A Galois connection between partial formal contexts and attribute sets

Francisco Pérez-Gámez[1,*], Pablo Cordero[1], Manuel Enciso[2] and Ángel Mora[1]

[1]Depto. de Matemática Aplicada, Universidad de Málaga, Spain
[2]Depto. de Lenguajes y Ciencias de la Computación, Universidad de Málaga, Spain

Abstract
This paper establishes an ordering between partial formal contexts, which are trivalued contexts. We use three values to represent the presence or absence of a certain property or its unknown value. We establish a Galois connection between this ordered set and the Boolean algebra of attribute sets. Finally, we discuss the interpretation of this Galois connection.

Keywords
Unknown information, Formal concept analysis, Concept lattice

1. Introduction

Formal Concept Analysis (FCA) [1] extracts knowledge from a binary relationship $\mathbb{K} = (G, M, I)$ where G is a set of objects, M is a set of attributes, and I is a relation between G and M (called incidence). See [2] for readers not habituated to FCA. The standard interpretation only considers the information in pairs $(g, m) \in I$ where g is an object and m is an attribute, named positive information.

The classical approach in FCA only considers the knowledge extracted using the positive information. If we take the view of the relationship as a table, the positive values are set by the crosses of the table. But, we emphasise that really, we are missing out a lot of information that can generate richer knowledge. What about the information provided by the table positions in where there is no cross (blank cells - negative information)? We can gain significant knowledge by taking them into account; for instance, if an attribute is "switch on" and we have a blank cell, then we can affirm that the object is "switch off". This point of view was taken into account in [3] and [4] among others. In [4] they duplicate the attributes to consider each attribute and its negation. Nevertheless, Missaoui et al. proved in [5] that this approach is inefficient because it does not assume that the negative and positive information is complementary and mutually exclusive, generating redundant information. In [6] were introduced the so-called

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*Corresponding author.
franciscoperezgamez@uma.es (F. Pérez-Gámez); pcordero@uma.es (P. Cordero); enciso@uma.es (M. Enciso); amora@uma.es (Á. Mora)
0000-0001-7518-1828 (F. Pérez-Gámez); 0000-0002-5506-6467 (P. Cordero); 0000-0002-0531-4055 (M. Enciso); 0000-0003-4548-8030 (Á. Mora)

mixed concepts to extend FCA with positive and negative. The authors presented a new Galois connection that takes advantage of the relationship between the positive and negative attributes.

There are some occasions when we can not work out what those blank cells mean; that is, the blanks are unknown information. There is quite a lot of recent work on processing unknown information through FCA [7, 8, 9]. In [7] a new value that means unknown information is added to obtain a 3-valued logic. This new value induces an order considering a "borderline" and is located between positive and negative values. This work interprets T as "we have information that it is true" and F as "we have information that it is false". Then, the third truth-value can be seen as "unknown", i.e. we do not know its truthfulness. For instance, in a book with examination marks, the variable "final exam" matches this interpretation since the unknown value can be when we do not have the information at all (maybe we have not done the exam or we have not corrected it yet).

In [10], the authors present another point of view where the unknown value appears. When we have a "large" amount of data in a formal context, and we would like to reduce the number of objects to handle it more efficiently, we can compact groups of objects in a single row by summarizing the information about them.

Considering the following, given an attribute in the original formal context: the new object will have the attribute if all the packed objects had the attribute; it will not have the attribute if none of the compacted objects had the attribute; and finally, in other cases, the relation between the new object and the attribute will be unkown. We want to merge the work [7] with this idea. We present a new Galois connection that allows building a concept lattice that can be considered to work with the notion of granularity.

2. Preliminaries

2.1. Trivalued formal contexts

We consider the $\wedge$-semilattice $\underline{\mathbf{3}} = (\mathbf{3}, \leq)$ where $\mathbf{3} = \{+, -, \circ\}$ and $\leq$ is the smallest order such that $\circ \leq +$ and $\circ \leq -$ (see Fig. 2a). Then, a $\mathbf{3}$-set on a universal set U is a mapping $A : U \to \mathbf{3}$ where, for each $u \in U$, $A(u)$ represents the knowledge about the membership of u to A: $A(u) = +$ means that it is known that u belongs to A (we call it positive information), $A(u) = -$ means that it is known that u does not belong to A (we call it negative information), and $A(u) = \circ$ denotes the absence of information about the membership of u (which is called unknown information). As usual, the algebraic structure of $\underline{\mathbf{3}}$ is powered to $\mathbf{3}^U$, the set of $\mathbf{3}$-sets on U, by considering the pointwise ordering:

$$A \sqsubseteq B \text{ iff } A(u) \leq B(u) \text{ for all } u \in U.$$

This new structure is denoted by $\underline{\mathbf{3}}^U = (\mathbf{3}^U, \sqsubseteq)$.

Given $A \in \mathbf{3}^U$, we call support of A to the set of the elements of A which we have information about, i.e. $\mathrm{Spp}(A) = \{u \in U\colon\ A(u) \neq \circ\}$. In addition, any $\mathbf{3}$-set A establishes a partition of the universal set (the quotient set) in three parts:

$$\mathrm{Pos}(A) = \{u \in U\colon\ A(u) = +\} = A^{-1}(+)$$
$$\mathrm{Neg}(A) = \{u \in U\colon\ A(u) = -\} = A^{-1}(-)$$

$\mathbb{P}$	a	b	c
1	$+$	$\circ$	$-$
2	$\circ$	$\circ$	$+$
3	$-$	$-$	$\circ$

Figure 1: Partial formal context $\mathbb{P}$

$$\text{Unk}(A) = \{u \in U\colon\ A(u) = \circ\} = A^{-1}(\circ)$$

Thus, two of these sets determine the third one and, therefore, we can consider that we are dealing with a pair of sets (Pos, Neg) with $Pos \cap Neg = \varnothing$. It corresponds to the equivalent formalisation provided by the so-called orthopairs [11].

When the support of a **3**-set A is finite, we express it as a sequence of elements (with no delimiters). It can be seen as an ordered re-writing of the concatenation of the elements of the orthopair, where the atoms in N are capped. For instance, $A = u_1\bar{u}_5u_7$ means that $A(u_1) = A(u_7) = +$, $A(u_5) = -$, and $A(x) = \circ$ in other case. In particular, when $\text{Spp}(A) = \varnothing$, we notate $A = \varepsilon$ (i.e. the empty chain).

It is not difficult to see that $\underline{\mathbf{3}}^U$ is a $\wedge$-semilattice (see, for instance, Figure 3a) whose maximal elements are those holding $\text{Unk}(A) = \varnothing$. They are called the full sets and coincide with those orthopairs (Pos, Neg) such that $Pos \cup Neg = U$. The set of full sets is denoted by $\mathfrak{F}\text{ull}(U)$.

Considering this three-valued structure, a *partial formal context* is defined as a triple $\mathbb{P} = (G, M, I)$ being G and M sets and I a **3**-set on $G \times M$. As usually, the elements of G are named *objects*, the elements of M *attributes*, and $I\colon\ G \times M \to \mathbf{3}$ *incidence relation.* For $g \in G$ and $m \in M$, the semantics of this incidence relation is the following: $I(g, m) = +$ means that the object g has the attribute m ; $I(g, m) = -$ means that the object g doesn't have the attribute m ; and $I(g, m) = \circ$ means that we do not know whether the object g has the attribute m or not. As in the classical case, these contexts are shown as tables and, given $g \in G$, we define the **3**-set $I(g,\ )$ by currying the mapping I, i.e. $I(g,\ ) \in \mathbf{3}^M$ is those **3**-set such that $I(g,\ )(m) = I(g, m)$ for each $m \in M$. For instance, for the partial formal context depicted in Figure 1, we have $I(1,\ ) = a\bar{c}$.

A partial formal context $\mathbb{P} = (G, M, I)$ such that $I \in \mathfrak{F}\text{ull}(G \times M)$ is named *total formal context.* On the other hand, following the classical interpretation, a (classical) formal context $\mathbb{K} = (G, M, I)$ is a partial formal context $\mathbb{P} = (G, M, I')$ where $I'(g, m) = +$ iff $g\ I\ m$, and $I'(g, m) = \circ$ otherwise.

2.2. Inconsistencies: extending the underlying structure

In the formalism introduced by Ganter et.al. [10] a group of rows are packed considering that an unkown value is introduced for an attribute, if in the source table for these rows, the attribute has a positive value in some rows and a negative one for others. This fact corresponds with a disjunctive interpretation of the attribute sets. However, in our framework proposed in [7], and given the interpretation of the concepts, we need a conjunctive interpretation of the attribute sets. Thus, for example, when obtaining the intent of a concept, we look for the attributes shared by "all" objects in the extent.

If we generalise this idea in the framework of partial contexts, we may find that, in a set of

(a) ∧-semilattice $\underline{\mathbf{3}}$ (b) lattice $(\mathbf{4}, \leq)$

Figure 2: Truthfulness's values

objects, some have a certain attribute and others do not, i.e. we may find inconsistencies (due to conjunctive interpretation). This takes us from a trivalued model to a tetravalued model.

Hence, we introduce a fourth element representing inconsistent or contradictory information. This new element, which is denoted ι and called *oxymoron*, will be the maximum completion of $\mathbf{3}$ to be a lattice. This lattice $(\mathbf{4}, \leq)$ is shown in Fig. 2b and corresponds with the so-called "information ordering" in the Belnap's lattice [12]. The posets $\mathbf{3}$ and $\mathbf{4}$ are respectively denoted by $\mathbf{1} \oplus \overline{\mathbf{2}}$ and $\mathbf{1} \oplus \overline{\mathbf{2}} \oplus \mathbf{1}$ in [13].

In the same way that we did for $\mathbf{3}^U$, we consider its extension $\mathbf{4}^U$, the set of mappings $A: U \to \mathbf{4}$ or $\mathbf{4}$-sets, and the pointwise order $\sqsubseteq$. Notice that $(\mathbf{4}^U, \sqsubseteq)$ is a lattice whose infimum is ε and supremum is the $\mathbf{4}$-set that maps any $u \in U$ to ι, which is named *oxymoron* and denoted by i. The $\mathbf{4}$-sets can be seen as paraconsistent orthopairs [14], where the condition $Pos \cap Neg = \varnothing$ is omitted.

However, as in the classical propositional logic, we semantically consider that, when any contradiction appears, we can derive anything. Thus, we identify all inconsistent orthopairs and use i as the representative element of this class. To formalise it, we define the following closure operator:

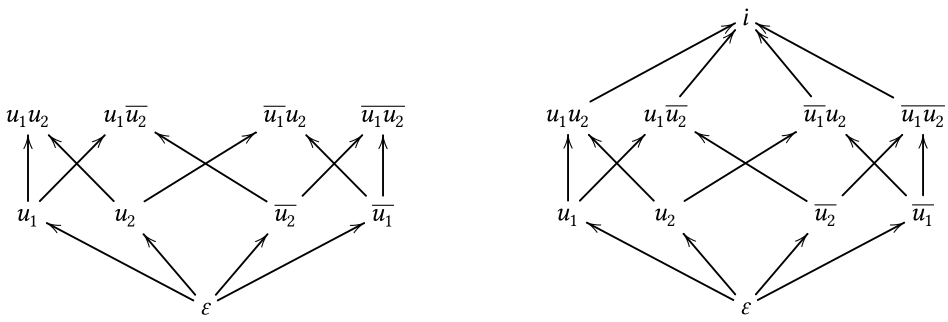
$$\mathcal{O}: \mathbf{4}^U \to \mathbf{4}^U \text{ being } \mathcal{O}(A) = \begin{cases} A & \text{if } A \in \mathbf{3}^U, \\ i & \text{otherwise.} \end{cases}$$

We denote by $\dot{\mathbf{3}}^U$ its codomain $\mathcal{O}(\mathbf{4}^U) = \mathbf{3}^U \cup \{i\}$, which is a closure system in $(\mathbf{4}^U, \sqsubseteq)$. Therefore, $(\dot{\mathbf{3}}^U, \sqsubseteq)$ is, on the one hand, a complete lattice (see Fig.3) and, on the other hand, a ∧-subsemilattice of $(\mathbf{4}^U, \sqsubseteq)$ (but not a sublattice). This lattice will be denoted by $\underline{\dot{\mathbf{3}}}^U$. Notice that the super-atoms in $\underline{\dot{\mathbf{3}}}^U$ are the full sets. Since both infima coincide, they will be denoted by the same symbol: ∧. However, the supremum in $(\mathbf{4}^U, \sqsubseteq)$ is denoted by the ∨, whereas in $(\dot{\mathbf{3}}^U, \sqsubseteq)$ is denoted by ⊔. Thus, for all $\{A_j : j \in J\} \subseteq \dot{\mathbf{3}}^U$, we have that

$$\bigsqcup_{j \in J} A_j = \mathcal{O}(\bigvee_{j \in J} A_j)$$

and, in particular,

$$\bigsqcup_{j \in J} A_j \neq i \qquad \text{implies} \qquad \bigsqcup_{j \in J} A_j = \bigvee_{j \in J} A_j. \tag{1}$$



(a) The $\wedge$-semilattice $\underline{\mathbf{3}}^{\{u_1,u_2\}}$

(b) The lattice $\dot{\underline{\mathbf{3}}}^{\{u_1,u_2\}}$

Figure 3: Lattices from the set $\{u_1, u_2\}$

3. The lattice of partial formal contexts

As mentioned in the introduction, we have studied partial formal context in [7] with the idea of highlighting the knowledge that can be inferred from the context although it could no longer be true when new information is available. In this paper, we focus on a new Galois Connection which allows us to work with all the possible universe for the partial formal context. That is, we are interested in the extraction of knowledge that is necessarily true in all possible configurations after learning more information. The worst way to do this is to complete the partial context with all possible extensions (see, for instance, Figure 4). Thus, given a partial formal context $\mathbb{P} = (G, M, I)$, we define its completion as the total formal context $\mathbb{K}_*(\mathbb{P}) = (G', M, I')$ where

$$G' = \{(g, X) \in G \times \mathbf{3}^M : \ \mathrm{Pos}(X) \cup \mathrm{Neg}(X) = \mathrm{Unk}(I(g,\ )\}$$

and $I'((g, X),\ ) = I(g,\ ) \sqcup X$ for all $(g, X) \in G'$. Finally, this total formal context can be analyzed and managed with the tools introduced in [15]. The main problem of this approach is that the growth of the size of $\mathbb{K}_*(\mathbb{P})$ with respect to the initial $\mathbb{P}$ is exponential. Specifically,

$$|G'| = \sum_{g \in G} 2^{|\mathrm{Unk}(I(g,\ ))|}$$

An important feature of FCA is that, although the concept lattice has an exponential size with respect to the context, concepts can be computed lazily with algorithms whose cost is "polynomial delay". In the following, we describe how to extend this idea to partial formal context by partially computing the concepts of $\mathbb{K}_*(\mathbb{P})$ in a lazy way without having to have previously calculated $\mathbb{K}_*(\mathbb{P})$. To do it, we introduce a lattice of partial contexts on which we will navigate in the search for concepts.

Given two partial formal contexts $\mathbb{P}_1 = (G_1, M_1, I_1)$ and $\mathbb{P}_2 = (G_2, M_2, I_2)$, we say that $\mathbb{P}_1$ is a refinement of $\mathbb{P}_2$ (denoted by $\mathbb{P}_1 \preceq \mathbb{P}_2$) if

$$G_1 \subseteq G_2,\ M_1 = M_2, \ \text{ and } I_2(g,\ ) \sqsubseteq I_1(g,\ ) \text{ for all } g \in G_1 \tag{2}$$

In the Figure 5, a chain of partial formal contexts is shown.

$\mathbb{P}$	a	b	c
1	$+$	$\circ$	$-$
2	$\circ$	$\circ$	$+$
3	$-$	$-$	$\circ$

(a) $\mathbb{K} = (G, M, I)$

$\mathbb{K}_*$	a	b	c
$1.b$	$+$	$+$	$-$
$1.\bar{b}$	$+$	$-$	$-$
$2.ab$	$+$	$+$	$+$
$2.\bar{a}b$	$-$	$+$	$+$
$2.a\bar{b}$	$+$	$-$	$+$
$2.\bar{a}\bar{b}$	$-$	$-$	$+$
$3.c$	$-$	$-$	$+$
$3.\bar{c}$	$-$	$-$	$-$

(b) $\mathbb{K}_*(\mathbb{P}) = (G', M, I')$

Figure 4: Completion of a partial formal context.

	a	b	c
1	$+$	$\circ$	$-$
2	$\circ$	$\circ$	$+$
3	$-$	$-$	$\circ$

$\succeq$

	a	b	c
1	$+$	$\circ$	$-$
2	$+$	$\circ$	$+$

$\succeq$

	a	b	c
1	$+$	$-$	$-$
2	$+$	$-$	$+$

$\succeq$

	a	b	c
1	$+$	$-$	$-$

Figure 5: A chain of partial formal contexts.

Theorem 1. *Let* $\mathbb{P}_0$ *be a partial formal context and* $\mathfrak{P}(\mathbb{P}_0) = \{\mathbb{P} : \mathbb{P} \preceq \mathbb{P}_0\}$*. Then* $\underline{\mathfrak{P}}(\mathbb{P}_0) = (\mathfrak{P}(\mathbb{P}_0), \preceq)$ *is a complete lattice.*

The infimum and the supremum in the complete lattice $\underline{\mathfrak{P}}(\mathbb{P}_0)$ are defined as follow:

- The infimum of $\{\mathbb{P}_j = (G_j, M, I_j) : j \in J\} \subseteq \mathfrak{P}(\mathbb{P}_0)$ is $\bigwedge_{j\in J} \mathbb{P}_j = (G, M, I)$ with

$$G = \Big\{g \in \bigcap_{j\in J} G_j : \bigsqcup_{j\in J} I_j(g,\ ) \neq i\Big\} \text{ and, for all } g \in G,\ I(g,\ ) = \bigsqcup_{j\in J} I_j(g,\ )$$

- The supremum of $\{\mathbb{P}_j = (G_j, M, I_j) : j \in J\} \subseteq \mathfrak{P}(\mathbb{P}_0)$ is $\bigvee_{j\in J} \mathbb{P}_j = (G, M, I)$ with

$$G = \bigcup_{j\in J} G_j \text{ and, for all } g \in G,\ I(g,\ ) = \bigsqcap_{j\in J_g} I_j(g,\ )$$

  being $J_g = \{j \in J : g \in G_j\}$.

In addition, the upper bound and the lower bound of $\underline{\mathfrak{P}}(\mathbb{P}_0)$ are $\mathbb{P}_0$ and $(\emptyset, M, \varepsilon)$ respectively.

4. A Galois connection between partial formal contexts and 3-sets of attributes

Now we present the Galois connection that will allow us to collect the formal concepts in a lazy way. Given a partial formal context $\mathbb{P}_0 = (G_0, M, I_0)$, we define two derivation operators as follows:

- $\Uparrow\,:\,\mathfrak{P}(\mathbb{P}_0) \to \dot{\mathbf{3}}^M$ that maps any $\mathbb{X} = (G, M, I) \in \mathfrak{P}(\mathbb{P}_0)$ to $\mathbb{X}^{\Uparrow} = \bigsqcap_{g \in G} I(g,\ )$.
- $\Downarrow\,:\,\dot{\mathbf{3}}^M \to \mathfrak{P}(\mathbb{P}_0)$ that maps any $\dot{\mathbf{3}}$-set $A \in \dot{\mathbf{3}}^M$ to $A^{\Downarrow} = (G, M, I)$ where

$$G = \{g \in G_0 :\ I_0(g,\ ) \sqcup A \neq i\} \text{ and } I(g,\ ) = I_0(g,\ ) \sqcup A,$$

  for each $g \in G$.

Example 1. Given the following partial formal context $\mathbb{P}_0$ and $\mathbb{X}_1, \mathbb{X}_2 \in \mathfrak{P}(\mathbb{P}_0)$

$\mathbb{P}_0$	a	b	c
1	$+$	$\circ$	$-$
2	$-$	$+$	$\circ$

$\mathbb{X}_1$	a	b	c
1	$+$	$+$	$-$
2	$-$	$+$	$\circ$

$\mathbb{X}_2$	a	b	c
1	$+$	$\circ$	$-$

we have ${\mathbb{X}_1}^{\Uparrow} = b$ and $a\bar{c}^{\Downarrow} = \mathbb{X}_2$.

Theorem 2. *The pair* $(\Uparrow, \Downarrow)$ *is a Galois connection between* $\underline{\mathfrak{P}}(\mathbb{P}_0)$ *and* $\underline{\dot{\mathbf{3}}}^M$.

Proof. First, assume $\mathbb{X}_1 = (G_1, M, I_1) \preceq \mathbb{X}_2 = (G_2, M, I_2)$, i.e. $G_1 \subseteq G_2$ and $I_2(g,\ ) \sqsubseteq I_1(g,\ )$ for all $g \in G_1$. Then

$${\mathbb{X}_2}^{\Uparrow} = \bigsqcap_{g \in G_2} I_2(g,\ ) \sqsubseteq \bigsqcap_{g \in G_1} I_2(g,\ ) \sqsubseteq \bigsqcap_{g \in G_1} I_1(g,\ ) = {\mathbb{X}_1}^{\Uparrow}$$

and, therefore, $\Uparrow$ is an antitone mapping.

Let's prove that $\Downarrow$ is also antitone. Assume that $A_1, A_2 \in \dot{\mathbf{3}}^M$ satifiy $A_1 \sqsubseteq A_2$, and let ${A_1}^{\Downarrow} = (G_1, M, I_1)$ and ${A_2}^{\Downarrow} = (G_2, M, I_2)$. On the one hand, since $A_1 \sqsubseteq A_2$, we straightforwardly have that

$$G_2 = \{g \in G :\ I_0(g,\ ) \sqcup A_2 \neq i\} \subseteq \{g \in G :\ I_0(g,\ ) \sqcup A_1 \neq i\} = G_1.$$

On the other hand, for all $g \in G_2 \subseteq G_1$, we have that

$$I_1(g,\ ) = I_0(g,\ ) \sqcup A_1 \sqsubseteq I_0(g,\ ) \sqcup A_2 = I_2(g,\ ).$$

Therefore, ${A_2}^{\Downarrow} \preceq {A_1}^{\Downarrow}$.

Now we prove that $\mathbb{X}_1 \preceq {\mathbb{X}_1}^{\Uparrow\Downarrow}$ for all $\mathbb{X}_1 \in \mathfrak{P}(\mathbb{P}_0)$. If $\mathbb{X}_1 = (G_1, M, I_1)$ and ${\mathbb{X}_1}^{\Uparrow\Downarrow} = \mathbb{X}_2 = (G_2, M, I_2)$, we have that

$$G_2 = \left\{ g \in G_0 :\ I_0(g,\ ) \sqcup \bigsqcap_{g_1 \in G_1} I_1(g_1,\ ) \neq i \right\}.$$

Then, for all $g \in G_1$, we have

$$I_0(g,\ ) \sqcup \bigsqcap_{g_1 \in G_1} I_1(g_1,\ ) \sqsubseteq I_0(g,\ ) \sqcup I_1(g,\ ) = I_1(g,\ )$$

and, therefore $g \in G_2$ and $I_2(g,\ ) \sqsubseteq I_1(g,\ )$.

Finally, let's prove that $A \sqsubseteq A^{\Downarrow\Uparrow}$ for all $A \in \dot{\mathbf{3}}^M$. Since $A^{\Uparrow} = (G_1, M, I_1)$ where $G_1 = \{g \in G :\ I_0(g,\ ) \sqcup A \neq i\}$ and $I_1(g,\ ) = I_0(g,\ ) \sqcup A$ for each $g \in G_1$, we have that $A \sqsubseteq \bigsqcap_{g \in G_1} I_1(g,\ ) = A^{\Downarrow\Uparrow}$. $\square$ $\square$

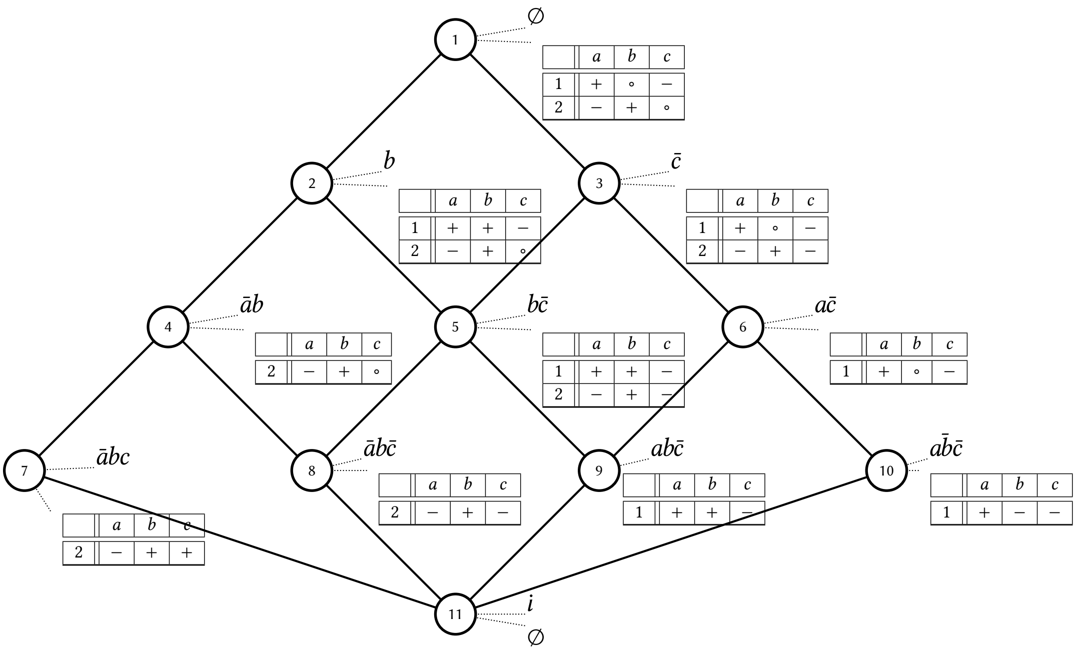


Figure 6: Lattice $\underline{\mathfrak{S}}(\mathbb{P})$

As a consequence, we have that both compositions of these maps are closure operators and their fixed points provide dually isomorphic lattices.

Corollary 1. *Given a partial formal context* $\mathbb{P}_0 = (G_0, M, I_0)$*, the set*

$$\mathfrak{S}(\mathbb{P}_0) = \{(\mathbb{X}, Y) \in \mathfrak{P}(\mathbb{P}_0) \times \dot{\mathbf{3}}^M : \ \mathbb{X}^{\Uparrow} = Y \ \textit{and} \ Y^{\Downarrow} = \mathbb{X}\}$$

with the order

$$(\mathbb{X}_1, Y_1) \preceq (\mathbb{X}_2, Y_2) \ \textit{iff} \ \mathbb{X}_1 \preceq \mathbb{X}_2 \ (\textit{or equivalently,} \ \textit{iff} \ Y_2 \sqsubseteq Y_1)$$

form a complete lattice denoted by $\underline{\mathfrak{S}}(\mathbb{P}_0)$*.*

The couples $(\mathbb{X}, Y) \in \mathfrak{S}(\mathbb{P}_0)$ are named *formal concept* on $\mathbb{P}_0$, and its components $\mathbb{X}$ and Y are named *extent* and *intent* of the concept, respectively.

Example 2. In Figure 6 we present the lattice $\underline{\mathfrak{S}}(\mathbb{P}_0)$ obtained from the following partial formal context $\mathbb{P}_0$

$\mathbb{P}_0$	a	b	c
1	+	∘	−
2	−	+	∘

Given a partial fomral context $\mathbb{P} = (G, M, I)$, the set of atoms of $\underline{\mathfrak{S}}(\mathbb{P})$ is $\{(A^{\Downarrow}, A) : \ A \in \mathscr{M}(\mathbb{P})\}$ where

$$\mathscr{M}(\mathbb{P}) = \{A \in \mathfrak{F}\text{ull}(M) : \ I(g, \ ) \sqsubseteq A \text{ for some } g \in G\}$$

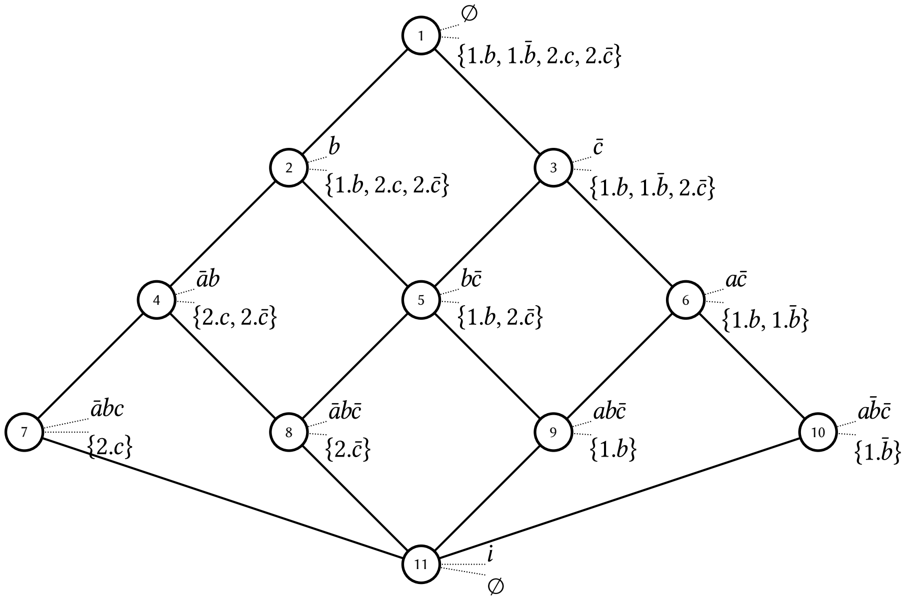


Figure 7: The mixed concept lattice defined by $\mathbb{K}_*(\mathbb{P}_0)$

In addition, if the completion of $\mathbb{P}$ is $\mathbb{K}_*(\mathbb{P}) = (G', M, I')$ then

$$\mathscr{M}(\mathbb{P}) = \{I'((g, X), \ ) : \ (g, X) \in G'\}$$

and the lattice $\underline{\mathfrak{S}}(\mathbb{P})$ is isomorphic to the mixed concept lattice obtained from $\mathbb{K}_*(\mathbb{P})$, which was defined in [6].

Example 3. For the partial formal context $\mathbb{P}_0$ defined in Example 2 provided in Section 4, the atoms of the lattice $\underline{\mathfrak{S}}(\mathbb{P}_0)$ are $\mathscr{M}(\mathbb{P}) = \{\bar{a}bc, \bar{a}b\bar{c}, ab\bar{c}, a\bar{b}\bar{c}\}$ (see Fig. 6) and, from the completion of $\mathbb{P}_0$,

$\mathbb{K}_*(\mathbb{P}_0)$	a	b	c
$1.b$	$+$	$+$	$-$
$1.\bar{b}$	$+$	$-$	$-$
$2.c$	$-$	$+$	$+$
$2.\bar{c}$	$-$	$+$	$-$

we obtain the mixed concept lattice depicted in Figure 7.

5. Conclusions and future works

In this work, we have presented a Galois connection that allows us to combine the work presented in [7] with the idea of granularity that appears in [10]. We present a concept lattice that can be used to explore at the granules. We present an order with the Partial formal contexts that can be seen as the order having less granularity or more unknown information. We want to discuss the different uses that this lattice can contribute to work with unknown information and granularity.

In the future, we want to extend this work by presenting implications that are fulfilled in any possible universe, that is, in any of the completions of the partial formal context or, which is equivalent in granularity terms when we explore any grains obtaining the missing information.

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