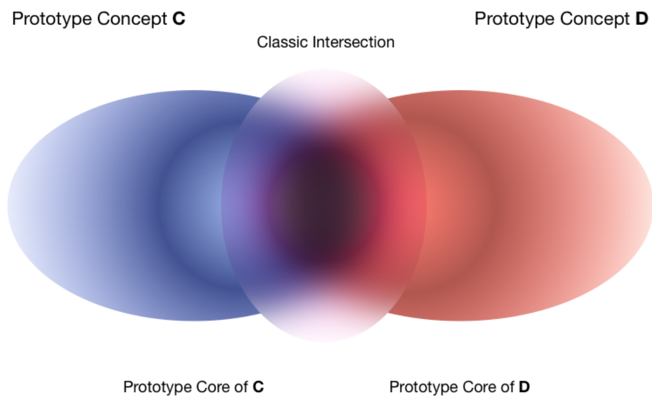

Fig. 1. The interaction of two graded concepts defined via prototype theory: the circular areas inside C, D correspond to the cores of the concepts (the most prototypical instances); the classic intersection is inside the dark overlapping part.

more expressive, replacing extremely large disjunctions with short tooth-based expressions may be arguably seen as a more cognitively adequate modelling. Another, related point, is that the relative weights of the participating features are being made explicit.

The $\mathbb{W}$ operators can represent the cognitive approach to concepts based on *prototypes*, cf. Porello et al. [2019]. Following Smith et al. [1988], a "prototype is a prestored representation of the usual properties associated with the concept's instances"[Smith et al., 1988, p.487]. In our setting, we can introduce a concept C by using a $\mathbb{W}$ operator that directly considers the usual properties associated to its instances:

$$\mathsf{C} = \mathbb{W}^t((P_1, w_1), \dots, (P_m, w_m)) \tag{6}$$

The classification under C applies to the objects that have "enough" features in common with the prototype, so to exceed the threshold t. This is a very simple way to model the idea of being *close enough* with respect to the prototype. We can also individuate the prototypical instances of C as the objects (if they exist) that satisfy *all* the P_j in C. A different and more liberal approach is to collect all those instances that reach the maximal possible value confirming a number of the P_j: we call this set the **core of C** and compute it as $\mathbb{W}^{max}(\mathsf{C})$.

In the following paragraphs, we assume that each concept is defined by means of a prototype, cf. Eq. (6). We focus on two concepts, which are thus defined by: $\mathsf{C} = \mathbb{W}^s((P_1, w_1), \dots, (P_m, w_m))$ and $\mathsf{D} = \mathbb{W}^t((Q_1, v_1), \dots, (Q_n, v_n))$.

4.1 Simple intersection

The logical conjunction of two concepts, which provides the intersection of the extension of the concepts, can be simply defined by means of the $\sqcap$ operator of $\mathcal{ALC}$:

$$\overline{\mathbb{W}}^s((P_1, w_1), \ldots, (P_m, w_m)) \sqcap \overline{\mathbb{W}}^t((Q_1, v_1), \ldots, (Q_m, v_m)) \quad (7)$$

Example 2. Suppose now we define the two concepts Sport and Game as follows:

$$\mathsf{Sport} = \overline{\mathbb{W}}^s((\mathsf{DoneForFun}, w_1), (\mathsf{IsCompetitive}, w_2), (\mathsf{DoneToGetFit}, w_3), (\mathsf{NeedsAthleticAbilities}, w_4)) \quad (8)$$

$$\mathsf{Game} = \overline{\mathbb{W}}^t((\mathsf{DoneForFun}, w_1), (\mathsf{IsCompetitive}, w_2), (\mathsf{NotSeriousGoal}, w_5), (\mathsf{InvolvesConcentration}, w_6)) \quad (9)$$

Let us further assume the following knowledge base:

$$\mathcal{K}_1 = \{\mathsf{DoneForFun(f)},\ \mathsf{IsCompetitive(f)}, \mathsf{NeedsAthleticAbilities(f)}, \mathsf{NotSeriousGoal(f)}\}$$

It allows one to categorise 'football' (f) as an instance of both Sport and Game (i.e., $w_1 + w_2 + w_4 \geq s$ and $w_1 + w_2 + w_5 \geq t$). In this case, football would get both the weight for being a Sport and the weight for being a Game, therefore belonging to the simple intersection of the two concepts as defined above.

On the other hand, consider a different knowledge base, about beerpong (bp):

$$\mathcal{K}_2 = \{\mathsf{DoneForFun(bp)}, \mathsf{IsCompetitive(bp)}, \mathsf{NotSeriousGoal(bp)}\}$$

In this case, we could imagine that $w_1 + w_2 + w_5 \geq t$ but $w_1 + w_2 \leq s$: beerpong would then not enter in the simple intersection of Sport and Game.

4.2 Modelling Overextensions

A combination of concepts that returns a set larger than the intersection of the two concepts can be approached in a number of ways. We discuss two strategies. Firstly, we can lower the threshold for the classification under C or under D. That is, in the combination of C and D the importance of the list of features defining C and D is relaxed. So, more instances shall fall under either concepts, returning a larger intersection. For example, when combining *boat* and *house*, into a *house boat* we are not requiring that the instance of the combined concept satisfy all the attributes of the composing concepts, only a sufficient number.

Given $\mathsf{C} = \overline{\mathbb{W}}^s((P_1, w_1), \ldots, (P_m, w_m))$ and $\mathsf{D} = \overline{\mathbb{W}}^t((Q_1, v_1), \ldots, (Q_n, v_n))$, a family of *overextending compositions* of C and D can be introduced:

$$\mathsf{C} \sqcap^{\mathsf{O}} \mathsf{D} = \overline{\mathbb{W}}^x((P_1, w_1), \ldots, (P_m, w_m)) \sqcap \overline{\mathbb{W}}^y((Q_1, v_1), \ldots, (Q_n, v_n)) \quad (10)$$

where $s \geq x$ or $t \geq y$ or both. In this case, as for the standard intersection, we assume that the (marginal) contribution of the two concepts C and D is equal (they both have to be satisfied). This reflects that the overextending composition is in fact a composition of the two concepts.

The effect of lowering the threshold for classifying an entity under the composing concepts has the consequence of *raising* the importance of a (number of) features for the composed concept (whether it is their salience or their diagnosticity). Globally lowering the threshold amounts to assume no information about which specific feature is important in the combination, whereas directly raising the weight of a number of features entails knowing that those are the important bits, when combining the concepts. Thus, a second strategy for combining concepts that overextends the intersection is by raising the weights of some of the features in C and D.

Example 3. Suppose we define again the two concepts Sport and Game as in Equation 8 and Equation 9. Let us also assume the following knowledge base:

$$\mathcal{K}_3 = \{\mathsf{DoneForFun}(\mathsf{w}), \mathsf{IsCompetitive}(\mathsf{w}), \mathsf{NeedsAthleticAbilities}(\mathsf{w})\}$$

which allows one to categorise 'wrestling' (individual w) as an instance of Sport (i.e., $w_1 + w_2 + w_4 \geq s$) but does not allow to categorise it as a Game (i.e., $w_1 + w_2 < t$). The simple intersection would not allow to represent 'wrestling' as an instance of the intersection of Sport and Game. To overextend the intersection of the two concepts in our setting, one should represent the intersection as follow:

$$\begin{aligned}\mathsf{Sport} \sqcap^{\mathsf{O}} \mathsf{Game} = \nabla\!\!\!\!\nabla^{x}\big((\mathsf{DoneForFun}, w_1), (\mathsf{IsCompetitive}, w_2), (\mathsf{DoneToGetFit}, w_3),\\ (\mathsf{NeedsAthleticAbilities}, w_4)\big) \sqcap \nabla\!\!\!\!\nabla^{y}\big((\mathsf{DoneForFun}, w_1), (\mathsf{IsCompetitive}, w_2),\\ (\mathsf{NonSeriousGoals}, w_3), (\mathsf{InvolvesConcentration}, w_4)\big) \quad (11)\end{aligned}$$

where the threshold y is set in a way that allows to classify 'wrestling' as a Game.

4.3 Modelling Underextensions

In a similar manner, we can treat the case of underextensions, that is, by raising the threshold required for classifying the composing concepts. One interpretation of this case is that when combining two concepts, only instances that are more typical of the two concepts can enter the combination of the two concepts. By means of the $\nabla\!\!\!\!\nabla^{max}$, we can define underextension operators as follows. Given $\mathsf{C} = \nabla\!\!\!\!\nabla^{s}((P_1, w_1), \ldots, (P_m, w_m))$ and $\mathsf{D} = \nabla\!\!\!\!\nabla^{t}((Q_1, v_1), \ldots, (Q_n, v_n))$, we set:

$$\mathsf{C} \sqcap^{\mathsf{U}} \mathsf{D} = \nabla\!\!\!\!\nabla^{max}((P_1, w_1), \ldots, (P_m, w_m)) \sqcap \nabla\!\!\!\!\nabla^{max}((Q_1, v_1), \ldots, (Q_n, v_n)) \quad (12)$$

Example 4. Suppose we define the two concepts Sport and Game as before in Equation 8 and Equation 9. Let us also assume that both 'basketball' (individual bb) and 'ping pong' (individual pp) are in the intersection of Sport and Game, according to the following knowledge base

$$\begin{aligned}\mathsf{KB} = \{&\mathsf{DoneForFun}(\mathsf{bb}), \mathsf{IsCompetitive}(\mathsf{bb}), \mathsf{NeedsAthleticAbilities}(\mathsf{bb}),\\ &\mathsf{DoneToGetFit}(\mathsf{bb}), \mathsf{NotSeriousGoal}(\mathsf{bb}), \mathsf{InvolvesConcentration}(\mathsf{bb}),\\ &\mathsf{DoneForFun}(\mathsf{pp}), \mathsf{IsCompetitive}(\mathsf{pp}), \mathsf{NeedsAthleticAbilities}(\mathsf{pp}),\\ &\mathsf{NotSeriousGoal}(\mathsf{pp}), \mathsf{InvolvesConcentration}(\mathsf{pp})\}\end{aligned}$$

Suppose that we define underextending composition of Sport and Game as in Equation (12). Then, basketball would be selected by our operator, as it maximises both Sport and Game, while ping pong will not, as it does not maximise Sport (by failing to satisfy $\mathsf{DoneToGetFit}$).

Another way of introducing underextension is to constrain the intersection of C and D by imposing that the instances must maximise the combination of the features of C and D: $\mathsf{C} \sqcap \mathsf{D} \sqcap \nabla\!\!\!\!\nabla^{max}\big((P_1, w_1), \ldots, (P_m, w_m), (Q_1, w_1), \ldots, (Q_n, v_n)\big)$.

This definition is non-equivalent to (12) only in the case of mutually incompatible attributes defining C and D. We leave this aspect for future work.

4.4 Modelling Dominance Effects

In the case of dominance, as we have seen, the head of the combination outweighs the other composing concepts. Given $\mathsf{C} = \nabla\!\!\!\!\nabla^s((P_1, w_1), \ldots, (P_m, w_m))$ and $\mathsf{D} = \nabla\!\!\!\!\nabla^t((Q_1, v_1), \ldots, (Q_n, v_n))$, we present three strategies to model *dominance operators*. A first strategy is to define dominance operators as follows:

$$\mathsf{C} \sqcap_1^{\mathsf{D}} \mathsf{D} = \nabla\!\!\!\!\nabla^{s'}\big((P_1, w_1), \ldots, (P_m, w_m)\big) \sqcap \nabla\!\!\!\!\nabla^{t'}\big((Q_1, v_1), \ldots, (Q_n, v_n)\big) \tag{13}$$

where $s' \geq s$ and $t' \leq t$.[5] In this case, globally raising the threshold of the dominant concept amounts to include the typical instances of C into the composition, whereas globally lowering the threshold of the non-dominant concept includes possibly non-D instances. This means that a *sport that is also a game* must be primarily a *sport*.

A second strategy is to unpack (or flatten) the definitions of C and D and raise the weights of the most salient features $\varnothing \neq S(\mathsf{C}) \subseteq \{1, \ldots, m\}$ of the dominant concept C for the composition, whilst including some required elements $R(\mathsf{D}) \subseteq \{1, \ldots, n\}$ of D to compute a new minimal threshold. We may set:

$$\mathsf{C} \sqcap_2^{\mathsf{D}} \mathsf{D} = \nabla\!\!\!\!\nabla^{\pi(C,D)}\big((P_1, w_1^{\pi}), \ldots, (P_m, w_m^{\pi}), (Q_1, v_1^{\pi}), \ldots, (Q_n, v_n^{\pi})\big) \tag{14}$$

where π is a function that re-assigns weights such that $\pi(w_i) = w_i^{\pi} > w_i$ for all $i \in S(\mathsf{C})$ (and $\pi(x) = x$ otherwise) and $\pi(C, D) = \Sigma_{i \in S}\pi(w_i) + \Sigma_{j \in R}\pi(v_j)$ is a new threshold that accumulates the inflated weights for the salient features of C and the required weights from D.

A third strategy to define a dominant concept would be to collect those instances that maximise the sum of weights of the dominant concept (using e.g. the $\nabla\!\!\!\!\nabla^{max}$ operator).

Notice that these types of compositions are generally non-commutative, as the first combined concept (the head) is more important than the second.

Example 5. Suppose we define again the two concepts Sport and Game as in Equation 8 and Equation 9, and consider again the knowledge base in Example

[5] Here the choice of making the first concept dominant is merely formal, we are not making any assumption on the linguistic phrasing of this complex concept.

3. If we consider the concept Sport as the dominant one, we could apply Equation 13 to include wrestling in the intersection of the concepts Sport and Game, simply lowering the threshold for being included in Game. This is again a case of overextension, but the idea behind this choice here is that having the most important features of the concept Sport 'count more' to enter in the conjunctive concept.
We have a similar result if we apply Equation 14:

$$\begin{aligned}\mathsf{Sport} \sqcap_2^{\mathsf{D}} \mathsf{Game} = \nabla\!\!\!\!\nabla^{\pi(\mathsf{Sport},\mathsf{Game})}\big((\mathsf{DoneForFun}, w_1^{\pi}), (\mathsf{IsCompetitive}, w_2^{\pi}),\\ (\mathsf{DoneToGetFit}, w_5^{\pi}), (\mathsf{NeedsAthleticAbilities}, w_6^{\pi})(\mathsf{DoneForFun}, v_1^{\pi}),\\ (\mathsf{IsCompetitive}, v_2^{\pi}), (\mathsf{NotSeriousGoal}, v_3^{\pi}), (\mathsf{InvolvesConcentration}, v_4^{\pi})\big) \quad (15)\end{aligned}$$

where we set $\pi(\mathsf{Sport}, \mathsf{Game}) = \pi(w_1)+\pi(w_2)+\pi(w_6)+\pi(v_1)+\pi(v_2)$. It is worth noticing here that just the weights associated to the concept Sport are raised by the function π, as required in Eq. 14: since Sport is the dominant concept, its salient features are considered the most important in this combination.

5 Conclusions and Future Work

We studied three cognitively and psychologically interesting and important phenomena that address classification problems with a focus on conjunctive operations on concepts, namely overextension, underextension, and dominance. To address these phenomena from a formal modelling point of view, we employed the logic $\mathcal{ALC}_{\nabla\!\!\!\nabla}$ introduced in [Porello et al., 2019], where a class of concept-forming operators was introduced which can define complex concepts by means of accumulating the weights of defining attributes and comparing this value to a given threshold. We here introduced basic modelling approaches for the three classification problems, discussed several relevant examples, and outlined the remaining challenges for the formal approach.

Future work will be dedicated to further investigate the logical properties of the operators and their natural extensions, and to employ them in a systematic fashion to the modelling of salient cognitive features of concepts, concept combinations in general, and conceptual blending in particular [Eppe et al., 2018; Confalonieri and Kutz, 2019]. Moreover, the landscape of over/under-extension and dominance should be analysed in a more finegrained manner, considering interdependencies, special cases, and the introduction of further (conjunctive) classification problems. Eventually, we will aim to justify the formal fine-theory of conjuctive (and disjunctive etc.) classification in a detailed empirical study.

Another line of research will deepen the comparison with the formal studies on typicality (e.g. Lieto and Pozzato [2018]) and the relation and combination with similarity frameworks based on a notion of distance (e.g. Confalonieri et al. [2017]; Sheremet et al. [2007]; Hois and Kutz [2008]).

Further, we hope to pursue further a deeper exploration of the correspondence theory between tooth logic, statistical learning and classification models, and concept learning in DL, as begun in Galliani et al. [2019]. Such a correspondence will not only contribute to these fields individually, but will allow to more systematically *learn* the relevant thresholds for complex tooth concepts, and it will support hybrid frameworks where e.g. statistically learned concepts can be naturally represented in DL knowledge bases, thereby contributing also to neural-symbolic integration and to the AI explainability challenge [Garcez et al., 2019].

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