On the Domination Number of t-Constrained de Bruijn Graphs
(Short Paper)*

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Abstract
Motivated by the work on the domination number of de Bruijn graphs and some of its generalizations, we introduce a natural generalization of de Bruijn graphs (directed and undirected), namely t-*constrained de Bruijn* graphs, where t is a positive integer, and then study the domination number of these graphs.

Within the definition of t-constrained de Bruijn graphs, de Bruijn and Kautz graphs correspond to 1-constrained and 2-constrained de Bruijn graphs, respectively. This generalization inherits many structural properties of de Bruijn graphs and may have similar applications in interconnection networks or bioinformatics.

We establish upper and lower bounds for the domination number on t-constrained de Bruijn graphs both in the directed and in the undirected case. These bounds are often very close and in some cases we are able to find the exact value.

Keywords
domination number, de Bruijn graph, Kautz graph

In graph theory, the study of domination and dominating sets plays a prominent role. This topic has been extensively studied for more than 30 years [1, 2] due to its applications in several areas, *e.g.* wireless networks [3], protein-protein interaction networks [4], social networks [5]. For a comprehensive treatment of domination and its variations, we refer to [2].

In an undirected graph, a vertex *dominates* itself and all its neighbors. The concept of domination can be naturally transferred to directed graphs, where a vertex dominates itself and all of its outgoing neighbors. A *dominating set* of a (direct or undirected) graph is a subset S of vertices such that every vertex in the graph is dominated by at least one vertex in S. The domination number of a graph G is the minimum cardinality of a dominating set of G, and is denoted by $\gamma(G)$. Finding a minimum dominating set for general graphs is widely known to be

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NP-hard [6] and hence it is a challenge to determine classes of graphs for which $\gamma(G)$ can be exactly computed. A closed formula for the domination number has been exactly determined only for few specific classes of graphs such as de Bruijn graphs [7], Kautz graphs [8], generalized Petersen graphs [9], Cartesian product of two directed paths [10] and graphs defined by two levels of the n-cube [11]. Furthermore, close bounds are provided for some generalizations of the previous classes [12, 13].

Here we focus on the well-known de Bruijn graphs which have various applications in different areas as for example bioinformatics [14, 15], interconnection networks [16] and peer-to-peer systems [17]. De Bruijn graphs are defined as follows:

Definition 1. *Given an alphabet Σ and a positive integer n, the* de Bruijn graph *of order n on Σ is defined as follows:*
- vertices are associated to all the sequences in Σ^n
- edges are the ordered pair of the form $\big((a_1, \dots, a_n), (a_2, \dots, a_n, a_{n+1})\big)$, where $a_1, \dots$ and a_{n+1} are in Σ.

In Fig. 2.a we show a de Bruijn graph of order 2 on $\Sigma = \{1, 2, 3\}$. Particular induced subgraphs of de Bruijn graphs have also been studied due to their applications related to both DNA assembly and high-performance or fault-tolerant computing [18]. These subgraphs can be defined by choosing particular subsets of vertices. For instance, the subgraph induced by the sequences of Σ^n that do not contain equal adjacent characters corresponds to the well-known Kautz graph [19] which has many properties that are desirable in computer networks [18, 20]. In Fig. 2.b we show a Kautz graph of order 2 on $\Sigma = \{1, 2, 3\}$.

The undirected version of de Bruijn and Kautz graphs can be easily obtained by simply ignoring the direction of the edges and removing loops and multiple edges (see Fig. 2.c as the undirected version of the Kautz graph in Fig. 2.b).

Here we propose a new natural generalization of the Kautz graphs, obtained by extending the constraint on the sequences labeling the vertices, from adjacent positions to an arbitrary distance t.

Definition 2. *Given a sequence $\mathbf{x} = (x_1, \dots, x_n) \in \Sigma^n$ we say that $\mathbf{x}$ is* t-constrained*, for some integer t, if for all $1 \leq i, j \leq n$, whenever $x_i = x_j$ it holds $|i - j| \geq t$.*

Note that every sequence in Σ^n is trivially 1-constrained while the sequences labeling Kautz graphs are 2-constrained. For any t-constrained $\mathbf{x} \in \Sigma^n$ with $|\Sigma| = \sigma$ it holds that $1 \leq t \leq \min\{\sigma, n\}$. We denote by $V(\sigma, t, n)$ the set of *all* t-constrained sequences from the set Σ^n. We now introduce t-constrained de Bruijn graphs.

Definition 3. *Given an alphabet Σ, with $|\Sigma| = \sigma$, and two positive integers n and t, with $1 \leq t \leq \min\{\sigma, n\}$, we define the* directed t-constrained de Bruijn graph of order n on Σ *as the subgraph of the directed de Bruijn graph of order n on Σ induced by the set $V(\sigma, t, n)$.*

We denote a t-constrained directed de Bruijn graph with $cDB(\sigma, t, n)$ and its undirected version with $cDB^u(\sigma, t, n)$. Clearly, directed de Bruijn graphs and directed Kautz graphs coincide with $cDB(\sigma, 1, n)$ and $cDB(\sigma, 2, n)$, respectively.

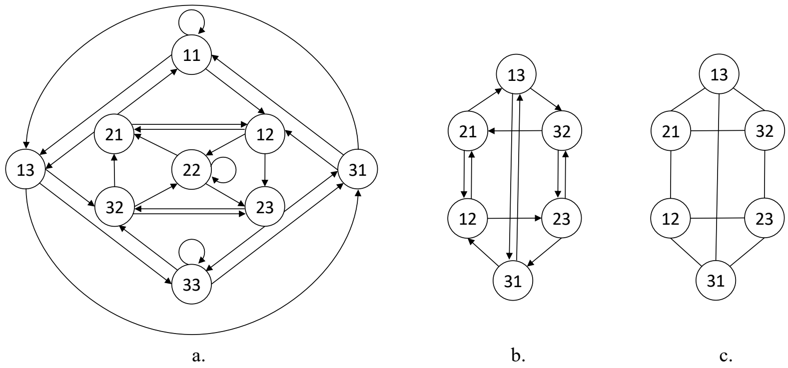

Figure 1: For $\Sigma = \{1, 2, 3\}$ we show: (a) the directed de Bruijn graph $cDB((,\sigma,,)1,2)$; (b) the directed Kautz graph $cDB(\sigma, 2, 2)$; (c) the undirected Kautz graph $cDB^u(\sigma, 2, 2)$.

In addition to their theoretical interest, t-constrained de Bruijn graphs may also find applications in interconnection networks due to their sparsity and connectivity. Moreover, in some cases, these graphs could be more suitable than de Bruijn graphs in modelling problems. For example, in genome rearrangement, which is an important part of bioinformatics research, permutations of integers (*i.e.* n-constrained sequences) are used to represent genomes. Thus, n-constrained de Bruijn graphs could find applications in genome assembly [21, 22]. More generally, t-constrained sequences for $t < n$ may be used to model genomes where duplication of genes is allowed only at a certain distance in the genome.

In the full version of this extended abstract we provide a systematic study of the domination number of t-constrained de Bruijn graphs in both the directed and the undirected case.

Although the domination numbers for de Bruijn and Kautz graphs have been exactly determined (see [7] and [8, 23]), the exact values in the undirected cases are still missing. We are able to provide close upper and lower bounds on the value of $\gamma(cDB^u(\sigma, 1, n))$ and $\gamma(cDB^u(\sigma, 2, n))$. Furthermore, in the particular case when the sequences labeling the vertices are permutations (*i.e.* $t = n$), we determine the exact value of the domination number in both the directed and undirected case.

We also consider the case where the sequences are partial n-permutations on the set of symbols when $|\Sigma| = \sigma = n + c$, for some integer c. In this case we are able to provide close upper and lower bounds for $\gamma(cDB^u(n + c, n, n))$ and $\gamma(cDB(n + c, n, n))$.

Finally, we are able to provide upper and lower bounds for the domination number of $cDB(\sigma, t, n)$. Concerning the value of $cDB^u(\sigma, t, n)$ it remains an open problem to find an upper bound that is asymptotically better than the one trivially derived by the directed case. The results announced in this extended abstract are summarized in Table 1 and Table 2 for the directed and the undirected cases, respectively.

All the proofs can be found in the full version of the paper [24].

We conclude this extended abstract with a comparison between the undirected and the directed cases.

A dominating set for a directed graph is *a fortiori* also a dominating set for its underlying undirected graph. Thus, $\gamma(cDB^u(\sigma, 1, n)) \leq \gamma(cDB(\sigma, 1, n))$. However, we improved this

Graph G	$\gamma(G)$
$cDB(\sigma,1,n)$	$\gamma(G) = \left\lceil \frac{\sigma^n}{d+1} \right\rceil$ [7]
$cDB(\sigma,2,n)$	$\gamma(G) = (\sigma-1)^{n-1}$ [8]
$cDB(\sigma,3,n)$	$\gamma(G) = \sigma(\sigma-2)^{n-2}$ if σ even $\sigma(\sigma-2)^{n-2} \leq \gamma(G) \leq \left(1+\Theta\left(\frac{1}{\sigma^2}\right)\right)\sigma(\sigma-2)^{n-2}$ if σ odd
$cDB(\sigma,t,n)$	$\frac{\sigma!}{(d-t)!}\frac{(\sigma-t+1)^{n-t}}{(\sigma-t+2)} \leq \gamma(G) \leq \left(1+\Theta\left(\frac{t}{\sigma(\sigma-t+1)}\right)\right)\frac{\sigma!}{(\sigma-t)!}\frac{(\sigma-t+1)^{n-t}}{(\sigma-t+2)}$
$cDB(n,n,n)$	$\gamma(G) = \left\lceil \frac{n}{2} \right\rceil (n-1)!$
$cDB(n+c,n,n)$	$\frac{1}{c+2}\frac{(n+c)!}{c!} \leq \gamma(G) \leq \left(1+\Theta\left(\frac{1}{c}\right)\right)\frac{1}{c+2}\frac{(n+c)!}{c!}$

Table 1
Summary of the results for t-constrained de Bruijn graphs. Results without a reference are proved in the full version of this extended abstract.

Graph G	$\gamma(G)$
$cDB^u(\sigma,1,2)$ $cDB^u(\sigma,1,3)$ $cDB^u(\sigma,1,n), n \geq 4$	$\gamma(G) = \sigma - 1$ $\gamma(G) = \sigma\left\lceil \frac{d}{2} \right\rceil$ $\frac{\sigma^n}{2\sigma+1} \leq \gamma(G) \leq \left(2-\Theta\left(\frac{1}{\sigma}\right)\right)\frac{\sigma^n}{2\sigma+1}$
$cDB^u(\sigma,2,2)$ $cDB^u(\sigma,2,3)$ $cDB^u(\sigma,2,n), n \geq 4$	$\gamma(G) = \sigma - 1$ $\frac{\sigma(\sigma-1)}{2}\gamma(G) \leq \left\lfloor \frac{\sigma^2}{2} \right\rfloor$ $\frac{\sigma(\sigma-1)^{n-1}}{2\sigma-1} \leq \gamma(G) \leq \left(2-\Theta\left(\frac{1}{\sigma}\right)\right)\frac{\sigma(\sigma-1)^{n-1}}{2\sigma-1}$
$cDB^u(\sigma,3,n)$	$\frac{\sigma(\sigma-1)(\sigma-2)^{n-2}}{2\sigma-3} \leq \gamma(G) \leq \left(2-\Theta\left(\frac{1}{\sigma}\right)\right)\frac{\sigma(\sigma-1)(\sigma-2)^{n-2}}{2\sigma-3}$
$cDB^u(n,n,n)$	$\gamma(G) = \left\lceil \frac{n}{3} \right\rceil (n-1)!$
$cDB^u(n+c,n,n)$	$\frac{1}{2c+3}\frac{(n+c)!}{c!} \leq \gamma(G) \leq \left(1+\Theta\left(\frac{1}{c}+\frac{1}{n}\right)\right)\frac{1}{2c+3}\frac{(n+c)!}{c!}$

Table 2
Summary of the results for undirected t-constrained de Bruijn graphs. Results without a reference are proved in the full version of this extended abstract.

bound. Namely, for $n = 2$ and $n = 3$, we deduce $\gamma(cDB^u(\sigma,1,2)) \leq \sigma - 1 + \frac{1}{\sigma+1}$ and $\gamma(cDB^u(\sigma,1,3)) \leq \sigma^2 - \sigma + 1$, respectively, that are worse than our results $\gamma(cDB^u(\sigma,1,2)) = \sigma - 1$ and $\gamma(cDB^u(\sigma,1,3)) = \sigma\left\lceil \frac{\sigma}{2} \right\rceil$, respectively. For $n \geq 4$ the upper bound provided for the directed case is $\left\lceil \frac{\sigma^n}{d+1} \right\rceil$ and the one from the undirected case is $\left\lceil \left(1 - \frac{1}{\sigma^2}\right)\frac{\sigma^n}{\sigma+1} \right\rceil$.

Regarding $t = 2$, the result proved in [8], gives us $\gamma(cDB^u(\sigma,2,n)) \leq \gamma(cDB(\sigma,2,n)) = (\sigma-1)^{n-1}$. Our results show that, for $n = 2$, $\gamma(cDB^u(\sigma,2,n)) = \gamma(cDB(\sigma,2,n)) = d - 1$. For $n \geq 3$ the upper bound of $(\sigma-1)^{n-1}$ derived from the directed case is improved to

$(\sigma-1)^{n-1}-(\sigma-2)(\sigma-1)^{n-4}$.

Since $\gamma(cDB^u(\sigma,3,n)) \leq \gamma(cDB(\sigma,3,n))$, for σ even we have an upper bound of $\sigma(\sigma-2)^{n-2}$ and in the undirected case we improve this result to $(\sigma-1)(\sigma-2)^{n-2}=\sigma(\sigma-2)^{n-2}-(\sigma-2)^{n-2}$. For σ odd, we have $(\sigma-1)^2(\sigma-2)^{n-3}$ in the directed case and the improvement to $(\sigma-1)(\sigma-2)^{n-2}=(\sigma-1)^2(\sigma-2)^{n-3}-(\sigma-1)(\sigma-2)^{n-3}$ in the undirected case.

Finally, we have $\gamma(cDB^u(n+c,n,n)) \leq \frac{(n+c-1)(n+c-1)!}{(c+1)!} = \frac{n+c-1}{(n+c)(c+1)}\frac{(n+c)!}{c!}$ which is worse that the undirected result when n goes to infinity. Furthermore, for small values of n the results show that, while for $n=2$, $\gamma(cDB^u(\sigma,2,n))=\gamma(cDB(\sigma,2,n))=\sigma-1$, for $n\geq 3$ the upper bound of $(\sigma-1)^{n-1}$ derived from the directed case can be already improved to $(\sigma-1)^{n-1}-(\sigma-2)(\sigma-1)^{n-4}$.

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