Dihypergraph decomposition: application to closure system representations ⋆

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Abstract. Closure systems and their representations are essential in numerous fields of computer science. Among representations, dihypergraphs (or attribute implications) and meet-irreducible elements (reduced context) are widely used in the literature. Translating between the two representations is known to be harder than hypergraph dualization, a well-known open problem. In this paper we are interested in enumerating the meet-irreducible elements of a closure system from a dihypergraph. To do so, we use a partitioning operation of a dihypergraph which gives a recursive characterization of its meet-irreducible elements. From this result, we deduce an algorithm which computes meet-irreducible elements in a divide-and-conquer way and puts the light on the major role of dualization in closure systems. Using hypergraph dualization, this strategy can be applied in output quasi-polynomial time to particular classes of dihypergraphs, improving at the same time previous results on ranked convex geometries.

Keywords: Dihypergraphs · Decomposition · Closure systems · Meet-irreducible elements

1 Introduction

Closure systems play a major role in several areas of computer science and mathematics such as database [9, 18, 19], Horn logic [16], lattice theory [6, 7] or Formal Concept Analysis (FCA) [12] where they are known as concept lattice.

Due to the exponential size of a closure system, several compact representations have been studied over the last decades [12, 14, 16, 20]. Among all possible representations, there are two prominent candidates: *implicational bases* and *meet-irreducible elements*. The former consists in set a of rules $B \rightarrow h$ over the ground set where B is the body and h the head of the rule. A rule depicts a causality relation between the elements of B and h, i.e., whenever a set contains B, it must also contain h. As several implicational bases can represent the same closure system, numerous bases with "good" properties have been studied. Among them, the Duquenne-Guigues base [13] being minimum or the canonical direct base [5] are worth mentioning. Like closure systems, implicational bases are ubiquitous in computer science. They appear for instance as Horn theories in propositional logic [16], attribute implications in FCA [12], functional dependencies in databases theory [9, 18] and they are conveniently expressed by

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directed hypergraphs (*dihypergraphs* for short) [1,11] where an implication $B \to h$ corresponds to an arc (B, h). A nice survey on the topic can be found in [23].

The second representation for a closure system is a (minimum) subset of its elements from which it can be reconstructed. These elements are known as *meet-irreducible elements* [7]. In Horn logic, they are known as the characteristic models [16]. In FCA, they are written as a binary relation: the context [12]. They appear in the Armstrong relation [19] in database theory.

The problem of translating between these representations has been widely studied in the literature [2, 4, 8, 16, 19, 23]. Even though the two directions of the translation are equivalent [16], computing meet-irreducible elements from a set of implications has been less studied. This problem can be equivalently reformulated in FCA terms as follows: given a set of attribute implications, find an associated (reduced) context. Algorithms for this problem are used in databases to build relations satisfying a set of functional dependencies [19]. Furthermore, some tasks such as a abduction [16] are easier with meet-irreducible elements than implications. On the negative side, it has been shown in [16] that this problem is harder than enumerating minimal transversals of a hypergraph, also known as *hypergraph dualization* for which the best algorithm runs in output quasi-polynomial time [10]. Furthermore, Kavvadias et al. [15] have shown that enumerating maximal meet-irreducible elements cannot be done in output-polynomial time unless $\mathsf{P} = \mathsf{NP}$. On the positive side, exponential time algorithms have been given in [19, 22]. More recently, output quasi-polynomial time algorithms have been given for some classes of closure systems [4, 8].

In this paper we seek to push further the understanding of this problem, based on previous works such as [8, 17]. We use a hierarchical decomposition method introduced in [21] for dihypergraphs representing implicational bases. To achieve this decomposition we use a restricted version of a *split*, a partitioning operation of the ground set [21]. We call this restriction an *acyclic split*. An acyclic split of $\mathcal{H}$ is a bipartition of its ground set V into two non-trivial parts $\mathrm{V}_1, \mathrm{V}_2$ such that any arc (B, h) (i.e., any implication) is either fully contained in one of the two parts or the body B is in V_1 while the head h is in V_2. Intuitively, $\mathcal{H}$ is divided in three subhypergraphs, $\mathcal{H}[\mathrm{V}_1], \mathcal{H}[\mathrm{V}_2]$ and a *bipartite dihypergraph* $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$ which models interactions from V_1 to V_2. Clearly, some dihypergraphs do not admit such splits. An acyclic split yields a decomposition of the underlying closure system into *projections* (or *traces*) and provide a recursive characterization of its meet-irreducible elements. Therefore, we propose an algorithm which compute meet-irreducible elements of a dihypergraph from a hierarchical decomposition using acyclic splits.

The paper is presented as follows. In Section 2 we recall definitions about directed hypergraphs and closure systems. Section 3 introduces acyclic split of a dihypergraph $\mathcal{H}$ and presents an example to illustrate our contribution. In Section 4 we study the construction of the underlying closure system and we give a characterization of its meet-irreducible elements. This characterization suggests a recursive algorithm which computes meet-irreducible elements of $\mathcal{H}$ in a divide-and-conquer way with acyclic splits, discussed in Section 5. We obtain new classes of dihypergraphs for which computing meet-irreducible elements can be done in output quasi-polynomial time using hypergraph dualization, thus generalizing recent works on ranked convex geometries [8].

2 Preliminaries

All the objects considered in this paper are finite. If V is a set, 2^{V} denotes its powerset. For $n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, \dots, n\}$. Sometimes we will denote by $x_1 \dots x_n$ the set $\{x_1, \dots, x_n\}$.

We begin with notions on lattices and closure systems [6,7]. A *closure system* on V is a set system $\mathcal{F} \subseteq 2^{\mathrm{V}}$ such that $\mathrm{V} \in \mathcal{F}$ and for any $F_1, F_2 \in \mathcal{F}$, $F_1 \cap F_2 \in \mathcal{F}$. An element F of $\mathcal{F}$ is called a *closed set*. The number of closed sets in $\mathcal{F}$ represents its size, written $|\mathcal{F}|$. When ordered by set-inclusion, $(\mathcal{F}, \subseteq)$ is a *lattice*. Let $F \in \mathcal{F}$. The *ideal* of F, denoted $\downarrow F$ is the collection of closed sets of $\mathcal{F}$ included in F, namely $\downarrow F = \{F' \in \mathcal{F} \mid F' \subseteq F\}$. The *filter* $\uparrow F$ is defined dually. For a subset $\mathcal{B}$ of $\mathcal{F}$, we put $\downarrow \mathcal{B} = \bigcup_{F \in \mathcal{B}} \downarrow F$ and dually $\uparrow \mathcal{B} = \bigcup_{F \in \mathcal{B}} \uparrow F$. Let $F_1, F_2 \in \mathcal{F}$. We say that F_1 and F_2 are *incomparable* if $F_1 \not\subseteq F_2$ and $F_2 \not\subseteq F_1$. Assume $F_1 \subseteq F_2$. Then F_2 is a *cover* of F_1, written $F_1 \prec F_2$, if for any other $F' \in \mathcal{F}$, $F_1 \subseteq F' \subseteq F_2$ implies $F_1 = F'$ or $F_2 = F'$. A closed set M of $\mathcal{F}$ is a *meet-irreducible element* if for any $F_1, F_2 \in \mathcal{F}$, $M = F_1 \cap F_2$ implies $M = F_1$ or $M = F_2$. The ground set V is not a meet-irreducible element. Equivalently, M is a meet-irreducible element of $\mathcal{F}$ if and only if it has a unique cover. The set of meet-irreducible elements of $\mathcal{F}$ is written $\mathcal{M}(\mathcal{F})$ or simply $\mathcal{M}$ when clear from the context. A subset $\mathcal{B}$ of $\mathcal{F}$ is an *antichain* if elements of $\mathcal{B}$ are pairwise incomparable. Let $U \subseteq \mathrm{V}$. The *trace* (or *projection*) of $\mathcal{F}$ on U, denoted $\mathcal{F} \colon U$, is obtained by intersecting each closed set of $\mathcal{F}$ with U, i.e., $\mathcal{F} \colon U = \{F \cap U \mid F \in \mathcal{F}\}$. If $\mathcal{F}' \subseteq \mathcal{F}$ is a closure system, it is a *meet-sublattice* of $\mathcal{F}$. Let $\mathcal{F}_1, \mathcal{F}_2$ be two closure systems on disjoint $\mathrm{V}_1, \mathrm{V}_2$ respectively. The *direct product* of $\mathcal{F}_1$ and $\mathcal{F}_2$, denoted $\mathcal{F}_1 \times \mathcal{F}_2$, is given by $\mathcal{F}_1 \times \mathcal{F}_2 = \{F_1 \cup F_2 \mid F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$.

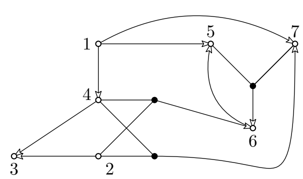
In this paper, we suppose that implicational bases are given as directed hypergraphs. Directed hypergraphs are a convenient representation for attribute implications of FCA, Horn clauses, functional dependencies [1,11,23]. We mainly refer to papers [1,11] for definitions of dihypergraphs. A *(directed) hypergraph* (*dihypergraph* for short) $\mathcal{H}$ is a pair $(\mathrm{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ where $\mathrm{V}(\mathcal{H})$ is its set of vertices, and $\mathcal{E}(\mathcal{H}) = \{e_1, \dots, e_n\}$, $n \in \mathbb{N}$, its set of *arcs*. An arc $e \in \mathcal{E}(\mathcal{H})$ is a pair $(B(e), h(e))$, where $B(e)$ is a non-empty subset of V called the *body* of e and $h(e) \in \mathrm{V} \setminus B$ called the *head* of e. When clear from the context, we write V, $\mathcal{E}$ and (B, h) instead of $\mathrm{V}(\mathcal{H})$, $\mathcal{E}(\mathcal{H})$ and $(B(e), h(e))$ respectively. An arc $e = (B, h)$ is written as the set $e = B \cup \{h\}$ when no confusion can arise. Whenever a body B is reduced to a single vertex b, we shall write (b, h) instead of $(\{b\}, h)$ for clarity. In this case, the arc (b, h) is called a *unit arc*. A dihypergraph where all edges are unit is a digraph. Let $\mathcal{H} = (\mathrm{V}, \mathcal{E})$ be a dihypergraph and $U \subseteq \mathrm{V}$. The subhypergraph $\mathcal{H}[U]$ *induced* by U is the pair $(U, \mathcal{E}(\mathcal{H}[U]))$ where $\mathcal{E}(\mathcal{H}[U])$ is the set of arcs of $\mathcal{E}$ contained in U, namely $\mathcal{E}(\mathcal{H}[U]) = \{e \in \mathcal{E} \mid e \subseteq U\}$. A *bipartite dihypergraph* is a dihypergraph in which the ground set can be partitioned into two parts $(\mathrm{V}_1, \mathrm{V}_2)$ such that for any $(B, h) \in \mathcal{E}$, $B \subseteq \mathrm{V}_1$ or $B \subseteq \mathrm{V}_2$. We denote a bipartite dihypergraph by $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$. A *split* [21] of a dihypergraph

$\mathcal{H}$ is a non-trivial bipartition $(\mathrm{V}_1, \mathrm{V}_2)$ of V such that for any arc (B, h) of $\mathcal{H}$, either $B \subseteq \mathrm{V}_1$ or $B \subseteq \mathrm{V}_2$. A split $(\mathrm{V}_1, \mathrm{V}_2)$ partitions $\mathcal{H}$ into three arc disjoint subhypergraphs $\mathcal{H}[\mathrm{V}_1], \mathcal{H}[\mathrm{V}_2]$ and a bipartite dihypergraph $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$.

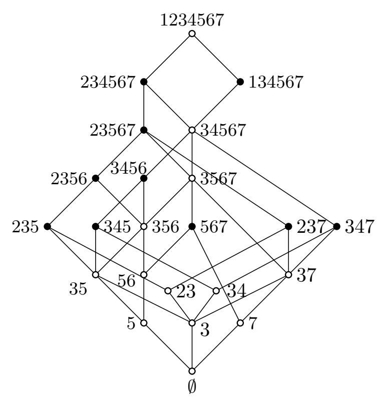
The closure system associated to a dihypergraph $\mathcal{H}$ is obtained with the *forward chaining algorithm*. It starts from a subset X of V and constructs a chain $X = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_k = X^{\mathcal{H}}$ such that for any $i = 1, \ldots, k$ we have $X_i = X_{i-1} \cup \{h \mid \exists (B, h) \in \mathcal{E} \text{ s.t. } B \subseteq X_{i-1}\}$. The operation $(.)^{\mathcal{H}}$ is a *closure operator*, that is for any $X, Y \subseteq \mathrm{V}$, we have $X \subseteq X^{\mathcal{H}}$, $X \subseteq Y \implies X^{\mathcal{H}} \subseteq Y^{\mathcal{H}}$ and $(X^{\mathcal{H}})^{\mathcal{H}} = X^{\mathcal{H}}$. A set X is closed if $X = X^{\mathcal{H}}$. Note that X is closed for $\mathcal{H}$ if and only if for any arc $(B, h) \in \mathcal{E}$, $B \subseteq X$ implies $h \in X$. We say that X *satisfies* an arc (B, h) if $B \subseteq X \implies h \in X$. The collection $\mathcal{F}(\mathcal{H}) = \{X^{\mathcal{H}} \mid X \subseteq \mathrm{V}\}$ of closed sets of $\mathcal{H}$ is a closure system. For clarity, we may write $\mathcal{F}$ instead of $\mathcal{F}(\mathcal{H})$. Our definition of a dihypergraph implies $\emptyset \in \mathcal{F}$, without loss of generality.

3 Acyclic split and illustration on an example

In this section we introduce acyclic splits and we illustrate our approach to compute meet-irreducible elements from a dihypergraph on a toy example. Let $\mathrm{V} = [7]$ and $\mathcal{H} = (\mathrm{V}, \{(2,3), (4,3), (6,5), (57,6), (24,6), (24,7), (1,4), (1,5), (1,7)\})$. It is represented in Figure 1 (a). To represent an arc (B, h) with $|B| \geq 2$ we use a black vertex connecting every elements of B from which starts an arrow towards h. The closure system $\mathcal{F}$ associated to $\mathcal{H}$ is given in Figure 1 (b).



(a) A dihypergraph $\mathcal{H}$



(b) The closure system $\mathcal{F}$ associated to $\mathcal{H}$, where meet-irreducible elements are darkened.

Fig. 1: The dihypergraph $\mathcal{H}$ and its closure system $\mathcal{F}$

The idea is to split $\mathcal{H}$ into three subhypergraphs $\mathcal{H}[\mathrm{V}_1], \mathcal{H}[\mathrm{V}_2]$ and $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$ as in [21]. However we use a restricted version of a split we call an *acyclic split*. A split is acyclic if for any arc (B, h) of $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$, $B \subseteq \mathrm{V}_1$ and $h \in \mathrm{V}_2$. A dihypergraph which does not have any acyclic split is *indecomposable*. A maximum subhypergraph of $\mathcal{H}$ which has no acyclic split is a *c-factor* (cyclic factor) of $\mathcal{H}$. If a c-factor $\mathcal{H}'$ of $\mathcal{H}$ is reduced to a vertex, i.e., $\mathcal{H}' = (\{x\}, \emptyset)$, it is a *singleton c-factor* of $\mathcal{H}$.

For instance in $\mathcal{H}$, the bipartition $\mathrm{V}_1 = \{1,2,3\}$, $\mathrm{V}_2 = \{4,5,6,7\}$ is not a split because the body of $(24,6)$ has elements from both V_1 and V_2. If we fix $\mathrm{V}_1 = \{1,2,4,6\}$ and $\mathrm{V}_2 = \{3,5,7\}$, then the bipartition is a split but not acyclic since the arc $(6,5)$ goes from V_1 to V_2 and $(57,6)$ from V_2 to V_1. An acyclic split is $\mathrm{V}_1 = \{1,2,3,4\}$ and $\mathrm{V}_2 = \{5,6,7\}$. It induces the three subhypergraphs $\mathcal{H}[\mathrm{V}_1] = (\mathrm{V}_1, \{(4,3),(1,4),(2,3)\})$, $\mathcal{H}[\mathrm{V}_2] = (\mathrm{V}_2, \{(6,5),(57,6)\}$ and $\mathcal{H}[\mathrm{V}_1,\mathrm{V}_2] = (\mathrm{V}, \{(24,6),(24,7),(1,5),(1,7)\})$. Observe that $\mathcal{H}[\mathrm{V}_2]$ is indecomposable: the unique split of V_1 is $\mathrm{V}_1' = \{5,7\}$ and $\mathrm{V}_2' = \{6\}$, which is not acyclic. Hence, $\mathcal{H}[\mathrm{V}_2]$ is a c-factor of $\mathcal{H}$. Closure systems $\mathcal{F}_1$, $\mathcal{F}_2$ of $\mathcal{H}[\mathrm{V}_1]$ and $\mathcal{H}[\mathrm{V}_2]$ are given in Figure 2. Note that $\mathcal{F}$ is a meet-sublattice of $\mathcal{F}_1 \times \mathcal{F}_2$.

(a) $\mathcal{F}_1$, the closure lattice of $\mathcal{H}[\mathrm{V}_1]$

(b) $\mathcal{F}_2$, the closure lattice of $\mathcal{H}[\mathrm{V}_2]$

Fig. 2: Closure lattices of $\mathcal{H}[\mathrm{V}_1]$ and $\mathcal{H}[\mathrm{V}_2]$, meet-irreducible are darkened.

The splitting operation provides a partition of $\mathcal{M}(\mathcal{F})$ into two classes. The first class contains meet-irreducible elements of $\mathcal{F}_1$ to which we added V_2. This is the case for example of 234567 and 567, which are the meet-irreducible elements 234 and $\emptyset$ of $\mathcal{F}_1$. The second class contains meet-irreducible which are inclusion-wise maximal closed sets of $\mathcal{F}$ whose trace on V_2 is meet-irreducible in $\mathcal{F}_2$. For instance, 235 and 345 are inclusion-wise maximal closed sets of $\mathcal{F}$ whose intersection with V_2 rise 5, a meet-irreducible element of $\mathcal{F}_2$.

Thus, every meet-irreducible element of $\mathcal{F}$ belongs to exactly one of these two classes. Observe that any other $F \in \mathcal{F}$ cannot be part of $\mathcal{M}(\mathcal{F})$. As $\mathcal{F} \subseteq \mathcal{F}_1 \times \mathcal{F}_2$, every $M \in \mathcal{M}$ arise from the combination of some $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$. Let $F \in \mathcal{F}$ be outside of those two class. If $\mathrm{V}_2 \subseteq F$, then $F \cap \mathrm{V}_1$ cannot be meet-irreducible in $\mathcal{F}_1$. In this case, covers of $F \cap \mathrm{V}_1$ in $\mathcal{F}_1$ can be used to produce distinct covers of F in $\mathcal{F}$. If however $\mathrm{V}_2 \nsubseteq F$, then covers of $F \cap \mathrm{V}_2$ in $\mathcal{F}_2$ yield covers of F in $\mathcal{F}$. In the case where $F \cap \mathrm{V}_2$ is meet-irreducible in $\mathcal{F}_2$, there will be a closed set F_1 in $\mathcal{F}_1$ such that $F \cap \mathrm{V}_1 \subset F_1$ and $F_1 \cup (F \cap \mathrm{V}_2)$ will be closed in $\mathcal{F}$ by assumption. This can be used to find another cover of F in $\mathcal{F}$.

This characterization suggests to recursively find meet-irreducible elements of $\mathcal{F}$. If $\mathcal{H}$ is indecomposable, we computes $\mathcal{M}$ with known algorithms [4, 19]. Otherwise, we find an acyclic split $(\mathrm{V}_1, \mathrm{V}_2)$ and recursively applies on $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$. Then, we compute $\mathcal{M}$ using $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$, $\mathcal{M}_1$ and $\mathcal{M}_2$. In Figure 3, we give the trace of a decomposition for $\mathcal{H}$ using acyclic splits. This strategy is particularly interesting for cases where c-factors of $\mathcal{H}$ are all of the form $(\{x\}, \emptyset)$ for $x \in \mathrm{V}$, since the unique meet-irreducible element in this case is $\emptyset$.

Thus, the steps we will follow are the following. Given a dihypergraph $\mathcal{H}$ and its closure system $\mathcal{F}$, we will study the construction of $\mathcal{F}$ with respect to an acyclic split. This will lead us to a characterization of $\mathcal{M}$. Recursively applying this characterization, we will get an algorithm to compute $\mathcal{M}$ from $\mathcal{H}$.

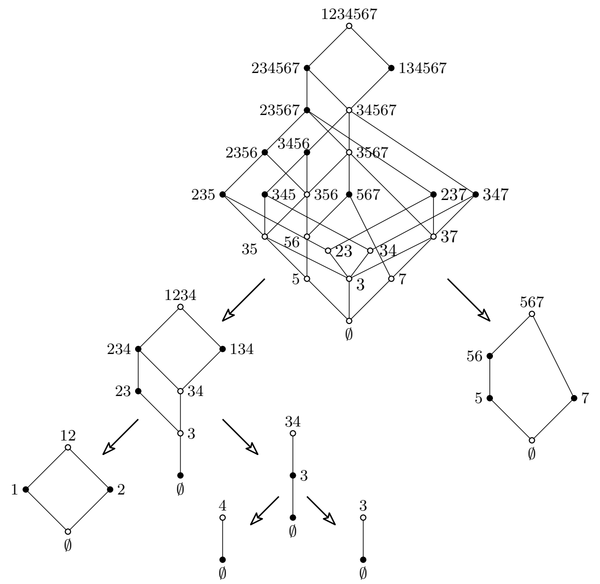


Fig. 3: Hierarchical decomposition of $\mathcal{F}$ using acyclic splits. Meet-irreducible elements are darkened.

4 The closure system induced by an acyclic split

In this section, we show the construction of a closure system with respect to an acyclic split. We give a characterization of its closed sets and meet-irreducible elements $\mathcal{M}$. Let $\mathcal{H} = (\mathrm{V}, \mathcal{E})$ be a dihypergraph and $(\mathrm{V}_1, \mathrm{V}_2)$ an acyclic split of $\mathcal{H}$. Let $\mathcal{F}_1, \mathcal{F}_2$ be the closure systems associated to $\mathcal{H}[\mathrm{V}_1]$ and $\mathcal{H}[\mathrm{V}_2]$ respectively. Similarly, $\mathcal{M}_1, \mathcal{M}_2$ are their meet-irreducible elements. We show how to construct $\mathcal{F}$ from $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$. We begin with the following theorem from [21]:

Theorem 1 (Theorem 3 of [21]). *Let $(\mathrm{V}_1, \mathrm{V}_2)$ be a split of $\mathcal{H}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ the closure systems corresponding to $\mathcal{H}[\mathrm{V}_1]$ and $\mathcal{H}[\mathrm{V}_2]$ respectively. Then,*

1. *If $F \in \mathcal{F}_{\mathcal{H}}$ then $F_i = F \cap \mathrm{V}_i \in \mathcal{F}_i, i = \{1, 2\}$. Moreover, $\mathcal{F}_{\mathcal{H}} \subseteq \mathcal{F}_1 \times \mathcal{F}_2$.*
2. *If $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$ has no arc then $\mathcal{F}_{\mathcal{H}} = \mathcal{F}_1 \times \mathcal{F}_2$.*
3. *If $B \subseteq \mathrm{V}_1$ for any arc (B, h) of $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$, then $\mathcal{F}_{\mathcal{H}} : \mathrm{V}_i = \mathcal{F}_i$ for $i \in \{1, 2\}$.*
4. *If $B \subseteq \mathrm{V}_2$ for any arc (B, h) of $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$, then $\mathcal{F}_{\mathcal{H}} : \mathrm{V}_i = \mathcal{F}_i$ for $i \in \{1, 2\}$.*

The first item states that $\mathcal{F}$ is a meet-sublattice of $\mathcal{F}$. From item *2* we can derive a characterization of meet-irreducible elements of the direct product $\mathcal{F}_1 \times \mathcal{F}_2$. This result has already been formulated in lattice theory, for instance in [7]. We reprove it in our framework for self-containment.

Proposition 1. *Let* $\mathcal{H}$ *be a dihypergraph an* $(\mathrm{V}_1, \mathrm{V}_2)$ *an acyclic split of* $\mathcal{H}$ *where* $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$ *has no arcs. Then* $\mathcal{M} = \{M_1 \cup \mathrm{V}_2 \mid M_1 \in \mathcal{M}_1\} \cup \{M_2 \cup \mathrm{V}_1 \mid M_2 \in \mathcal{M}_2\}$.

Proof. Let $M \in \mathcal{M}$, $i \in \{1, 2\}$ and $M_i = M \cap \mathrm{V}_i$. As $M \neq \mathrm{V}$, $\mathrm{V}_i \nsubseteq M$ for at least one of $i \in \{1, 2\}$. Suppose it holds for V_1 and V_2. Then, there exists $M_i' \in \mathcal{F}_i$, such that $M_i \prec M_i'$ in $\mathcal{F}_i$. However, by Theorem 1, $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$. Hence $M_1 \cup M_2'$ and $M_1' \cup M_2$ belong to $\mathcal{F}$. Furthermore they are incomparable and we have $M \prec M_1 \cup M_2'$ and $M \prec M_1' \cup M_2$ which contradicts $M \in \mathcal{M}$. Therefore, either $\mathrm{V}_1 \subseteq M$ or $\mathrm{V}_2 \subseteq M$. Assume without loss of generality that $\mathrm{V}_1 \subseteq M$. Let M'' be the unique cover of M in $\mathcal{F}$. Then, $\mathrm{V}_1 \subseteq M''$ and it follows that $M_2 \prec M'' \cap \mathrm{V}_2$ in $\mathcal{F}_2$. As M'' is the unique cover of M in $\mathcal{F}$, we conclude that $M'' \cap \mathrm{V}_2$ is the unique cover of M_2 in $\mathcal{F}_2$ and $M_2 \in \mathcal{F}_2$.

Let $M_1 \in \mathcal{M}_1$ and consider $M_1 \cup \mathrm{V}_2 \in \mathcal{F}_2$. Let M_1' be the unique cover of M_1 in $\mathcal{F}_1$. As $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ by Theorem 1, we have that $M_1 \cup \mathrm{V}_2 \prec M_1' \cup \mathrm{V}_2$ is in $\mathcal{F}$. Let F be any closed set such that $M_1 \cup \mathrm{V}_2 \subset F$. We have $F \cap \mathrm{V}_2 = \mathrm{V}_2$ and hence $M_1 \subset F \cap \mathrm{V}_1$. Since $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$, we get $F \cap \mathrm{V}_1 \in \mathcal{F}_1$. As $M_1 \prec M_1'$ in $\mathcal{F}_1$ and $M_1 \in \mathcal{M}_1$, we conclude that $M_1' \subseteq F \cap \mathrm{V}_1$ and hence that $M_1' \cap \mathrm{V}_2 \subseteq F$. Therefore, $M_1 \cup \mathrm{V}_2 \in \mathcal{M}$. Similarly we obtain $M_2 \cup \mathrm{V}_1 \in \mathcal{M}$, for $M_2 \in \mathcal{M}_2$. ◻

Item *3* of Theorem 1 considers the case where the split is acyclic (as item *4*). In particular, the proof of Theorem 1 shows that $\mathcal{F}_2 \subseteq \mathcal{F}$ in this case. Since $\mathcal{F}$ is a meet-sublattice of $\mathcal{F}_1 \times \mathcal{F}_2$, and both $\mathcal{F}_1$, $\mathcal{F}_2$ appear as traces of $\mathcal{F}$, we have that $|\mathcal{F}| \geq |\mathcal{F}_1|$ and $|\mathcal{F}| \geq |\mathcal{F}_2|$. As $\mathcal{F}_2 \subseteq \mathcal{F}$, for each $F_2 \in \mathcal{F}_2$, it may exists several closed sets of $\mathcal{F}_1$ which extend F_2 to another element of $\mathcal{F}$.

Definition 1. *Let* $\mathcal{H}$ *be a dihypergraph with acyclic split* $(\mathrm{V}_1, \mathrm{V}_2)$. *Let* $F_1 \in \mathcal{F}_1$, $F_2 \in \mathcal{F}_2$. *We say that* $F_1 \cup F_2$ *is an* extension *of* F_2 *if it belongs to* $\mathcal{F}$. *We denote by* $\mathsf{Ext}(F_2)$ *the set of extensions of* F_2, *namely* $\mathsf{Ext}(F_2) = \{F \in \mathcal{F} \mid F \cap \mathrm{V}_2 = F_2\}$.

We denote by $\mathsf{Ext}(F_2) \colon \mathrm{V}_1$ the set of closed sets of $\mathcal{F}_1$ which make extensions of F_2. Hence, any closed set F of $\mathcal{F}$ can be seen as the extension of some $F_2 \in \mathcal{F}_2$ so that $\mathcal{F}$ results from the union of extensions of closed sets in $\mathcal{F}_2$:

$$\mathcal{F} = \bigcup_{F_2 \in \mathcal{F}_2} \mathsf{Ext}(F_2)$$

Extensions of $F_2 \in \mathcal{F}_2$ can be characterized using $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$ as follows.

Lemma 1. *Let* $F_1 \in \mathcal{F}_1$, $F_2 \in \mathcal{F}_2$. *Then* $F_1 \cup F_2$ *is an extension of* F_2 *if and only if for any arc* (B, h) *in* $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$, $B \subseteq F_1$ *implies* $h \in F_2$.

Proof. We begin with the only if part. Let F_1 be a closed set of $\mathcal{F}_1$ such that $F_1 \cup F_2$ is an extension of F_2 and let $(B, h) \in \mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$. If $B \subseteq F_1$, then it must be that $h \in F_2$ since otherwise we would contradict $F_1 \cup F_2 \in \mathcal{F}$.

We move to the if part. Let F_1 be a closed set of $\mathcal{F}_1$ and F_2 a closed set of $\mathcal{F}_2$ such that for any arc $(B, h) \in \mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$, $B \subseteq F_1$ implies $h \in F_2$. We have to show that $F_1 \cup F_2$ is closed. Let (B, h) be an arc of $\mathcal{H}$. As $(\mathrm{V}_1, \mathrm{V}_2)$ is an acyclic split of V, we have two cases for (B, h): either (B, h) is in $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$ or it is not. In the second case, assume it is in $\mathcal{H}[\mathrm{V}_1]$. As $B \subseteq F_1 \cup F_2$, we have

$B \subseteq F_1$. Furthermore, F_1 is closed for $\mathcal{H}[\mathrm{V}_1]$. Hence, $h \in F_1 \subseteq F_1 \cup F_2$. The same reasoning can be applied if (B, h) is in $\mathcal{H}[\mathrm{V}_2]$. Now assume (B, h) is in $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$. We have that $B \subseteq \mathrm{V}_1$ by definition of an acyclic split. In particular we have $B \subseteq F_1$ which entails $h \in F_2$ by assumption. In any case, $F_1 \cup F_2$ already contains h for any arc (B, h) such that $B \subseteq F_1 \cup F_2$ and $F_1 \cup F_2$ is closed. □

Observe that for the particular case $F_2 = \mathrm{V}_2$, we have $\mathsf{Ext}(\mathrm{V}_2)\colon \mathrm{V}_1 = \mathcal{F}_1$ because any arc (B, h) of $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$ satisfies $h \in \mathrm{V}_2$. A consequence of Lemma 1 is that the extension is hereditary, as stated by the following lemma.

Lemma 2. *Let $F_1 \in \mathcal{F}_1$, $F_2 \in \mathcal{F}_2$. If $F_1 \cup F_2$ is an extension of F_2, then for any closed set F_1' of $\mathcal{F}_1$ such that $F_1' \subseteq F_1$, $F_1' \cup F_2$ is also an extension of F_2.*

Proof. Let $F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2$ such that $F_1 \cup F_2 \in \mathcal{F}$. Let $F_1' \in \mathcal{F}_1$ such that $F_1' \subseteq F_1$. As $F_1 \cup F_2$ is an extension of F_2, for any arc (B, h) of $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$ such that $B \subseteq F_1$, we have $h \in F_2$ by Lemma 1. Since $F_1' \subseteq F_1$, this condition holds in particular for any arc (B, h) of $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$ such that $B \subseteq F_1' \subseteq F_1$. Applying Lemma 1, we have that $F_1' \cup F_2$ is closed. □

As $\mathcal{F}$ is a meet-sublattice of $\mathcal{F}_1 \times \mathcal{F}_2$ by Theorem 1, it follows from Lemma 2 that for any $F_2 \in \mathcal{F}_2$, the set $\mathsf{Ext}(F_2)\colon \mathrm{V}_1$ is an ideal of $\mathcal{F}_1$. Thus, it is uniquely determined by its maximal elements. They are inclusion-wise maximal closed sets of $\mathcal{F}_1$ satisfying the condition of Lemma 1.

Example 1. We consider the introductory example $\mathcal{H}$ and the acyclic split $\mathrm{V}_1 = \{1, 2, 3, 4\}$ and $\mathrm{V}_2 = \{5, 6, 7\}$. We have for instance $\mathsf{Ext}(7) = \{7, 37, 237, 347\}$ which corresponds to the ideal $\{\emptyset, 3, 23, 24\}$ of $\mathcal{F}_1$ illustrated on the left of Figure 2 representing $\mathcal{F}_1$.

Now we are interested in the characterization of meet-irreducible elements $\mathcal{M}$ of $\mathcal{F}$. The strategy is to identify for each $F_2 \in \mathcal{F}_2$, which closed sets of $\mathsf{Ext}(F_2)$ are meet-irreducible elements of $\mathcal{F}$.

Proposition 2. *Let $F = F_1 \cup F_2 \in \mathcal{F}$. Let $F_2' \in \mathcal{F}_2$ such that $F_2 \prec F_2'$. Then $F_2' \cup F_1$ is closed in $\mathcal{F}$ and $F \prec F_2' \cup F_1$ in $\mathcal{F}$.*

Proof. Let $F = F_1 \cup F_2 \in \mathcal{F}$. Let $F_2' \in \mathcal{F}_2$ such that $F_2 \prec F_2'$. As $F_1 \cup F_2$ is an extension of F_2, for every arc (B, h) of $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$ such that $B \subseteq F_1$, we have $h \in F_2 \subseteq F_2'$ by Lemma 1. Therefore, $F_1 \cup F_2'$ is an extension of F_2'.

Now we show that $F_1 \cup F_2'$ is a cover of F. Let $F'' \in \mathcal{F}$ such that $F \subseteq F'' \subseteq F_1 \cup F_2'$. As $F \cap \mathrm{V}_1 = F_1 = (F_1 \cup F_2') \cap \mathrm{V}_1$, we have that $F'' \cap \mathrm{V}_1 = F_1$. Recall from Theorem 1 that $\mathcal{F} \subseteq \mathcal{F}_1 \times \mathcal{F}_2$. Therefore $F'' \cap \mathrm{V}_2$ is a closed set of $\mathcal{F}_2$ and $F_2 \subseteq F'' \cap \mathrm{V}_2 \subseteq F_2'$. As $F_2 \prec F_2'$ in $\mathcal{F}_2$, we have either $F_2 = F'' \cap \mathrm{V}_2$ or $F_2' = F'' \cap \mathrm{V}_2$. Consequently, $F'' = F$ or $F'' = F_1 \cup F_2'$ which entails $F \prec F_1 \cup F_2'$ in $\mathcal{F}$, concluding the proof. □

A consequence of this proposition is that for any $F_2, F_2' \in \mathcal{F}_2$ such that $F_2 \subseteq F_2'$, one has $\mathsf{Ext}(F_2)\colon \mathrm{V}_1 \subseteq \mathsf{Ext}(F_2')\colon \mathrm{V}_1$. In particular, if $F_2 \prec F_2'$ in $\mathcal{F}_2$, then each extension F of F_2 is covered by the unique extension F' of F_2' such that $F \cap \mathrm{V}_1 = F' \cap \mathrm{V}_1$. This leads us to the following lemmas.

Lemma 3. *Let $F_2 \in \mathcal{F}_2, F_2 \neq \mathrm{V}_2$ and $F_1 \in \mathcal{F}_1$ such that $F_1 \cup F_2$ is a non-maximal extension of F_2. Then $F_1 \cup F_2 \notin \mathcal{M}$.*

Proof. Let $F_2 \in \mathcal{F}_2, F_2 \neq \mathrm{V}_2$ and $F_1 \in \mathcal{F}_1$ such that $F_1 \cup F_2$ is a non-maximal extension of F_2. As $F_2 \neq \mathrm{V}_2$, there exists at least one closed set $F_2' \in \mathcal{F}_2$ such that $F_2 \prec F_2'$. By Proposition 2 we have that $F_1 \cup F_2 \prec F_1 \cup F_2'$ in $\mathcal{F}$. Furthermore, $F_1 \cup F_2$ is not a maximal extension of F_2. Therefore, there exists a closed set F_1' in $\mathcal{F}_1$ such that $F_1 \prec F_1'$ and $F_1' \cup F_2 \in \mathcal{F}$. As $\mathcal{F} \subseteq \mathcal{F}_1 \times \mathcal{F}_2$ by Theorem 1 and extension is hereditary by Lemma 2, it follows that $F_1 \cup F_2 \prec F_1' \cup F_2$ in $\mathcal{F}$ with $F_1 \cup F_2' \neq F_1' \cup F_2$. Therefore $F_1 \cup F_2$ is not a meet-irreducible element of $\mathcal{F}$. $\square$

Lemma 4. *Let $F_2 \in \mathcal{F}_2$ such that $F_2 \neq \mathrm{V}_2$ and $F_2 \notin \mathcal{M}_2$. Then $F \notin \mathcal{M}$ for any $F \in \mathsf{Ext}(F_2)$.*

Proof. Let $F_2 \in \mathcal{F}_2$ such that $F_2 \neq \mathrm{V}_2$ and $F_2 \notin \mathcal{M}_2$. Let $F \in \mathsf{Ext}(F_2)$ and $F_1 = F \cap \mathrm{V}_1$. As $F_2 \notin \mathcal{M}_2$, it has at least two covers F_2', F_2'' in $\mathcal{F}_2$. By Proposition 2, it follows that both $F_2' \cup F_1$ and $F_2'' \cup F_1$ are covers of F in $\mathcal{F}$. Hence $F \notin \mathcal{M}$. $\square$

These lemmas suggest that meet-irreducible elements of $\mathcal{F}$ arise from maximal extensions of meet-irreducible elements of $\mathcal{F}_2$. They might also come from meet-irreducible extensions of V_2 since $\mathsf{Ext}(\mathrm{V}_2)\colon \mathrm{V}_1 = \mathcal{F}_1$. As V_2 has no cover in $\mathcal{F}_2$, Proposition 2 cannot apply. These ideas are proved in the following theorem which characterize meet-irreducible elements $\mathcal{M}$ of $\mathcal{F}$.

Theorem 2. *Let $\mathcal{H} = (\mathrm{V}, \mathcal{E})$ be a dihypergraph with an acyclic split $(\mathrm{V}_1, \mathrm{V}_2)$. Meet-irreducible elements $\mathcal{M}$ of $\mathcal{F}$ are given by the following equality:*

$$\mathcal{M} = \{M_1 \cup \mathrm{V}_2 \mid M_1 \in \mathcal{M}_1\} \cup \{F \in \mathsf{max}_{\subseteq}(\mathsf{Ext}(M_2)) \mid M_2 \in \mathcal{M}_2\}$$

Proof. First we show that $\{M_1 \cup \mathrm{V}_2 \mid M_1 \in \mathcal{M}_1\} \subseteq \mathcal{M}$. Let $M_1 \in \mathcal{M}_1$. By Lemma 1, we have that $M_1 \cup \mathrm{V}_2 \in \mathcal{F}$, as $h \in \mathrm{V}_2$ for any (B, h) in $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$. Let F', F'' be two covers of $M_1 \cup \mathrm{V}_2$ in $\mathcal{F}$. First, observe that F' and F'' differ from $M_1 \cup \mathrm{V}_2$ only in V_1 as they both contain V_2. By Theorem 1, $\mathcal{F} \subseteq \mathcal{F}_1 \times \mathcal{F}_2$, so $F' \cap \mathrm{V}_1$ and $F'' \cap \mathrm{V}_1$ are closed sets of $\mathcal{F}_1$. Furthermore $\mathsf{Ext}(\mathrm{V}_2)\colon \mathrm{V}_1 = \mathcal{F}_1$ by Lemmas 2 and 1. Therefore, both $F' \cap \mathrm{V}_1$ and $F'' \cap \mathrm{V}_1$ cover M_1 in $\mathcal{F}_1$. Since M_1 is a meet-irreducible element of $\mathcal{F}_1$, we conclude that $F' = F''$ and $M_1 \cup \mathrm{V}_2 \in \mathcal{M}$.

Next, we prove that $\{F \in \mathsf{max}_{\subseteq}(\mathsf{Ext}(M_2)) \mid M_2 \in \mathcal{M}_2\} \subseteq \mathcal{M}$. Let $M_2 \in \mathcal{M}_2$ and $F \in \mathsf{max}_{\subseteq}(\mathsf{Ext}(M_2))$ with $F = F_1 \cup M_2$. Since $M_2 \in \mathcal{F}_2$, it has a unique cover M_2' in $\mathcal{F}_2$. By Proposition 2, we get $F \prec M_2' \cup F_1$ in $\mathcal{F}$. Let $F'' \in \mathcal{F}$ such that $F \subset F''$. Recall that $\mathcal{F} \subseteq \mathcal{F}_1 \times \mathcal{F}_2$ by Theorem 1, so that $F'' \cap \mathrm{V}_1 \in \mathcal{F}_1$ and $F'' \cap \mathrm{V}_2 \in \mathcal{F}_2$. Furthermore, $F \in \mathsf{max}_{\subseteq}(\mathsf{Ext}(M_2))$, therefore $F \subset F''$ implies that $M_2 \subset F'' \cap \mathrm{V}_2$ and hence that $M_2' \subseteq F'' \cap \mathrm{V}_2$ as $M_2 \in \mathcal{F}_2$. Since $F_1 \subseteq F'' \cap \mathrm{V}_1$, we get $F \prec M_2' \cup F_1 \subseteq F''$ and $F \in \mathcal{M}$ as it has a unique cover.

Now we prove the other side of the equation. Let $M \in \mathcal{M}$. As $\mathcal{F} \subseteq \mathcal{F}_1 \times \mathcal{F}_2$, $M \cap \mathrm{V}_2 \in \mathcal{F}_2$ and we can distinguish two cases. Either $M \cap \mathrm{V}_2 = \mathrm{V}_2$ or $M \cap \mathrm{V}_2 \subset \mathrm{V}_2$. Let us study the first case and let $M_1 = M \cap \mathrm{V}_1$. Let M' be the unique cover of M in $\mathcal{F}$. We show that $M_1' = M' \cap \mathrm{V}_1$ is the unique cover of M_1 in $\mathcal{F}_1$. By Theorem 1 and Lemma 2, we have that $M_1 \prec M_1'$ in $\mathcal{F}_1$. Let F_1 be any closed set of $\mathcal{F}_1$ with $M_1 \subset F_1$. Recall that $\mathsf{Ext}(\mathrm{V}_2)\colon \mathrm{V}_1 = \mathcal{F}_1$ by Lemmas 1 and 2. Hence

$F_1 \cup \mathrm{V}_2$ is closed and $M \subseteq F_1 \cup \mathrm{V}_2$. As $M \in M$, we also deduce $M' \subseteq F_1 \cup \mathrm{V}_2$. Therefore, $M'_1 \subseteq F_1$, and M'_1 must be the unique cover of M_1 in $\mathcal{F}_1$. So, $M_1 \in \mathcal{M}_1$ and for any $M \in \mathcal{M}$ such that $\mathrm{V}_2 \subseteq M$, we have $M \in \{M_1 \cup \mathrm{V}_2 \mid M_1 \in \mathcal{M}_1\}$.

Now assume that $M \cap \mathrm{V}_2 \subset \mathrm{V}_2$. Let $M_1 = M \cap \mathrm{V}_1$ and $M_2 = M \cap \mathrm{V}_2$. Then by contrapositive of Lemma 3 we have that $M \in \mathsf{max}_{\subseteq}(\mathsf{Ext}(M_2))$ as $M_2 \neq \mathrm{V}_2$. Similarly we get $M_2 \in \mathcal{M}_2$ by Lemma 4. □

This theorem hints a strategy to compute meet-irreducible elements in a recursive manner, using a hierarchical decomposition of $\mathcal{H}$ with acyclic splits, as proposed in the next section.

5 Recursive application of acyclic splits

In this section, we discuss an algorithm to compute $\mathcal{M}$ from a dihypergraph $\mathcal{H}$ based on Theorem 2. First, note that we have both $|\,\mathcal{M}\,| \geq |\,\mathcal{M}_1\,|$ and $|\,\mathcal{M}\,| \geq |\,\mathcal{M}_2\,|$. Furthermore, each $M \in \mathcal{M}$ arise from a unique element of $M' \in \mathcal{M}_1 \cup \mathcal{M}_2$, and each $M' \in \mathcal{M}_1 \cup \mathcal{M}_2$ is used to construct at least one new meet-irreducible element $M \in \mathcal{M}$. Therefore, we deduce an algorithm whose output is precisely $\mathcal{M}$, where each $M \in \mathcal{M}$ is given only once. Furthermore, the space needed to store intermediate solutions is bounded by the size of the output $\mathcal{M}$ which prevents an exponential blow up during the execution. The algorithm proceeds as follows. For c-factors of $\mathcal{H}$, we use algorithms such as in [19] to compute $\mathcal{M}$. When c-factors are singletons, the unique meet-irreducible to find is $\emptyset$ and hence no call to other algorithm is required. Otherwise, we find an acyclic split $(\mathrm{V}_1, \mathrm{V}_2)$ of $\mathcal{H}$ and we recursively call the algorithm on $\mathcal{H}[\mathrm{V}_1]$ and $\mathcal{H}[\mathrm{V}_2]$. Then, we compute $\mathcal{M}$ using $\mathcal{M}_1, \mathcal{M}_2$ and Theorem 2.

Computing $\mathcal{M}$ from $\mathcal{M}_1$, $\mathcal{M}_2$ requires to find maximal extensions of every meet-irreducible element $M_2 \in \mathcal{M}_2$. We will show that finding maximal extensions of a closed set is equivalent to a *dualization problem* in closure systems. First, we state the extension problem:

> **Problem:** Find Maximal Extensions with Acyclic Split (FMEAS)
> **Input:** A triple $\mathcal{H}[\mathrm{V}_1]$, $\mathcal{H}[\mathrm{V}_2]$, $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$ given by an acyclic split of a dihypergraph $\mathcal{H}$, meet-irreducible elements $\mathcal{M}_1, \mathcal{M}_2$, and a closed set F_2 of $\mathcal{H}[\mathrm{V}_2]$.
> **Output:** The maximal extensions of F_2 in $\mathcal{F}$, i.e., $\mathsf{max}_{\subseteq}(\mathsf{Ext}(F_2))$.

Let $\mathcal{B}^+$, $\mathcal{B}^-$ be two antichains of $\mathcal{F}$. The dualization in lattices asks if two antichains $\mathcal{B}^-$, $\mathcal{B}^+$ are *dual* in $\mathcal{F}$, that is if

$$\downarrow \mathcal{B}^+ \cup \uparrow \mathcal{B}^- = \mathcal{F} \text{ and } \uparrow \mathcal{B}^- \cap \downarrow \mathcal{B}^+ = \emptyset.$$

Note that $\mathcal{B}^-$ and $\mathcal{B}^+$ are dual if either $\mathcal{B}^+ = \mathsf{max}_{\subseteq}\{F \in \mathcal{F} \mid F \notin \uparrow \mathcal{B}^-\}$ or $\mathcal{B}^- = \mathsf{min}_{\subseteq}\{F \in \mathcal{F} \mid F \notin \downarrow \mathcal{B}^+\}$. If $\mathcal{F}$ is given, the question can be answered in polynomial time. In our case however, $\mathcal{F}$ is implicitly given by $\mathcal{M}$ and $\mathcal{H}$. More precisely we use the next generation problem:

> **Problem:** Dualization with Dihypergraph and meet-irreducible (DMDual)

Input: A dihypergraph $\mathcal{H} = (\mathrm{V}, \mathcal{E})$, the meet-irreducible elements $\mathcal{M}$ of $\mathcal{F}$, and an antichain $\mathcal{B}^-$ of $\mathcal{F}$.
Output: The dual antichain $\mathcal{B}^+$ of $\mathcal{B}^-$.

This problem has been introduced in [3] in its decision version, where authors show that it is not harder than finding a (minimum) dihypergraph from a set of meet-irreducible elements. In general however, the problem is open. When $\mathcal{H}$ has no arcs, DMDual is equivalent to hypergraph dualization as there are $|\mathrm{V}|$ meet-irreducible elements which can easily be computed by taking $\mathrm{V} \setminus \{x\}$ for any $x \in \mathrm{V}$. This latter problem can be solved in output quasi-polynomial time using the algorithm of Fredman and Khachiyan [10].

We show that FMEAS and DMDual are equivalent under polynomial reduction. First, we relate maximal extensions of a closed set with dualization. Let $F_2 \in \mathcal{F}_2$. Recall that $\mathsf{Ext}(F_2)\colon \mathrm{V}_1$ is an ideal of $\mathcal{F}_1$. Hence, the antichain $\mathsf{max}_{\subseteq}(\mathsf{Ext}(F_2)\colon \mathrm{V}_1)$ has a dual antichain $\mathcal{B}^-(F_2)$ in $\mathcal{F}_1$, i.e., $\mathcal{B}^-(F_2) = \mathsf{min}_{\subseteq}\{F_1 \in \mathcal{F}_1 \mid F_1 \notin \mathsf{Ext}(F_2)\colon \mathrm{V}_1\}$.

Proposition 3. *Let $F_2 \in \mathcal{F}_2$, and $F_1 \in \mathcal{F}_1$. Then, $F_1 \in \mathcal{B}^-(F_2)$ if and only if $F_1 \in \mathsf{min}_{\subseteq}\{B^{\mathcal{H}[\mathrm{V}_1]} \mid (B,h) \in \mathcal{H}[\mathrm{V}_1, \mathrm{V}_2], h \notin F_2\}$.*

Proof. We show the if part. Let $F_1 \in \mathsf{min}_{\subseteq}\{B^{\mathcal{H}[\mathrm{V}_1]} \mid (B,h) \in \mathcal{H}[\mathrm{V}_1, \mathrm{V}_2], h \notin F_2\}$. We show that for any closed set $F_1' \subsetneq F_1$ in $\mathcal{F}_1$, F_1' contributes to an extension of F_2. It is sufficient to show this property to the case where $F_1' \prec F_1$ as $\mathsf{Ext}(F_2)\colon \mathrm{V}_1$ is an ideal of $\mathcal{F}_1$. Hence consider a closed set F_1' in $\mathcal{F}_1$ such that $F_1' \prec F_1$. Note that such F_1' exists since $\emptyset \in \mathcal{F}_1$ and no arc (B,h) in $\mathcal{H}$ has $B = \emptyset$ so that $\emptyset \subset B^{\mathcal{H}[\mathrm{V}_1]}$ for any arc (B,h) of $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$ such that $h \notin F_2$. Then, by construction of F_1', for any (B,h) in $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$ such that $h \notin F_2$, we have $B^{\mathcal{H}[\mathrm{V}_1]} \nsubseteq F_1'$. As $(.)^{\mathcal{H}[\mathrm{V}_1]}$ is a closure operator, it is monotone and $B^{\mathcal{H}[\mathrm{V}_1]} \nsubseteq {F_1'}^{\mathcal{H}[\mathrm{V}_1]} = F_1'$ entails $B \nsubseteq F_1'$ for any such arc (B,h). Therefore $F_1' \in \mathsf{Ext}(F_2)\colon \mathrm{V}_1$ and $F_1 \in \mathcal{B}^-(F_2)$.

We prove the only if part. We use contrapositive. Assume $F_1 \notin \mathsf{min}_{\subseteq}\{B^{\mathcal{H}[\mathrm{V}_1]} \mid (B,h) \in \mathcal{H}[\mathrm{V}_1, \mathrm{V}_2], h \notin F_2\}$. Then we have two cases. First, for any arc (B,h) in $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$ such that $h \notin F_2$, $B^{\mathcal{H}[\mathrm{V}_1]} \nsubseteq F_1$. As $(.)^{\mathcal{H}[\mathrm{V}_1]}$ is a closure operator, it is monotone, and since F_1 is closed in $\mathcal{F}_1$, we have $B \nsubseteq F_1$ and $F_1 \in \mathsf{Ext}(F_2)\colon \mathrm{V}_1$ by Lemma 1. Hence $F_1 \notin \mathcal{B}^-(F_2)$. In the second case, there is an arc (B,h) with $h \notin F_2$ in $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$ such that $B^{\mathcal{H}[\mathrm{V}_1]} \subseteq F_1$ which implies $F_1 \notin \mathsf{Ext}(F_2)\colon \mathrm{V}_1$. If $B^{\mathcal{H}[\mathrm{V}_1]} \subset F_1$, then clearly $F_1 \notin \mathcal{B}^-(F_2)$ as $B^{\mathcal{H}[\mathrm{V}_1]} \in \mathcal{F}_1$ and $B^{\mathcal{H}[\mathrm{V}_1]} \notin \mathsf{Ext}(F_2)\colon \mathrm{V}_1$. Hence, assume that $F = B^{\mathcal{H}[\mathrm{V}_1]}$. Since $F_1 \notin \mathsf{min}_{\subseteq}\{B^{\mathcal{H}[\mathrm{V}_1]} \mid (B,h) \in \mathcal{H}[\mathrm{V}_1, \mathrm{V}_2], h \notin F_2\}$ by hypothesis, there exists another arc $(B', h') \in \mathcal{E}(\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2])$ such that $h \notin F_2$ and ${B'}^{\mathcal{H}[\mathrm{V}_1]} \subset F_1$. Hence ${B'}^{\mathcal{H}[\mathrm{V}_1]} \notin \mathsf{Ext}(F_2)\colon \mathrm{V}_1$ and $F_1 \notin \mathcal{B}^-(F_2)$ as it is not an inclusion-wise minimum closed set which does not belong to $\mathsf{Ext}(F_2)\colon \mathrm{V}_1$. ◻

Observe that for any $F_2 \in \mathcal{F}_2$, $\mathcal{B}^-(F_2)$ can easily be computed using $\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2]$ and Lemma 1. Therefore we prove the following theorem.

Theorem 3. FMEAS *and* DMDual *are polynomially equivalent.*

Proof. First we show that DMDual is harder than FMEAS. Let $\mathcal{H} = (\mathrm{V}, \mathcal{E})$ be a dihypergraph, and $(\mathcal{H}[\mathrm{V}_1], \mathcal{H}[\mathrm{V}_2], \mathcal{H}[\mathrm{V}_1, \mathrm{V}_2], \mathcal{M}_1, \mathcal{M}_2, F_2)$ be an instance of FMEAS. By Proposition 3, finding $\mathsf{max}_{\subseteq}(\mathsf{Ext}(F_2))$ amounts to find the dual antichain of $\mathcal{B}^-(F_2) = \mathsf{min}_{\subseteq}\{B^{\mathcal{H}[\mathrm{V}_1]} \mid (B, h) \in \mathcal{H}[\mathrm{V}_1, \mathrm{V}_2], h \notin F_2\}$ in $\mathcal{F}_1$. Note that $\mathcal{B}^-(F_2)$ can be computed in polynomial time in the size of $\mathcal{H}[\mathrm{V}_1]$ and $|\,\mathcal{B}^-(F_2)| \leq |\,\mathcal{E}(\mathcal{H}[\mathrm{V}_1, \mathrm{V}_2])|$. Therefore, the instance of FMEAS reduces to the instance $(\mathcal{H}[\mathrm{V}_1], \mathcal{M}_1, \mathcal{B}^-(F_2))$ of DMDual.

Now we show that FMEAS is harder than DMDual. Let $(\mathcal{H}, \mathcal{M}, \mathcal{B}^-)$ be an instance of DMDual. Let z be a new gadget vertex and consider the bipartite dihypergraph $\mathcal{H}[\mathrm{V}, \{z\}] = (\mathrm{V} \cup \{z\}, \{(B, z) \mid B \in \mathcal{B}^-\})$. Let $\mathcal{H}_{new} = \mathcal{H} \cup \mathcal{H}[\mathrm{V}, \{z\}]$. Clearly, $\mathcal{H}_{new}$ has an acyclic split $(\mathrm{V}, \{z\})$ such that $\mathcal{H}_{new}[\mathrm{V}] = \mathcal{H}$, $\mathcal{H}_{new}[\{z\}] = (\{z\}, \emptyset)$ and $\mathcal{H}_{new}[\mathrm{V}, \{z\}] = \mathcal{H}[\mathrm{V}, \{z\}]$. The closure system associated to $\mathcal{H}_{new}[\{z\}]$ has only 2 elements: its unique meet-irreducible element $\emptyset$ and $\{z\}$. We obtain an instance FMEAS where the input is $\mathcal{H}$, $\mathcal{H}_{new}[\{z\}]$, $\mathcal{H}[\mathrm{V}, \{z\}]$, $\mathcal{M}$, $\{\emptyset\}$ and where the closed set of interest is $\emptyset$. Moreover this reduction is polynomial in the size of $(\mathcal{H}, \mathcal{M}, \mathcal{B}^-)$ as we create a unique new element and $|\,\mathcal{B}^-\,|$ arcs. According to Proposition 3, maximal extensions of $\emptyset$ are given by the antichain dual to $\mathcal{B}^-(\emptyset) = \mathsf{min}_{\subseteq}\{B^{\mathcal{H}} \mid (B, z) \in \mathcal{H}[\mathrm{V}, \{z\}]\}$. However, we have $\mathcal{B}^-(\emptyset) = \mathcal{B}^-$, so that maximal extensions of $\emptyset$ are precisely elements of the dual antichain $\mathcal{B}^+$ of $\mathcal{B}^-$. □

We can deduce a class of dihypergraphs where our strategy can be applied to obtain meet-irreducible elements in output quasi-polynomial time. Let us assume that $\mathcal{H}$ can be decomposed as follows. Its c-factors are singletons. If $\mathcal{H}$ is not itself a singleton, it has an acyclic split $(\mathrm{V}_1, \mathrm{V}_2)$ with $\mathcal{H}[\mathrm{V}_1] = (\mathrm{V}_1, \emptyset)$. Hence, DMDual reduces to hypergraph dualization and can be solved in output-quasi polynomial time using the algorithm of [10]. Recursively applying hypergraph dualization, we get $\mathcal{M}$ for $\mathcal{H}$ in output-quasi polynomial time. This class of dihypergraph generalizes ranked convex geometries of [8].

The closure system represented by a dihypergraph $\mathcal{H}$ is a ranked convex geometry if there exists a full partition $\mathrm{V}_1, \ldots, \mathrm{V}_n$, of V such that $\mathcal{H}[\mathrm{V}_i] = (\mathrm{V}_i, \emptyset)$ for any $1 \leq i \leq n$ and for any arc (B, h) in $\mathcal{H}$ there is a $j < k$ such that $B \subseteq \mathrm{V}_j$ and $h \in \mathrm{V}_{j+1}$. All c-factors of $\mathcal{H}$ are singletons. Choosing the acyclic split $(\mathrm{V}_i, \bigcup_{j=i+1}^{n} \mathrm{V}_j)$ at the i-th step of the algorithm yields a decomposition which satisfies conditions of the previous paragraph.

6 Conclusion

In this paper we investigated the problem of finding meet-irreducible elements of a closure system represented by a dihypergraph. In general, the complexity of this problem is unknown and harder than hypergraph dualization. Using a partitioning operation called an acyclic split on the dihypergraph, we gave a characterization of its associated meet-irreducible elements. Acyclic splits lead to a recursive algorithm to find meet-irreducible elements from a dihypergraph.

With our algorithm, we reach new classes of dihypergraphs for which meet-irreducible elements can now be computed in output quasi-polynomial time. In particular, we improve previous results on ranked convex geometries [8].

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