Convergence of Adaptive Methods for Equilibrium Problems in Hadamard Spaces

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Abstract

In this paper we consider equilibrium problems in metric Hadamard spaces. We propose and study new adaptive algorithms for their approximate solution. For pseudomonotone bifunctions of Lipschitz type, theorems on the weak convergence of sequences generated by the algorithms are proved. The proofs are based on the use of Fejer properties of algorithms with respect to the set of solutions to the problem. A new regularized adaptive extraproximal algorithm is also proposed and studied. To regularize the basic extraproximal scheme, the classical Halpern scheme was used. The proposed algorithms are applicable to pseudomonotone variational inequalities in Hilbert spaces.

Keywords [1]

Equilibrium problems, Hadamard space, pseudomonotonicity, adaptability, regularization, convergence, extraproximal algorithm

1. Introduction

A popular direction of modern nonlinear analysis is the study of equilibrium problems (Ky Fan inequalities) of the form [1-9]:

$$\text{find } x \in C: \ F(x,y) \geq 0 \quad \forall y \in C, \tag{1}$$

where C is nonempty subset of vector space H (usually Hilbert space), $F : C \times C \to R$ is function such that $F(x,x) = 0 \ \ \forall x \in C$ (called bifunction). We can formulate mathematical programming problems, variational inequalities, and many game theory problems in form (1).

The study of algorithms for solving equilibrium and related problems is actively continuing [5-8, 10-30]. In this article, we will focus only on methods of the extraproximal type. The following analogue of G. Korpelevich extragradient method [15] for equilibrium problems [16] is called extraproximal

$$\begin{cases} y_n = \operatorname{prox}_{\lambda_n F(x_n,\cdot)} x_n, \\ x_{n+1} = \operatorname{prox}_{\lambda_n F(y_n,\cdot)} x_n, \end{cases}$$

where $\lambda_n \in (0,+\infty)$, $\operatorname{prox}_{\varphi}$ is proximal operator for function φ. In [19] two step proximal method for solving equilibrium problems in Hilbert space was proposed

$$\begin{cases} y_n = \operatorname{prox}_{\lambda_n F(y_{n-1},\cdot)} x_n, \\ x_{n+1} = \operatorname{prox}_{\lambda_n F(y_n,\cdot)} x_n, \end{cases}$$

where $\lambda_n \in (0,+\infty)$, which is adaptation for L. D. Popov method [20] for general equilibrium programming problems (see also [21, 22]). Note that a version of this algorithm for variational inequalities became known among machine learning specialists under the name "Extrapolation from the Past" [31].

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Recently, there has been an increased interest in the construction of theory and algorithms for solving mathematical programming problems in metric Hadamard spaces [32] (also known as $CAT(0)$ spaces). A strong motivation for studying these problems is the ability to rewrite some nonconvex problems in the form of convex (more precisely, geodesically convex) in a space with a specially selected metric structure [32, 33]. Some authors began to study equilibrium problems in Hadamard spaces [33-37]. For example, in [35], concluding from the results of [16], the authors proposed and substantiated an analogue of the extraproximal method for pseudomonotone equilibrium problems in Hadamard spaces.

In this paper, which continues and refines articles [36, 37], two new adaptive two-stage proximal algorithms for the approximate solution of equilibrium problems in Hadamard spaces are described and studied. The proposed rules for choosing the step size do not calculate the values of the bifunction at additional points and do not require knowledge of the Lipschitz constants of the bifunction.

For pseudo-monotone bifunctions of Lipschitz type, theorems on the weak convergence of sequences generated by the algorithms are proved. The proofs are based on the use of Fejer properties of algorithms with respect to the set of solutions to the problem. A new regularized adaptive extraproximal algorithm is also proposed and studied. To regularize the basic adaptive extraproximal scheme [37], the classical Halpern scheme [38] was used, a version of which for Hadamard spaces was studied in [32]. It is shown that the proposed algorithms are applicable to pseudomonotone variational inequalities in Hilbert spaces.

2. Preliminaries

Here are some concepts and facts related to metric Hadamard spaces. Details can be found in [32, 39, 40].

Let (X,d) be a metric space and x, $y \in X$. Geodesic path connecting points x and y is isometry $\gamma : [0, d(x,y)] \to X$ such that $\gamma(0) = x$, $\gamma(d(x,y)) = y$. Set $\gamma([0, d(x,y)]) \subseteq X$ is denoted by $[x,y]$ and called geodesic segment with ends x and y (or simply geodesic). Metric space (X,d) is called geodesic space if it is possible to connect any two points of X by geodesic and it is unique geodesic space if for any two points from X there exists exactly one geodesic to connect them. Geodesic space (X,d) is called $CAT(0)$ space if for any three points y_0, y_1, $y_2 \in X$ such that $d^2(y_1,y_0) = d^2(y_2,y_0) = \frac{1}{2} d^2(y_1,y_2)$ the following inequality holds:

$$d^2(x,y_0) \le \frac{1}{2} d^2(x,y_1) + \frac{1}{2} d^2(x,y_2) - \frac{1}{4} d^2(y_1,y_2) \quad \forall x \in X .$$

It is known that $CAT(0)$ space is unique geodesic [32]. For two points x and y from $CAT(0)$ space (X,d) and $t \in [0,1]$ we denote by $tx \oplus (1-t) y$ unique point z of $[x,y]$ such that $d(z,x) = (1-t) d(x,y)$ and $d(z,y) = td(x,y)$. Set $C \subseteq X$ is called convex if for all x, $y \in C$ and $t \in [0,1]$ holds $tx \oplus (1-t) y \in C$. The following inequality is useful property for $CAT(0)$ space (X,d)

$$d^2(tx \oplus (1-t) y, z) \le td^2(x,z) + (1-t) d^2(y,z) - t(1-t) d^2(x,y), \ \{x,y,z\} \in X , \ t \in [0,1]. \quad (2)$$

Important examples of $CAT(0)$ spaces are Euclidean spaces, R-trees, Hadamard manifolds (complete connected Riemannian manifolds of non-positive curvature) and Hilbert sphere with hyperbolic metric [32, 39, 40].

Complete $CAT(0)$ space is called Hadamard space.

As in a Hilbert space, the operator of metric projection onto a closed convex set is well defined in Hadamard spaces C [32]. For each $x \in X$ there exists unique element $P_C x$ from set C with the property $d\left(P_C x, x\right) = \min_{z \in C} d\left(z, x\right)$, moreover the characterization takes place [32]:

$$y = P_C x \iff y \in C \text{ and } d^2\left(y,z\right) \le d^2\left(x,z\right) - d^2\left(y,x\right) \quad \forall z \in C.$$

Let $\left(X,d\right)$ be a metric space and $\left(x_n\right)$ be a bounded sequence of elements from X. Let $r\left(x,\left(x_n\right)\right) = \overline{\lim_{n\to\infty}} d\left(x, x_n\right)$. The number $r\left(\left(x_n\right)\right) = \inf_{x\in X} r\left(x,\left(x_n\right)\right)$ is called asymptotical radius $\left(x_n\right)$ and set $A\left(\left(x_n\right)\right) = \left\{x \in X : r\left(x,\left(x_n\right)\right) = r\left(\left(x_n\right)\right)\right\}$ is asymptotic center $\left(x_n\right)$. It is known that in Hadamard space $A\left(\left(x_n\right)\right)$ it consists of one point [32]. Sequence $\left(x_n\right)$ of elements from Hadamard space $\left(X,d\right)$ converges weakly to an element $x \in X$ if $A\left(\left(x_{n_k}\right)\right) = \left\{x\right\}$ for any sequence $\left(x_{n_k}\right)$. It is known that any sequence of elements from closed convex bounded subset K of Hadamard space has subsequence which converges weakly to element from K [32, 39]. The well-known analogue of Opial lemma is useful in proving the weak convergence of sequences of elements of the Hadamard space.

Lemma 1 ([32, p. 60]). Let sequence $\left(x_n\right)$ of elements from Hadamard space $\left(X,d\right)$ converges weakly to an element $x \in X$. Then for all $y \in X \setminus \left\{x\right\}$ we have $\underline{\lim}_{n\to\infty} d\left(x_n, x\right) < \underline{\lim}_{n\to\infty} d\left(x_n, y\right)$.

Let $\left(X,d\right)$ be an Hadamard space. Function $\varphi : X \to \overline{R} = R \cup \left\{+\infty\right\}$ is called convex if for all points x, $y \in X$ and $t \in \left[0,1\right]$ holds

$$\varphi\left(tx \oplus \left(1-t\right)y\right) \le t\varphi\left(x\right) + \left(1-t\right)\varphi\left(y\right).$$

For example, in Hadamard space functions $y \mapsto d\left(y,x\right)$ are convex [32]. If there exists $\mu > 0$ such that for all x, $y \in X$ and $t \in \left[0,1\right]$ the following inequality is satisfied

$$\varphi\left(tx \oplus \left(1-t\right)y\right) \le t\varphi\left(x\right) + \left(1-t\right)\varphi\left(y\right) - \mu t\left(1-t\right)d^2\left(x,y\right),$$

then function φ is called strongly convex. It is known that for convex functions lower semicontinuity and weakly lower semicontinuity are equivalent [32, p. 64] and strongly convex semicontinuous function reaches its minimum at unique point.

For convex proper and lower semicontinuous function $\varphi : X \to \overline{R} = R \cup \left\{+\infty\right\}$ proximal operator is defined by [32]

$$\operatorname{prox}_{\varphi} x = \arg\min_{y\in X} \left(\varphi\left(y\right) + \tfrac{1}{2} d^2\left(y,x\right)\right).$$

Since functions $\varphi + \frac{1}{2} d^2\left(\cdot, x\right)$ are strongly convex the definition of proximal operator is correct, i.e. for all $x \in X$ there exists unique element $\operatorname{prox}_{\varphi} x \in X$.

3. Equilibrium problems in Hadamard space

Let $\left(X,d\right)$ be a Hadamard space. Consider an equilibrium problem for nonempty closed convex set $C \subseteq X$ and bifunction $F : C \times C \to R$ [34-37]:

$$\text{find } x \in C: \ F\left(x,y\right) \ge 0 \quad \forall\, y \in C. \tag{3}$$

Assume that following conditions are satisfied:

1. $F\left(x,x\right) = 0$ for all $x \in C$;

2. functions $F(x,\cdot):C\to R$ are convex and lower semicontinuous for all $x\in C$;
3. functions $F(\cdot,y):C\to R$ are upper weakly semicontinuous for all $y\in C$;
4. bifunction $F:C\times C\to R$ is pseudomonotone, i.e.
   for all x, $y\in C$ from $F(x,y)\ge 0$ it follows that $F(y,x)\le 0$.
5. bifunction $F:C\times C\to R$ is Lipschitz type, i.e. there exist $a>0$, $b>0$, such that

$$F(x,y)\le F(x,z)+F(z,y)+ad^2(x,z)+bd^2(z,y)\quad \forall\, x,\, y,\, z\in C. \tag{4}$$

Remark 1. If $F(x,y)=(Ax,y-x)$, where $A:C\to H$, C is nonempty subset of Hilbert space H, then problem (3) takes form of variational inequality

$$\text{find } x\in C:\ (Ax,y-x)\ge 0\quad \forall\, y\in C. \tag{5}$$

If set $C\subseteq H$ is convex and closed and operator $A:C\to H$ pseudomonotone, Lipschitz continuous and sequential weakly semicontinuous, then for (5) conditions 3–5 are satisfied.

Consider dual equilibrium problem:

$$\text{find } x\in C:\ F(y,x)\le 0\quad \forall\, y\in C. \tag{6}$$

We denote sets of solutions for problems (3) and (6) by S and S^*. If conditions 1–4 are satisfied we have $S=S^*$ [34]. Moreover, set S^* is closed and convex.

Further we assume that $S\ne\varnothing$.

4. Adaptive algorithms

For approximate solution of (3) we consider extraproximal algorithm with adaptive choice of step size [37].

Algorithm 1.

Initialization. Choose element $x_1\in C$, $\tau\in(0,1)$, $\lambda_1\in(0,+\infty)$. Set $n=1$.

Step 1. Calculate

$$y_n=\operatorname{prox}_{\lambda_n F(x_n,\cdot)}x_n=\arg\min_{y\in C}\left(F(x_n,y)+\tfrac{1}{2\lambda_n}d^2(y,x_n)\right).$$

If $x_n=y_n$, then stop and $x_n\in S$. Otherwise, go to step 2.

Step 2. Calculate

$$x_{n+1}=\operatorname{prox}_{\lambda_n F(y_n,\cdot)}x_n=\arg\min_{y\in C}\left(F(y_n,y)+\tfrac{1}{2\lambda_n}d^2(y,x_n)\right).$$

Step 3. Calculate

$$\lambda_{n+1}=\begin{cases}\lambda_n, & \text{if}\quad F(x_n,x_{n+1})-F(x_n,y_n)-F(y_n,x_{n+1})\le 0,\\ \min\left\{\lambda_n,\dfrac{\tau}{2}\dfrac{d^2(x_n,y_n)+d^2(x_{n+1},y_n)}{\left(F(x_n,x_{n+1})-F(x_n,y_n)-F(y_n,x_{n+1})\right)}\right\}, & \text{otherwise.}\end{cases}$$

Set $n:=n+1$ and go to step 1.

Remark 2. On each step of algorithm 1 we need to solve two convex problems with strongly convex functions.

In proposed algorithm parameter λ_{n+1} depends on location of points x_n, y_n, x_{n+1}, values $F(x_n,x_{n+1})$, $F(x_n,y_n)$ and $F(y_n,x_{n+1})$. No information about constants a and b from inequality

(4) is used. Obviously, the sequence (λ_n) is non-decreasing. Also, it is lower bounded by $\min\left\{\lambda_1, \frac{\tau}{2\max\{a,b\}}\right\}$. Indeed, we have

$$F(x_n, x_{n+1}) - F(x_n, y_n) - F(y_n, x_{n+1}) \le \max\{a,b\}\left(d^2(x_n, y_n) + d^2(x_{n+1}, y_n)\right).$$

Let us prove the important inequality.

Lemma 2. For $x \in C$ and $x^+ = \mathrm{prox}_{\lambda F(x,\cdot)} x$, where $\lambda > 0$, the following inequality takes place

$$F(x, x^+) - F(x, y) \le \frac{1}{2\lambda}\left(d^2(y, x) - d^2(x, x^+) - d^2(x^+, y)\right) \quad \forall y \in C. \tag{7}$$

Proof. From the definition $x^+ = \arg\min_{y\in C}\left(F(x,y) + \frac{1}{2\lambda} d^2(y,x)\right)$ it follows that

$$F(x, x^+) + \frac{1}{2\lambda} d^2(x^+, x) \le F(x, p) + \frac{1}{2\lambda} d^2(p, x) \quad \forall p \in C. \tag{8}$$

Setting in (8) $p = tx^+ \oplus (1-t)y$, $y \in C$, $t \in (0,1)$, we obtain

$$F(x, x^+) + \frac{1}{2\lambda} d^2(x^+, x) \le F(x, tx^+ \oplus (1-t)y) + \frac{1}{2\lambda} d^2(tx^+ \oplus (1-t)y, x) \le$$

$$\le tF(x, x^+) + (1-t)F(x,y) + \frac{1}{2\lambda}\left(td^2(x^+, x) + (1-t)d^2(y,x) - t(1-t)d^2(x^+, y)\right).$$

Thereby,

$$(1-t)F(x, x^+) - (1-t)F(x,y) \le$$

$$\le \frac{1}{2\lambda}\left(-(1-t)d^2(x^+, x) + (1-t)d^2(y,x) - t(1-t)d^2(x^+, y)\right). \tag{9}$$

Dividing in (9) by $1-t$ and passing to the limit as $t \to 1$ we obtain (7). ∎

From Lemma 2 it follows that for sequences (x_n), (y_n), generated by Algorithm 1 the following inequalities hold

$$F(x_n, y_n) - F(x_n, y) \le \frac{1}{2\lambda_n}\left(d^2(y, x_n) - d^2(x_n, y_n) - d^2(y_n, y)\right) \quad \forall y \in C. \tag{10}$$

$$F(y_n, x_{n+1}) - F(y_n, y) \le \frac{1}{2\lambda_n}\left(d^2(y, x_n) - d^2(x_n, x_{n+1}) - d^2(x_{n+1}, y)\right) \quad \forall y \in C. \tag{11}$$

Inequality (10) provides a justification for the stopping rule for Algorithm 1. Indeed, for $x_n = y_n$ from (10) it follows

$$-F(x_n, y) \le 0 \quad \forall y \in C,$$

i.e., $x_n \in S$.

Let us prove an important estimate relating the distances between the points generated by Algorithm 1 to an arbitrary element of the set of solutions S.

Lemma 3. For sequences (x_n), (y_n), generated by Algorithm 1, the following inequality takes place

$$d^2(x_{n+1},z) \le d^2(x_n,z) - \left(1-\tau\frac{\lambda_n}{\lambda_{n+1}}\right)d^2(x_{n+1},y_n) - \left(1-\tau\frac{\lambda_n}{\lambda_{n+1}}\right)d^2(y_n,x_n), \qquad (12)$$

where $z \in S$.

Proof. Let $z \in S$. From pseudomonotonicity of bifunction F it follows that

$$F(y_n,z) \le 0. \qquad (13)$$

From (13) and (11)

$$2\lambda_n F(y_n,x_{n+1}) \le d^2(z,x_n) - d^2(x_n,x_{n+1}) - d^2(x_{n+1},z). \qquad (14)$$

From the calculation rule for λ_{n+1} we conclude

$$F(x_n,x_{n+1}) - F(x_n,y_n) - F(y_n,x_{n+1}) \le \frac{\tau}{2\lambda_{n+1}}\left(d^2(x_n,y_n) + d^2(x_{n+1},y_n)\right). \qquad (15)$$

Evaluating the left side of (14) from below using (15), we get

$$2\lambda_n\left(F(x_n,x_{n+1}) - F(x_n,y_n)\right) - \tau\frac{\lambda_n}{\lambda_{n+1}}\left(d^2(x_n,y_n) + d^2(x_{n+1},y_n)\right) \le$$

$$\le d^2(z,x_n) - d^2(x_n,x_{n+1}) - d^2(x_{n+1},z). \qquad (16)$$

For a lower bound $2\lambda_n\left(F(x_n,x_{n+1}) - F(x_n,y_n)\right)$ in (16) we use (10). We have

$$d^2(x_n,y_n) + d^2(y_n,x_{n+1}) - d^2(x_{n+1},x_n) - \tau\frac{\lambda_n}{\lambda_{n+1}}\left(d^2(x_n,y_n) + d^2(x_{n+1},y_n)\right) \le$$

$$\le d^2(z,x_n) - d^2(x_n,x_{n+1}) - d^2(x_{n+1},z). \qquad (17)$$

By regrouping (17), we get (12). ■

To prove the convergence of Algorithm 1, we need an elementary lemma about number sequences.

Lemma 4. Let (a_n), (b_n) be two sequences of non-negative numbers which satisfy $a_{n+1} \le a_n - b_n$ for all $n \in N$. Then exists a limit $\lim_{n\to\infty} a_n$ and $(b_n) \in l_1$.

Let us formulate one of the main results of the work.

Theorem 1. Let (X,d) be an Hadamard space, $C \subseteq X$ be a non-empty convex closed set, for bifuntion $F: C\times C \to R$ conditions 1–5 are satisfied and $S \ne \varnothing$. Then sequences (x_n), (y_n) generated by Algorithm 1 converge weakly to the solution $z \in S$ of equilibrium problem (3), moreover, $\lim_{n\to\infty} d(y_n,x_n) = \lim_{n\to\infty} d(y_n,x_{n+1}) = 0$.

Proof. Let $z \in S$. Assume

$$a_n = d(z,x_n),\ \ b_n = \left(1-\tau\frac{\lambda_n}{\lambda_{n+1}}\right)d^2(x_{n+1},y_n) - \left(1-\tau\frac{\lambda_n}{\lambda_{n+1}}\right)d^2(y_n,x_n).$$

Inequality (12) takes form $a_{n+1} \le a_n - b_n$. Since there exists $\lim_{n\to\infty}\lambda_n > 0$,

$$1-\tau\frac{\lambda_n}{\lambda_{n+1}} \to 1-\tau \in (0,1),\ \ n\to\infty.$$

From Lemma 4 we conclude that exists a limit $\lim_{n\to\infty} d^2(z,x_n)$ and

$$\sum_{n=1}^{\infty}\left(d^2(x_{n+1},y_n) + d^2(y_n,x_n)\right) < +\infty.$$

Whence we get boundedness of the sequence (x_n) and

$$\lim_{n\to\infty} d\left(y_n, x_n\right) = \lim_{n\to\infty} d\left(x_{n+1}, y_n\right) = \lim_{n\to\infty} d\left(x_{n+1}, x_n\right) = 0 . \qquad (18)$$

Consider subsequence $\left(x_{n_k}\right)$ which converges weakly to the point $z \in C$. Then from (18) it follows that $\left(y_{n_k}\right)$ converges weakly to z. Let us show that $z \in S$. We have

$$F\left(y_{n_k}, y\right) \geq F\left(y_{n_k}, x_{n_k+1}\right) - \frac{1}{2\lambda_{n_k}}\left(d^2\left(y, x_{n_k}\right) - d^2\left(x_{n_k}, x_{n_k+1}\right) - d^2\left(x_{n_k+1}, y\right)\right) \geq$$

$$\geq F\left(x_{n_k}, x_{n_k+1}\right) - F\left(x_{n_k}, y_{n_k}\right) -$$

$$-\frac{\tau}{2\lambda_{n_k+1}}\left(d^2\left(x_{n_k}, y_{n_k}\right) + d^2\left(x_{n_k+1}, y_{n_k}\right)\right) - \frac{1}{2\lambda_{n_k}}\left(d^2\left(y, x_{n_k}\right) - d^2\left(x_{n_k}, x_{n_k+1}\right) - d^2\left(x_{n_k+1}, y\right)\right) \geq$$

$$\geq -\frac{1}{2\lambda_{n_k}}\left(d^2\left(x_{n_k+1}, x_{n_k}\right) - d^2\left(x_{n_k}, y_{n_k}\right) - d^2\left(y_{n_k}, x_{n_k+1}\right)\right) - \frac{\tau}{2\lambda_{n_k+1}}\left(d^2\left(x_{n_k}, y_{n_k}\right) + d^2\left(x_{n_k+1}, y_{n_k}\right)\right) -$$

$$-\frac{1}{2\lambda_{n_k}}\left(d^2\left(y, x_{n_k}\right) - d^2\left(x_{n_k}, x_{n_k+1}\right) - d^2\left(x_{n_k+1}, y\right)\right) \quad \forall y \in C . \qquad (19)$$

Passing to the limit in (19) taking into account (18) and weakly upper semicontinuity of function $F(\cdot, y): C \to R$, we get $F(z, y) \geq \overline{\lim_{k\to\infty}} F\left(y_{n_k}, y\right) \geq 0 \quad \forall y \in C$, i.e., $z \in S$.

Applying Opial lemma for Hadamard spaces (Lemma 1) we obtain the convergence of sequence $\left(x_n\right)$ to the point $z \in S$. Indeed, we argue by contradiction. Let exists the subsequence $\left(x_{m_k}\right)$, which converges weakly to some point $\overline{z} \in C$ and $\overline{z} \neq z$. It is clear that $\overline{z} \in S$. We have

$$\lim_{n\to\infty} d\left(x_n, z\right) = \lim_{k\to\infty} d\left(x_{n_k}, z\right) < \lim_{k\to\infty} d\left(x_{n_k}, \overline{z}\right) = \lim_{n\to\infty} d\left(x_n, \overline{z}\right) = \lim_{k\to\infty} d\left(x_{m_k}, \overline{z}\right) <$$

$$< \lim_{k\to\infty} d\left(x_{m_k}, z\right) = \lim_{n\to\infty} d\left(x_n, z\right),$$

which is impossible. Therefore $\left(x_n\right)$ converges weakly to $z \in S$. From (18) it follows that sequence $\left(y_n\right)$ also converges to $z \in S$. ■

Remark 3. We see from proof for Theorem 1 that for sequence $\left(x_n\right)$ starting from some number N Fejer condition is satisfied with respect to the set of solutions S.

In recent paper [36] for solution of problem (3) the following algorithm was proposed

$$\begin{cases} y_n = \operatorname{prox}_{\lambda_n F(y_{n-1},\cdot)} x_n = \arg\min_{y\in C}\left(F\left(y_{n-1}, y\right) + \frac{1}{2\lambda_n} d^2\left(y, x_n\right)\right), \\ x_{n+1} = \operatorname{prox}_{\lambda_n F(y_n,\cdot)} x_n = \arg\min_{y\in C}\left(F\left(y_n, y\right) + \frac{1}{2\lambda_n} d^2\left(y, x_n\right)\right), \end{cases} \qquad (20)$$

where values $\lambda_n > 0$ were set according to the requirement $\left\{\inf_n \lambda_n, \sup_n \lambda_n\right\} \subseteq \left(0, \frac{1}{2(2a+b)}\right)$. I.e. the information about constants from condition (4) was used. Based on the scheme (20) and works [28, 29, 37], we construct a two-stage proximal algorithm with adaptive choice of the value λ_n.

Algorithm 2.

Initialization. Choose element x_1, $y_0 \in C$, $\tau \in \left(0, \frac{1}{3}\right)$, $\lambda_1 \in (0, +\infty)$. Set $n = 1$.

Step 1. Calculate $y_n = \operatorname{prox}_{\lambda_n F(y_{n-1},\cdot)} x_n = \arg\min_{y\in C}\left(F\left(y_{n-1}, y\right) + \frac{1}{2\lambda_n} d^2\left(y, x_n\right)\right)$.

Step 2. Calculate $x_{n+1} = \operatorname{prox}_{\lambda_n F(y_n,\cdot)} x_n = \arg\min_{y\in C}\left(F\left(y_n, y\right) + \frac{1}{2\lambda_n} d^2\left(y, x_n\right)\right)$.

If $x_{n+1} = x_n = y_n$, then stop and $x_n \in S$. Otherwise, go to step 3.

Step 3. Calculate

$$\lambda_{n+1} = \begin{cases} \lambda_n, & \text{if } \quad F(y_{n-1}, x_{n+1}) - F(y_{n-1}, y_n) - F(y_n, x_{n+1}) \le 0, \\ \min\left\{ \lambda_n, \dfrac{\tau}{2} \dfrac{d^2(y_{n-1}, y_n) + d^2(x_{n+1}, y_n)}{\left(F(y_{n-1}, x_{n+1}) - F(y_{n-1}, y_n) - F(y_n, x_{n+1})\right)} \right\}, & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and go to the step 1.

Let us present the main results on the convergence of the Algorithm 2.

Lemma 5. For sequences (x_n), (y_n), generated by Algorithm 2 the following inequality takes place

$$d^2(x_{n+1}, z) \le d^2(x_n, z) - \left(1 - \tau \frac{\lambda_n}{\lambda_{n+1}}\right) d^2(x_{n+1}, y_n) -$$

$$- \left(1 - 2\tau \frac{\lambda_n}{\lambda_{n+1}}\right) d^2(y_n, x_n) + 2\tau \frac{\lambda_n}{\lambda_{n+1}} d^2(x_n, y_{n-1}),$$

where $z \in S$.

Theorem 2. Let (X, d) be a Hadamard space, $C \subseteq X$ be a nonempty convex closed set, for bifunction $F : C \times C \to R$ conditions 1–5 are satisfied and $S \ne \varnothing$. Then sequences (x_n), (y_n) generated by Algorithm 2 converge weakly to the solution $z \in S$ of problem (3).

5. Regularized adaptive algorithms

To ensure the convergence of the approximating sequences in the metric of space to the solution of the equilibrium problem (3), we consider the extraproximal Algorithm 1, regularized using the well-known Halpern scheme [32, 38], with adaptive choice of the step size.

Algorithm 3.

Initialization. Choose elements $a \in C$, $x_1 \in C$, numbers $\tau \in (0,1)$, $\lambda_1 \in (0, +\infty)$, and sequence (α_n), such that $\alpha_n \in (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$. Set $n = 1$.

Step 1. Calculate $y_n = \operatorname{prox}_{\lambda_n F(x_n, \cdot)} x_n = \arg\min_{y \in C} \left(F(x_n, y) + \frac{1}{2\lambda_n} d^2(y, x_n) \right)$.

Step 2. Calculate $z_n = \operatorname{prox}_{\lambda_n F(y_n, \cdot)} x_n = \arg\min_{y \in C} \left(F(y_n, y) + \frac{1}{2\lambda_n} d^2(y, x_n) \right)$.

Step 3. Calculate $x_{n+1} = \alpha_n a \oplus (1 - \alpha_n) z_n$.

Step 4. Calculate

$$\lambda_{n+1} = \begin{cases} \lambda_n, & \text{if } \quad F(x_n, z_n) - F(x_n, y_n) - F(y_n, z_n) \le 0, \\ \min\left\{ \lambda_n, \dfrac{\tau}{2} \dfrac{d^2(x_n, y_n) + d^2(z_n, y_n)}{\left(F(x_n, z_n) - F(x_n, y_n) - F(y_n, z_n)\right)} \right\}, & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and go to step 1.

The following known facts have an important role in proving the convergence of Algorithm 3.

Lemma 6 ([41]). Let sequence of numbers (a_n) has subsequence (a_{n_k}) with property $a_{n_k} < a_{n_k+1}$ for all $k \in N$. Then exists non-decreasing sequence (m_k) of natural numbers such that $m_k \to +\infty$ and $a_{m_k} \le a_{m_k+1}$, $a_k \le a_{m_k+1}$ for all $k \ge n_1$.

Lemma 7. Let (a_n) be a sequence of non-negative numbers satisfying the inequality

$$a_{n+1} \le (1-\alpha_n) a_n + \alpha_n \beta_n \quad \text{for all } n \in N,$$

where sequences (α_n) and (β_n) have properties: $\alpha_n \in (0,1)$, $\sum \alpha_n = +\infty$, $\overline{\lim_{n\to\infty}} \beta_n \le 0$. Then $\lim_{n\to\infty} a_n = 0$.

First, takes place

Lemma 8. For sequences (x_n), (y_n) and (z_n) generated by Algorithm 3 inequality holds

$$\begin{aligned} &d^2(x_{n+1}, z) - (1-\alpha_n) d^2(x_n, z) + \\ &+(1-\alpha_n)\left(1-\tau\frac{\lambda_n}{\lambda_{n+1}}\right) d^2(z_n, y_n) + (1-\alpha_n)\left(1-\tau\frac{\lambda_n}{\lambda_{n+1}}\right) d^2(y_n, x_n) \le \\ &\le \alpha_n d^2(a,z) - \alpha_n(1-\alpha_n) d^2(a, z_n), \end{aligned} \tag{21}$$

where $z \in S$.

Proof. Let $z \in S$. From $x_{n+1} = \alpha_n a \oplus (1-\alpha_n) z_n$ and inequality for strong convexity (2) the estimation follows

$$d^2(x_{n+1}, z) \le \alpha_n d^2(a,z) + (1-\alpha_n) d^2(z_n, z) - \alpha_n(1-\alpha_n) d^2(a, z_n).$$

For upper estimation $d^2(z_n, z)$ we use Lemma 3 and get (21). ■

Lemma 9. Sequences (x_n), (y_n) and (z_n) generated by Algorithm 3 are bounded.

Proof. Let $z \in S$. We have

$$d(x_{n+1}, z) = d(\alpha_n a \oplus (1-\alpha_n) z_n, z) \le \alpha_n d(a,z) + (1-\alpha_n) d(z_n, z).$$

Since exists $\lim_{n\to\infty} \lambda_n > 0$, then

$$1-\tau\frac{\lambda_n}{\lambda_{n+1}} \to 1-\tau \in (0,1), \quad n \to \infty.$$

Using inequality from Lemma 3, we obtain

$$d(x_{n+1}, z) \le \alpha_n d(a,z) + (1-\alpha_n) d(x_n, z) \le \max\{d(a,z), d(x_n, z)\}$$

for all $n \ge n_0$. Hence $d(x_{n+1}, z) \le \max\{d(a,z), d(x_{n_0}, z)\}$ for all $n \ge n_0$. Thereby sequence (x_n) is bounded. So from Lemma 3 we conclude that (y_n) and (z_n) are bounded.

Theorem 3. Let (X,d) be a Hadamard space, $C \subseteq X$ be a nonempty convex closed set, for bifunction $F: C \times C \to R$ conditions 1–5 are satisfied and $S \neq \varnothing$. Then sequences (x_n), (y_n) and (z_n) generated by Algorithm 3 converge to the element $P_S a$.

Proof. Consider element $z_0 = P_S a$. From Lemma 9 it follows that exists number $M > 0$ such that $\left| d^2(a, z_0) - (1-\alpha_n) d^2(a, z_n) \right| \le M$ for all $n \in \mathbb{N}$. Then from inequality of Lemma 8 we obtain the estimation

$$d^2(x_{n+1}, z_0) - (1-\alpha_n) d^2(x_n, z_0) + (1-\alpha_n)\left(1 - \tau \frac{\lambda_n}{\lambda_{n+1}}\right) d^2(z_n, y_n) +$$

$$+ (1-\alpha_n)\left(1 - \tau \frac{\lambda_n}{\lambda_{n+1}}\right) d^2(y_n, x_n) \le \alpha_n M. \tag{22}$$

Consider sequence $\left(d(x_n, z_0)\right)$. There are two options: a) there exists a number $\bar{n} \in N$ such that $d(x_{n+1}, z_0) \le d(x_n, z_0)$ for all $n \ge \bar{n}$; b) there exists increasing sequence of numbers (n_k) such that $d\left(x_{n_k+1}, z_0\right) > d\left(x_{n_k}, z_0\right)$ for all $k \in N$.

First, consider option a). In that case there exists $\lim\limits_{n\to\infty} d(x_n, z_0) \in R$. Since

$$d^2(x_{n+1}, z_0) - d^2(x_n, z_0) \to 0,\ \alpha_n \to 0 \text{ and } 1 - \tau \frac{\lambda_n}{\lambda_{n+1}} \to 1 - \tau \in (0,1),\ n \to \infty,$$

we have

$$d(x_n, y_n) \to 0, \tag{23}$$

$$d(z_n, y_n) \to 0. \tag{24}$$

Since (x_n) is bounded it follows that exists a subsequence $\left(x_{n_k}\right)$ which converges weakly to the point $w \in X$. Then from (23), (24) it follows that $\left(y_{n_k}\right)$ and $\left(z_{n_k}\right)$ converge weakly to w. So $w \in C$. Let us show that $w \in S$. We have

$$F\left(y_{n_k}, y\right) \ge F\left(y_{n_k}, z_{n_k}\right) - \frac{1}{2\lambda_{n_k}}\left(d^2\left(y, x_{n_k}\right) - d^2\left(x_{n_k}, z_{n_k}\right) - d^2\left(z_{n_k}, y\right)\right) \ge$$

$$\ge F\left(x_{n_k}, z_{n_k}\right) - F\left(x_{n_k}, y_{n_k}\right) -$$

$$- \frac{\tau}{2\lambda_{n_k+1}}\left(d^2\left(x_{n_k}, y_{n_k}\right) + d^2\left(z_{n_k}, y_{n_k}\right)\right) - \frac{1}{2\lambda_{n_k}}\left(d^2\left(y, x_{n_k}\right) - d^2\left(x_{n_k}, z_{n_k}\right) - d^2\left(z_{n_k}, y\right)\right) \ge$$

$$\ge -\frac{1}{2\lambda_{n_k}}\left(d^2\left(z_{n_k}, x_{n_k}\right) - d^2\left(x_{n_k}, y_{n_k}\right) - d^2\left(y_{n_k}, z_{n_k}\right)\right) - \frac{\tau}{2\lambda_{n_k+1}}\left(d^2\left(x_{n_k}, y_{n_k}\right) + d^2\left(z_{n_k}, y_{n_k}\right)\right) -$$

$$- \frac{1}{2\lambda_{n_k}}\left(d^2\left(y, x_{n_k}\right) - d^2\left(x_{n_k}, z_{n_k}\right) - d^2\left(z_{n_k}, y\right)\right) \quad \forall y \in C. \tag{25}$$

Passing to the limit in (25) taking into account (23), (24) and weak upper semicontinuity of function $F(\cdot, y): C \to R$, we get

$$F(z,y) \ge \overline{\lim_{k\to\infty}} F\left(y_{n_k}, y\right) \ge 0 \quad \forall y \in C,$$

i.e., $z \in S$.

Let us prove that

$$\overline{\lim_{n\to\infty}}\left(d^2(a,z_0)-(1-\alpha_n)d^2(a,z_n)\right)\le 0. \tag{26}$$

Consider subsequence $\left(z_{n_k}\right)$ such that

$$\lim_{k\to\infty}\left(d^2(a,z_0)-\left(1-\alpha_{n_k}\right)d^2\left(a,z_{n_k}\right)\right)=\overline{\lim_{n\to\infty}}\left(d^2(a,z_0)-(1-\alpha_n)d^2(a,z_n)\right).$$

We can also assume that $z_{n_k} \to w \in S$ weakly. Then, using the weak lower semicontinuity of the function $d^2(a,\cdot)$, we obtain

$$\lim_{k\to\infty}\left(d^2(a,z_0)-\left(1-\alpha_{n_k}\right)d^2\left(a,z_{n_k}\right)\right)\le d^2(a,z_0)-d^2(a,w). \tag{27}$$

Since $z_0 = P_S a = \arg\min_{w\in S} d(a,w)$, then from (27) follows (26).

Then from (26), inequality

$$d^2(x_{n+1},z_0)\le(1-\alpha_n)d^2(x_n,z_0)+\alpha_n\left(d^2(a,z_0)-(1-\alpha_n)d^2(a,z_n)\right),$$

which takes place for big n and Lemma 7 we conclude that $d(x_n,z_0)\to 0$. From (23), (24) we get $d(y_n,z_0)\to 0$ and $d(z_n,z_0)\to 0$.

Let us study option b). In that case consider sequence of numbers (m_k) with properties (Lemma 6): i) $m_k \to +\infty$; ii) $d\left(x_{m_k+1},z_0\right)\ge d\left(x_{m_k},z_0\right) \quad \forall k\ge n_1$; iii) $d\left(x_{m_k+1},z_0\right)\ge d\left(x_k,z_0\right) \quad \forall k\ge n_1$.

From inequality of Lemma 8 and ii) it follows

$$\alpha_{m_k}d^2\left(x_{m_k},z_0\right)+\left(1-\alpha_{m_k}\right)\left(1-\tau\frac{\lambda_{m_k}}{\lambda_{m_k+1}}\right)d^2\left(z_{m_k},y_{m_k}\right)+$$

$$+\left(1-\alpha_{m_k}\right)\left(1-\tau\frac{\lambda_{m_k}}{\lambda_{m_k+1}}\right)d^2\left(y_{m_k},x_{m_k}\right)\le\alpha_{m_k}d^2(a,z_0)-\alpha_{m_k}\left(1-\alpha_{m_k}\right)d^2\left(a,z_{m_k}\right)\le\alpha_{m_k}M.$$

From where $\lim_{k\to\infty} d\left(x_{m_k},y_{m_k}\right)=\lim_{k\to\infty} d\left(z_{m_k},y_{m_k}\right)=0$. Arguments similar to the above, show that the partial sequences weak limits $\left(x_{m_k}\right)$, $\left(y_{m_k}\right)$ and $\left(z_{m_k}\right)$ belongs to set S. As before, we get

$$\overline{\lim_{k\to\infty}}\left(d^2(a,z_0)-\left(1-\alpha_{m_k}\right)d^2\left(a,z_{m_k}\right)\right)\le 0.$$

For big numbers k we have

$$d^2\left(x_{m_k+1},z_0\right)\le\left(1-\alpha_{m_k}\right)d^2\left(x_{m_k},z_0\right)+\alpha_{m_k}\left(d^2(a,z_0)-\left(1-\alpha_{m_k}\right)d^2\left(a,z_{m_k}\right)\right)\le$$

$$\le\left(1-\alpha_{m_k}\right)d^2\left(x_{m_k+1},z_0\right)+\alpha_{m_k}\left(d^2(a,z_0)-\left(1-\alpha_{m_k}\right)d^2\left(a,z_{m_k}\right)\right).$$

Whence, taking into account iii), we obtain

$$d^2(x_k, z_0) \le d^2(x_{m_k+1}, z_0) \le d^2(a, z_0) - (1-\alpha_{m_k}) d^2(a, z_{m_k}).$$

Thereby

$$\overline{\lim_{k\to\infty}} d^2(x_k, z_0) \le \overline{\lim_{k\to\infty}} \left(d^2(a, z_0) - (1-\alpha_{m_k}) d^2(a, z_{m_k}) \right) \le 0.$$

So, $\lim_{n\to\infty} d(x_n, z_0) = 0$ and $\lim_{n\to\infty} d(y_n, z_0) = \lim_{n\to\infty} d(z_n, z_0) = 0$. ■

Using this technique and idea of work [36] we can construct regularized variant of Algorithm 2 with adaptive step.

Algorithm 4.

Initialization. Choose elements x_1, $y_0 \in C$, $\tau \in (0, \frac{1}{3})$, $\lambda_1 \in (0, +\infty)$ and sequence (α_n) such that $\alpha_n \in (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$. Set $n = 1$.

Step 1. Calculate $z_n = \alpha_n a \oplus (1-\alpha_n) x_n$.

Step 2. Calculate $y_n = \operatorname{prox}_{\lambda_n F(y_{n-1}, \cdot)} z_n = \arg\min_{y\in C} \left(F(y_{n-1}, y) + \frac{1}{2\lambda_n} d^2(y, z_n) \right)$.

Step 3. Calculate $x_{n+1} = \operatorname{prox}_{\lambda_n F(y_n, \cdot)} z_n = \arg\min_{y\in C} \left(F(y_n, y) + \frac{1}{2\lambda_n} d^2(y, z_n) \right)$.

Step 4. Calculate

$$\lambda_{n+1} = \begin{cases} \lambda_n, & \text{if} \quad F(y_{n-1}, x_{n+1}) - F(y_{n-1}, y_n) - F(y_n, x_{n+1}) \le 0, \\ \min\left\{ \lambda_n, \frac{\tau}{2} \frac{d^2(y_{n-1}, y_n) + d^2(x_{n+1}, y_n)}{\left(F(y_{n-1}, x_{n+1}) - F(y_{n-1}, y_n) - F(y_n, x_{n+1}) \right)} \right\}, & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and go to step 1.

Remark 4. Unfortunately, now we do not have a proof of the convergence of Algorithm 4 under the condition that the bifunction is pseudomonotone.

6. Modification of Algorithm 3 for variational inequalities

Consider a particular case of the equilibrium problem: the variational inequality in the real Hilbert space H:

$$\text{find } x \in C: \ (Ax, y - x) \ge 0 \quad \forall y \in C. \tag{28}$$

We assume that following conditions are satisfied

- $C \subseteq H$ is convex and closed;
- operator $A: C \to H$ is pseudomonotone, Lipschitz continuous, and sequentially weakly continuous;
- the set of solutions (28) is not empty.

Let P_C be a metric projection operator on convex closed set C, i.e. $P_C x$ is an unique element of set C with property

$$\|P_C x - x\| = \min_{z\in C} \|z - x\|.$$

For variational inequalities (28) Algorithm 3 takes the following form.

Algorithm 5.

Initialization. Choose elements $a \in C$, $x_1 \in C$, numbers $\tau \in (0,1)$, $\lambda_1 \in (0,+\infty)$ and sequence (α_n), such that $\alpha_n \in (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$. Set $n = 1$.

Step 1. Calculate $y_n = P_C(x_n - \lambda_n A x_n)$.

Step 2. Calculate $z_n = P_C(x_n - \lambda_n A y_n)$.

Step 3. Calculate $x_{n+1} = \alpha_n a + (1-\alpha_n) z_n$.

Step 4. Calculate

$$\lambda_{n+1} = \begin{cases} \lambda_n, & \text{if } \quad (Ax_n - Ay_n, z_n - y_n) \le 0, \\ \min\left\{ \lambda_n, \dfrac{\tau}{2} \dfrac{\|x_n - y_n\|^2 + \|z_n - y_n\|^2}{(Ax_n - Ay_n, z_n - y_n)} \right\}, & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and go to step 1.

From theorem 3 the following result follows.

Theorem 4. Let H be a Hilbert space, $C \subseteq X$ be an nonempty convex closed set, operator $A : C \to H$ pseudomonotone, Lipschitz continuous, sequentially weakly continuous and there are solutions (28). Then the sequences generated by Algorithm 5 (x_n), (y_n) and (z_n) strongly converge to projection of element a on the set of solutions (28).

Remark 5. If operator A is monotone, then the result of Theorem 4 is valid without the assumption of the sequential weak continuity of the operator A. Similar results take place for modifications of algorithms 1, 2, and 4.

7. Conclusions

In this paper, which continues and refines articles [36, 37], two new adaptive two-stage proximal algorithms for the approximate solution of equilibrium problems in Hadamard spaces are described and studied. The proposed rules for choosing the step size do not calculate the values of the bifunction at additional points and do not require knowledge of the Lipschitz constants of the bifunction. For pseudo-monotone bifunctions of Lipschitz type, theorems on the weak convergence of sequences generated by the algorithms are proved. A new regularized adaptive extraproximal algorithm is also proposed and studied. To regularize the basic adaptive extraproximal scheme [37], the classical Halpern scheme [38] was used, a version of which for Hadamard spaces was studied in [32]. It is shown that the proposed algorithms are applicable to pseudomonotone variational inequalities in Hilbert spaces. In the coming papers, we plan to consider more special versions of algorithms for variational inequalities and minimax problems on Hadamard manifolds (for example, on the manifold of symmetric positive definite matrices). The construction of randomized versions of algorithms is also of interest.

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