Partial formal contexts with degrees

Francisco Pérez-Gámez[1,*], Pablo Cordero[1], Manuel Enciso[1], Ángel Mora[1] and Manuel Ojeda-Aciego[1]

[1]University of Malaga, Bulevar Louis Pasteur 35, 29071 Málaga (Spain)

Abstract

Partial formal contexts are trivalued contexts that, besides allowing to establish whether a property is satisfied or not, allow to represent situations in which there is ignorance about whether a property is satisfied. This can be useful, not only for the cases in which the modeled phenomenon has intrinsically unknown information, but also when summarizing information from a formal context by grouping similar rows. In this paper, we prospect for its extension including degrees of knowledge.

Keywords

Implications, Unknown information, Formal concept analysis, Intuitionistic logic

1. Introduction

Trivalued logics are usually conceived as an extension of Classical Logic by adding an intermediate value to the set of Boolean truth-values $\{F, T\}$. This extra value enriches the expressive power of the logic and induces an order relation by considering an "intermediate" value to be located between the other two values. This is the case of Łukasiewicz logic [1] or Kleene logic [2]. In some sense, fuzzy logic also follows this idea, for it can be considered a generalisation which introduces a set of (infinitely many) values between the two Boolean truth values. The main difference between different 3-valued logics is the underlying meaning of the third value and, specifically the meaning of its negation.

The usual interpretation of the truth value F is "totally false", whereas T means "totally true". Then, the third truth value can be seen as an intermediate knowledge between these two situations.

We interpret T as "we do have information that shows it is true" and F as "we do have information showing that it is false". Then, the third truth-value can be interpreted as "unknown", i.e. we do not have any information either about whether it is true or it is false. For instance, in a health information system, the variable "being pregnant" matches this interpretation since the unknown value collects the situation where we don't have information at all (perhaps a test has not yet been carried out, or we do not have access to its result). An exhaustive review of this

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*Corresponding author.

franciscoperezgamez@uma.es (F. Pérez-Gámez); pcordero@uma.es (P. Cordero); enciso@uma.es (M. Enciso); amora@uma.es (Á. Mora); aciego@uma.es (M. Ojeda-Aciego)

0000-0001-7518-1828 (F. Pérez-Gámez); 0000-0002-5506-6467 (P. Cordero); 0000-0002-0531-4055 (M. Enciso); 0000-0003-4548-8030 (Á. Mora); 0000-0002-6064-6984 (M. Ojeda-Aciego)

topic can be found in [3]. In this paper we extend the research line in [4]. We present a graded extension of the algebraic framework presented. With this extension, we defined the graded partial formal contexts that are given by a triple (G, M, I) where G and M are the sets with the objects and the attributes (as usually) and $I : G \times M \to [0,1]^2$ expresses the degree to which we know that object g possesses attribute m and the degree to which we know that object g does not possess attribute m. Finally, Galois connections are given to capture the concepts of the graded formal context.

2. Preliminary definitions and results

2.1. An algebraic framework for unknown information

We consider the $\wedge$-semilattice $\underline{\mathbf{3}} = (\mathbf{3}, \leq)$ where $\mathbf{3} = \{+, -, \circ\}$ and $\leq$ is the reflexive closure of $\{(\circ, +), (\circ, -)\}$ (see Fig. 1a). Given a universe U, a $\mathbf{3}$-set in U is a mapping $A : U \to \mathbf{3}$ where, for each $u \in U$, $A(u)$ represents the knowledge about the membership of u to A. Thus, $+$ means that u belongs to A (we call it positive information), $-$ means that u does not belong to A (we call it negative information), and $\circ$ denotes the absence of information about the membership of u (which is called unknown information). As usual, the set of $\mathbf{3}$-sets on U inherits the $\wedge$-semilattice structure, denoted by $\underline{\mathbf{3}}^U = (\mathbf{3}^U, \sqsubseteq)$, by considering the componentwise ordering:

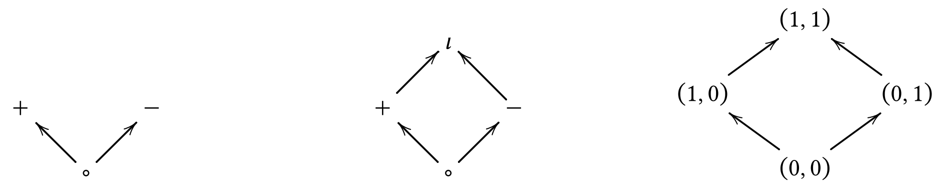
$$A \sqsubseteq B \text{ iff } A(u) \leq B(u) \text{ for all } u \in U.$$

Given a $\mathbf{3}$-set A, we call *support* of A to the set $\mathrm{Spp}(A) = \{u \in U \mid A(u) \neq \circ\}$. We define also the mappings $\mathrm{Neg}, \mathrm{Pos}, \mathrm{Unk} : \mathbf{3}^U \to \mathbf{2}^U$ where, for each $A \in \mathbf{3}^U$,

$$\mathrm{Neg}(A) = \{u \in U \mid A(u) = -\} = A^{-1}(-)$$
$$\mathrm{Pos}(A) = \{u \in U \mid A(u) = +\} = A^{-1}(+)$$
$$\mathrm{Unk}(A) = \{u \in U \mid A(u) = \circ\} = A^{-1}(\circ)$$

An equivalent formalisation of $\mathbf{3}$-sets can be found in [3] where only the known information (Positive or Negative) is given, and is represented as the so-called orthopairs by using sets (P, N) with $P \cap N = \varnothing$. When the support of a $\mathbf{3}$-set A is finite, we express it as a sequence of elements (with no delimiters). It can be seen as an ordered re-writing of the concatenation of the elements of the orthopair, where the atoms in N are overlined. For instance, $A = u_1\bar{u}_5u_7$ means that $A(u_1) = A(u_7) = +$, $A(u_5) = -$, and $A(x) = \circ$ otherwise. In particular, when $\mathrm{Spp}(A) = \varnothing$, we denote $A = \varepsilon$ (i.e. the empty sequence).

As mentioned above, we interpret a $\mathbf{3}$-valued set as the knowledge that we have about the properties of certain object. Thus, we have a conjunctive interpretation on these sets and, consequently, when we join two different $\mathbf{3}$-sets, we might find inconsistencies, that is, a property can be found positive in one of the sets and negative in the other set. Then, in the joined set, we have an inconsistent element. The sets $\mathbf{3}$ and $\mathbf{3}^U$ can be extended to model this situation by introducing a fourth element, denoted ι, which represents inconsistent or contradictory information. In order to obtain a lattice with the order of information, this new element will play the role of the (missing) maximum element of $\mathbf{3}$. This lattice of four elements



(a) ∧-semilattice $\underline{\mathbf{3}}$ (b) lattice $(\mathbf{4}, \leq)$ (c) Boolean Algebra $\underline{\mathbf{2}} \times \underline{\mathbf{2}}$

Figure 1: Different structures for truth values

is denoted $(\mathbf{4}, \leq)$ and is shown in Fig. 1b; it is isomorphic to the Boolean algebra $\underline{\mathbf{2}} \times \underline{\mathbf{2}}$ (see Fig. 1c). This algebraic structure is known as "information ordering" in the bilattice construction introduced by Belnap [5].

In addition, $\mathbf{4}^U$ denotes the set of mappings $A : U \to \mathbf{4}$ or $\mathbf{4}$-sets, and we assume that $\mathbf{3}^U$ is a subset of $\mathbf{4}^U$. In the same way that we did for $\mathbf{3}^U$, the order of $\mathbf{4}$ is componentwise extended to $\mathbf{4}^U$. Note that the infimum of $(\mathbf{4}^U, \leq)$ is ε (the constant mapping to $\circ$) and the supremum is the $\mathbf{4}$-set that maps any $u \in U$ to ι, which is called *oxymoron* and denoted by i.

The $\mathbf{4}^U$-sets can be seen as paraconsistent orthopairs [6] but, since in our framework *ex contradictione quodlibet* still holds, all the orthopairs containing a contradiction are identified and i is used as the canonical representative of this class. To formalise it, we define the following closure operator:

$$\mathcal{O} : \mathbf{4}^U \to \mathbf{4}^U \text{ being } \mathcal{O}(A) = \begin{cases} A & \text{if } A \in \mathbf{3}^U, \\ i & \text{otherwise.} \end{cases}$$

The codomain of $\mathcal{O}$ will be denoted $\dot{\mathbf{3}}^U$; it consists of the elements in $\mathbf{3}^U$, the *consistent* orthopairs, together with the oxymoron i. It is a closure system in $(\mathbf{4}^U, \leq)$ and, therefore, it is a ∧-subsemilattice of $(\mathbf{4}^U, \leq)$ (but not a sublattice) and $(\dot{\mathbf{3}}^U, \sqsubseteq)$ is a complete lattice (see Fig. 2). Since the infimum operation coincides in both structures, they will be denoted by the same symbol ∧, however the supremum in $(\mathbf{4}^U, \leq)$ will be denoted by ∨, whereas in $(\dot{\mathbf{3}}^U, \sqsubseteq)$ will be denoted by ⊔. Thus, for all $\{A_j : j \in J\} \subseteq \dot{\mathbf{3}}^U$, we have that

$$\bigsqcup_{j \in J} A_j = \mathcal{O}(\bigvee_{j \in J} A_j)$$

and, in particular,

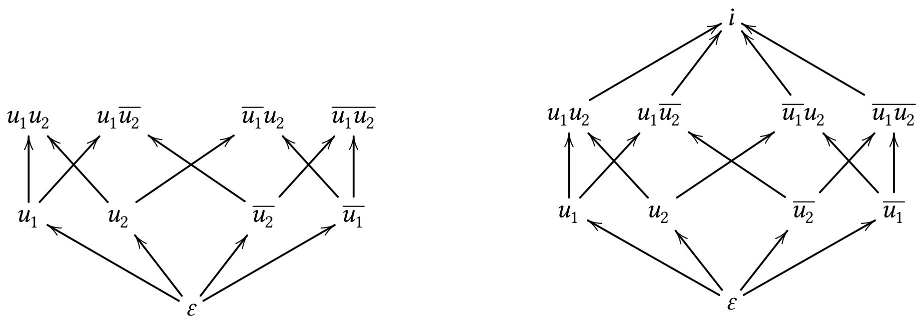
$$\bigsqcup_{j \in J} A_j \neq i \qquad \text{implies} \qquad \bigsqcup_{j \in J} A_j = \bigvee_{j \in J} A_j. \tag{1}$$

The maximal sets of $\underline{\mathbf{3}}^U$ are called *full sets*, and the set of all of them is denoted by $\mathcal{F}\text{ull}(U)$. They are the super-atoms of $(\dot{\mathbf{3}}^U, \sqsubseteq)$ and coincide with those orthopairs (P, N) such that $P \cup N = U$ and $P \cap N = \varnothing$.

We extend the mappings $\text{Neg}, \text{Pos}, \text{Unk} : \dot{\mathbf{3}}^U \to \mathbf{2}^U$ by considering $\text{Pos}(i) = \text{Neg}(i) = U$, $\text{Unk}(i) = \varnothing$.

Proposition 1 ([4]). *For any $A, B \in \dot{\mathbf{3}}^U$, we have that*

1. $A \sqsubseteq B$ *if and only if* $\text{Neg}(A) \subseteq \text{Neg}(B)$ *and* $\text{Pos}(A) \subseteq \text{Pos}(B)$.



(a) The $\wedge$-semilattice $\underline{\mathbf{3}}^{\{u_1,u_2\}}$ (b) The lattice $\dot{\underline{\mathbf{3}}}^{\{u_1,u_2\}}$

Figure 2: Lattices from the set $\{u_1, u_2\}$

2. $A \in \mathcal{F}\mathrm{ull}(U)$ *if and only if* $\mathrm{Unk}(A) = \varnothing$, *or equivalently* $\mathrm{Pos}(A) \cup \mathrm{Neg}(A) = U$ *and* $\mathrm{Pos}(A) \cap \mathrm{Neg}(A) = \varnothing$.
3. *The mappings* Pos *and* Neg *restricted to* $\mathcal{F}\mathrm{ull}(U)$ *are bijections in* $\mathbf{2}^U$.

Finally, we define the operation $\overline{(\,)}\colon\ \dot{\mathbf{3}}^U \to \dot{\mathbf{3}}^U$, which we call *opposite*, such that $\bar{i} = i$ and, for all $A \in \mathbf{3}^U$, and $u \in U$,

$$\overline{A}(u) = \begin{cases} + & \text{if } A(u) = - \\ - & \text{if } A(u) = + \\ \circ & \text{if } A(u) = \circ \end{cases}$$

Thus, $\mathrm{Neg}(\overline{A}) = \mathrm{Pos}(A)$, $\mathrm{Pos}(\overline{A}) = \mathrm{Neg}(A)$, and $\mathrm{Unk}(\overline{A}) = \mathrm{Unk}(A)$.

2.2. Formal concept analysis for unknown information

We start the extension of the FCA framework by presenting the notion of *partial formal context*, introduced by Ganter in [7], which is defined as a triple $\mathbb{P} = (G, M, I)$ being G and M non-empty sets and $I\colon\ G \times M \to \mathbf{3}$. We call the elements of G and M *objects* and *attributes* respectively. Given $g \in G$ and $m \in M$, the assignment $I(g, m) = +$ means that the object g has the attribute m ; $I(g, m) = -$ means that the object g has not the attribute m; and $I(g, m) = \circ$ means that we do not know whether the object g has the attribute m or not. As in the classical case, these contexts are shown as tables (see Figure 3, for instance).

$\mathbb{P}$	m_1	m_2	m_3
g_1	$+$	$\circ$	$-$
g_2	$\circ$	$\circ$	$+$
g_3	$-$	$-$	$\circ$

Figure 3: Partial formal context $\mathbb{P}$

A partial formal context $\mathbb{P} = (G, M, I)$ can induce the following (classical) formal contexts:

- $\mathbb{K}^+_{\mathbb{P}} = (G, M, I^+)$ where $I^+ = I^{-1}(+)$, that is gI^+m iff $I(g, m) = +$.

Their derivation operators are denoted by the symbol $\oplus$, that is, for all $X \subseteq G$ and $Y \subseteq M$

$$X^{\oplus} = \bigcap_{g \in X} gI^{+}(\ ) = \{m \in M \mid gI^{+}m, \forall g \in X\}$$
$$Y^{\oplus} = \bigcap_{m \in Y} (\ )I^{+}m = \{g \in G \mid gI^{+}m, \forall m \in Y\}$$

- $\mathbb{K}_{\mathbb{P}}^{-} = (G, M, I^{-})$ where $I^{-} = I^{-1}(-)$ and their derivation operators are denoted by the symbol $\ominus$ and defined in a similar way.

We use these formal contexts to define the derivation operators in the partial formal context as follows. The classical derivation operator is generalised by defining $(\ )^{\uparrow} : \mathbf{2}^{G} \to \dot{\mathbf{3}}^{M}$ and $(\ )^{\downarrow} : \dot{\mathbf{3}}^{M} \to \mathbf{2}^{G}$ as

$$X^{\uparrow} = \bigwedge_{g \in X} I(g,\ ), \qquad \text{and} \qquad Y^{\downarrow} = \mathrm{Pos}(Y)^{\oplus} \cap \mathrm{Neg}(Y)^{\ominus}$$

A pair $(A, B) \in \mathbf{2}^{G} \times \dot{\mathbf{3}}^{M}$ is said to be a *(formal) concept* if $A^{\uparrow} = B$ and $B^{\downarrow} = A$. As in the classical case, the concepts can be hierarchically ordered as

$$(A_1, B_1) \leq (A_2, B_2) \text{ if and only if } A_1 \subseteq A_2 \text{ (or, equivalently, iff } B_2 \sqsubseteq B_1).$$

Theorem 1 ([4]). *Let $\mathbb{P} = (G, M, I)$ be a partial formal context. The couple $(\uparrow, \downarrow)$ is a Galois connection between the lattices $(\mathbf{2}^{G}, \subseteq)$ and $(\dot{\mathbf{3}}^{M}, \sqsubseteq)$ and, therefore, concepts are fix-points of the Galois connection and, if $\mathfrak{B}_{\star}(\mathbb{P})$ is the set of all concepts, the pair $(\mathfrak{B}_{\star}(\mathbb{P}), \leq)$ is a complete lattice.*

In [4] we showed that a classical formal context can be seen as a partial formal context, and partial formal contexts without any unknown information were called total formal context. We also proved the existence of a classical formal context whose concept lattice is isomorphic to that of our partial formal context. Conversely, given a formal context, we proved the existence of a partial formal context whose concept lattice is isomorphic to that of the classic formal context.

For more details for the algebraic background, we refer to [8, 9], and for Formal Concept Analysis, we refer to [10, 11].

3. Graded extension of the algebraic framework

In this section, we extend the previous algebraic framework to a graded one following the idea of intuitionistic logic. Hereinafter, we consider a residuated lattice $\mathbf{I} = ([0,1], \wedge, \vee, \otimes, \to, 0, 1)$.

We generalise the lattice $\underline{\mathbf{4}}$ by considering the residuated lattice $\mathbf{I}^2 = \mathbf{I} \times \mathbf{I}$ where the ordering and the operations are componentwise extended. Thus, for a universal set U, the $\mathbf{4}$-sets are extended by using grades as $\mathbf{I}^2$-sets, i.e. mappings from U to $\mathbf{I}^2$. Equivalently, a $\mathbf{I}^2$-set $X : U \to \mathbf{I}^2$ can be seen as a graded paraconsistent orthopair (X^{+}, X^{-}) such that $X^{+}, X^{-} : U \to \mathbf{I}$ and $X(u) = (X^{+}(u), X^{-}(u))$ where $X^{+}(u)$ means *the degree in which we know that u belongs to X* and $X^{-}(u)$ means *the degree in which we know that u does not belong to X.*

The set of $\mathbf{I}^2$-sets will be denoted by $(\mathbf{I}^2)^{U}$. The operations can be extended from $\mathbf{I}^2$ to $(\mathbf{I}^2)^{U}$ as usual:

$$(X \star Y)(u) = X(u) \star Y(u) \text{ for all } u \in U \text{ and all } \star \in \{\wedge, \vee, \otimes, \rightarrow\}.$$

We also consider the order relation defined as

$$X_1 \sqsubseteq X_2 \text{ iff } X_1(u) \leq X_2(u) \text{ for all } u \in U,$$

or, equivalently,

$$X_1 \sqsubseteq X_2 \text{ iff } X_1^+(u) \leq X_2^+(u) \text{ and } X_1^-(u) \leq X_2^-(u) \text{ for all } u \in U.$$

It is easy to see that $\underline{(\mathbf{I}^2)^U} = ((\mathbf{I}^2)^U, \wedge, \vee, \otimes, \rightarrow, \varepsilon, i)$ is also a residuated lattice where ε and i are the $\mathbf{I}^2$-sets such that $\varepsilon(u) = (0, 0)$ and $i(u) = (1, 1)$ for all $u \in U$. The $\mathbf{I}^2$-set ε means that we have absolutely no knowledge about it, whereas the set i denotes total contradiction (we have knowledge that all elements belong to the set and, at the same time, do not belong to it).

This leads us to distinguish between consistent and inconsistent sets. We will say that a $\mathbf{I}^2$-set X on U is *consistent* if $X^+(u) \otimes X^-(u) = 0$ for all $u \in U$, i.e. its range is contained in $\mathbb{C} = \{(x_1, x_2) \in \mathbf{I}^2 : x_1 \otimes x_2 = 0\}$. Contrariwise, X is *inconsistent* if there exists $u \in U$ such that $X^+(u) \otimes X^-(u) > 0$.

The notion of consistent set is a generalization of the well-known Atanassov intuitionistic fuzzy set [12], which is a mapping $X: U \to [0, 1]^2$ such that, for all $u \in U$, if $X(u) = (\alpha, \beta)$ then $\alpha + \beta \leq 1$. It is the particular case in which the Łukasiewicz product is considered because $0 = \alpha \otimes \beta = \max\{0, \alpha + \beta - 1\}$ is equivalent to $\alpha + \beta \leq 1$.

Obviously, $\mathbb{C}$ is a $\wedge$-subsemilattice of $\underline{\mathbf{I}^2}$ that generalises $\underline{\mathbf{3}}$, in the same way that $\underline{\mathbf{I}^2}$ generalises $\underline{\mathbf{4}}$. The set of consistent sets on U, denoted $\mathbb{C}^U$, with the componentwise ordering is also a $\wedge$-semilattice that we will be denoted by $\underline{\mathbb{C}}^U$.

Following the same scheme as in the non-graded case, we consider equivalent all the inconsistent sets and we identify them with i. Thus, we define the closure operator

$$\mathcal{O}: (\mathbf{I}^2)^U \to (\mathbf{I}^2)^U \text{ where } \mathcal{O}(X) = \begin{cases} X & \text{if } X \in \mathbb{C}^U, \\ i & \text{otherwise.} \end{cases}$$

and denote by $\dot{\mathbb{C}}^U$ its range that is $\mathbb{C}^U \cup \{i\}$. Since this set is a closure system, we have that it is a $\wedge$-subsemilattice of $(\underline{\mathbf{I}^2})^U$, and also a complete lattice that we denote by $\underline{\dot{\mathbb{C}}}^U$. To distinguish the supremum of $(\underline{\mathbf{I}^2})^U$ from that of $\underline{\dot{\mathbb{C}}}^U$ we will use the symbols $\vee$ and $\sqcup$ respectively. Thus, given $\{X_j : j \in J\} \subseteq \dot{\mathbb{C}}^U$,

$$\bigsqcup_{j \in J} X_j = \mathcal{O}(\bigvee_{j \in J} X_j)$$

4. Graded partial formal contexts

We begin by extending the notion of partial formal context. A *graded partial formal context* is a triple $\mathbb{P} = (G, M, I)$ where G and M are sets whose elements are called *objects* and *attributes* respectively, and I is an $\mathbb{C}$-set on $G \times M$, i.e. $I: G \times M \to \mathbb{C}$. Thus, for each $(g, m) \in G \times M$, the degree $I^+(g, m)$ is those in which *we know that g has the attribute m* and $I^-(g, m)$ is the degree in which *we know that g does not have the attribute m*. As usual, eventually we use a currifying process and, given $g \in G$ and $m \in M$, we consider the $\mathbb{C}$-sets $I(g,.) \in \mathbb{C}^M$ and $I(.,m) \in \mathbb{C}^G$ defined as:

$$I(g,.)(x) = I(g,x) \text{ for all } x \in M; \quad I(.,m)(x) = I(x,m) \text{ for all } x \in G.$$

A first approach to extend the results given for partial formal contexts is to consider the following Galois connection.

Theorem 2. *Given a graded partial formal context* $\mathbb{P} = (G, M, I)$*, the derivation operators* $(\ )^{\uparrow} : \mathbf{2}^G \to \dot{\mathbb{C}}^M$ *and* $(\ )^{\downarrow} : \dot{\mathbb{C}}^M \to \mathbf{2}^G$ *defined as*

$$X^{\uparrow} = \bigwedge_{g \in X} I(g,\ ) \quad \textit{and} \quad Y^{\downarrow} = \{g \in G : Y \sqsubseteq I(g,\ )\}$$

form a Galois connection between the lattices $\underline{\mathbf{2}}^G$ *and* $\underline{\dot{\mathbb{C}}}^M$.

The concept lattice is defined in the usual way from the above Galois connection. Its minimum element is $(\varnothing, i)$ and, for any other formal concept (X, Y), if $Y(m) = (\alpha, \beta)$, the value α is a degree in which it is known that all the objects in X have the attribute m, whereas β is a degree in which it is known that all the objects in X have *not* the attribute m (rephrasing in more mathematical terms, α and β are, respectively, lower bounds of the degrees of each object of X having, respectively not having, the attribute m).

To have a more general framework, a second attempt could be done by considering fuzzy sets of objects. To do so, we must introduce some additional notation.

In a natural way, from a graded partial formal context $\mathbb{P} = (G, M, I)$, two fuzzy formal contexts, $\mathbb{K}^+_{\mathbb{P}} = (G, M, I^+)$ and $\mathbb{K}^-_{\mathbb{P}} = (G, M, I^-)$, can be induced. The derivation operators in $\mathbb{K}^+_{\mathbb{P}}$ and $\mathbb{K}^-_{\mathbb{P}}$ will be denoted by the symbols $\oplus$ and $\ominus$ respectively. Thus, given $X \in [0,1]^G$, the fuzzy sets $X^{\oplus}, X^{\ominus} \in [0,1]^M$ are defined as

$$X^{\oplus}(m) = \bigwedge_{g \in G} \big(X(g) \to I^+(g,m)\big) \text{ and } X^{\ominus}(m) = \bigwedge_{g \in G} (X(g) \to I^-(g,m))$$

and, given $Y \in [0,1]^M$, the fuzzy sets $Y^{\oplus}, Y^{\ominus} \in [0,1]^G$ are defined as

$$Y^{\oplus}(g) = \bigwedge_{m \in M} \big(Y(m) \to I^+(g,m)\big) \text{ and } Y^{\ominus}(g) = \bigwedge_{m \in M} (Y(m) \to I^-(g,m))$$

We use these formal contexts to define the derivation operators from a partial formal context as follows.

Theorem 3. *Given a graded partial formal context* $\mathbb{P} = (G, M, I)$*, the derivation operators* $(\ ) : [0,1]^G \to (\mathbf{I}^2)^M$ *and* $(\ ) : (\mathbf{I}^2)^M \to [0,1]^G$ *defined as*

$$X = (X^{\oplus}, X^{\ominus}) \quad \textit{and} \quad Y = (Y^+)^{\oplus} \cap (Y^-)^{\ominus}$$

form a Galois connection between the lattices $\underline{\mathbf{I}}^G$ *and* $\underline{\mathbf{I}^2}^M$.

From this theorem, which is the direct extension of Theorem 1, we can define formal concepts as fix pairs of the Galois connection and study the corresponding concept lattice. However, the main difference between this theorem and the previous ones is that X could be inconsistent.

5. Conclusions and further works

Partial formal contexts are useful to work with formal contexts that have been obtained via a granularisation process from a (classical) formal context, as described in [7]. This framework for the management of formal contexts in which there is missing or unknown information has been extended by considering graded knowledge about whether a property holds. Thus, we consider the degree in which is known that an object has certain property and, on the other hand, the degree in which it is known that this object does not have the property. We conclude with a pair of theorems that establish Galois connections and from which we can generalise the concept lattice and all the machinery of FCA.

A first approach to this extension is provided by Theorem 2 where the induced concepts, leaving aside the case of the minimum concept, are couples (X, Y) being X a classical set and Y a consistent $\mathbb{C}$-set of attributes.

The Galois connection established in Theorem 3 induces concepts (X, Y) where X is fuzzy set of objects and Y is a $\mathbf{I}^2$-set of attribute. Nevertheless, contrariwise to the previous cases (those obtained from Theorems 1 and 2), the $\mathbf{I}^2$-set Y could be inconsistent. As a consequence, a further refinement can be made to identify any concepts inconsistent with the pair $(i^{\downarrow}, i)$, in the same way that the closure operator $\mathcal{O}$ does. The problem of finding alternative Galois connection between the lattices $\underline{\mathbf{I}}^G$ and $\underline{\dot{\mathbb{C}}}^M$ that will allow to avoid the computation of inconsistent concepts is left as future work.

Further work in the short term is to situate the results obtained from Theorem 2 in the framework of the pattern structures introduced by Ganter and Kuznetsov [13], as well as the results arising from Theorem 3 with the works of Konecny [14] and Dubois *et al* [15].

The consistency issue is specially relevant when we work with attribute implications, where *ex contradictione quodlibet* principle must hold. In [4], not only the concept lattice of a partial formal context was introduced, but also the notion of attribute implication, and an axiomatic system proved to be sound and complete. Another interesting direction for future work will be to extend the previous study of implications to the framework defined from Theorem 2 by considering implications where premise and conclusion are $\dot{\mathbb{C}}$-sets of attributes.

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