An Error Performance Bound for Block Compressed Sensing

Xiuling Li, Bo Zhang, Lin Jiang*, Chengze Li

Communication NCO Academy Army Engineering University Chongqing 40035, China

Abstract

The permutation-based block compressed sensing (CS) scheme is a new CS-based image compression method, in which permutation strategies are used prior to sampling with purpose of balancing the sparsity levels among the blocks. Although it has been shown to be an efficient method to improve sampling efficiency, there remain several fundamental questions on both the theoretical and practical side of this scheme. This paper primarily concerns about one of these theoretical issues revolving around the error performance of block CS (BCS). In this paper, we analyze the error performance bound of BCS and a new error performance bound is established. It is revealed that better recovery quality can be achieved if the permutated 2D signal has smaller maximum block sparsity level.

Keywords

Block compressed sensing; matrix permutation; image compression; performance bound.

1. Introduction

Compressed sensing (CS) has received a lot of attention recently [1], [2], which is capable of efficiently capturing and recovering a signal through a few of linear measurements. Recently, image coding by using CS has become a hot topic in image processing field [3-10]. In order to use CS to encode 2D images, block CS (BCS) is developed for fast implementation [11-18], where the image is sampled block-by-block. However, the compression performance of traditional BCS scheme is poor.

To solve this problem, permutation-based BCS [11-18] schemes are proposed. In permutation-based BCS schemes, the wavelet coefficient matrix is scrambled by using some permutation strategies and then the permutated wavelet coefficient matrix is sampled by using BCS. A good permutation scheme can balance the nonzero elements among the blocks, thereby improving sampling efficiency. In practice, a lot of permutation strategies have been proposed in existing literatures [11-18].

Although the permutation-based BCS scheme has been shown to be an efficient method to improve sampling efficiency in practice applications, there remain several fundamental questions on both the theoretical and practical side of this scheme. This paper primarily concerns about one of these theoretical issues revolving around the error performance bound of block-based CS. In this paper, we analyze the error performance bound of block-based CS. It is revealed that better reconstruction performance can be achieved if the permutated 2D signal has smaller maximum block sparsity level. As far as we know, a similar theoretical work has been studied in [15]. But the theoretical result of [15] just proves that the CS reconstruction error is bound by the best k -term approximation error. In this paper, our further research shows that the best k -term approximation error can be bounded by an explicit function of k . Then, a new error performance of BCS for 2D signals is established.

2. PRELIMINARIES

2.1. Compressed sensing

The set of all K -sparse vectors can be denoted by

$$\Sigma_K = \{\boldsymbol{x} \in R^{N\times 1} \mid \|\boldsymbol{x}\|_0 \le K\} . \tag{1}$$

ICCEIC2022@3rd International Conference on Computer Engineering and Intelligent Control
EMAIL: Jianglin4901@qq.com (Lin Jiang), zhangboswjtu@163.com (Bo Zhang)

where $\|\cdot\|_0$ denoted l_0-norm.

For a K-sparse signal $\boldsymbol{x}$, the measurement vector can be obtained by

$$\boldsymbol{y} = \boldsymbol{\Phi x}. \tag{2}$$

where $\boldsymbol{\Phi} \in R^{M\times N}(M \ll N)$ is a measurement matrix and $\boldsymbol{y} \in R^{M\times 1}$ is the measurement vector of $\boldsymbol{x}$.

According to CS theory, $\boldsymbol{x}$ can be recovered from $\boldsymbol{y}$ if $\boldsymbol{\Phi} \in R^{M\times N}$ satisfies restricted isometry property (RIP) [1, 2]. $\boldsymbol{x}$ can be recovered by solving

$$\hat{\boldsymbol{x}} = \arg\min \|\boldsymbol{x}\|_1 \ \text{s.t.}\ \boldsymbol{y}=\boldsymbol{\Phi x}. \tag{3}$$

2.2. Block compressed sensing

For an image $\boldsymbol{D} \in R^{\sqrt{N}\times\sqrt{N}}$, it can be represented by

$$\boldsymbol{X} = \boldsymbol{\Psi D \Psi}^{\mathrm{T}}, \tag{4}$$

where $\boldsymbol{\Psi} \in R^{\sqrt{N}\times\sqrt{N}}$ is a wavelet basis and $\boldsymbol{X} \in R^{\sqrt{N}\times\sqrt{N}}$ is the wavelet coefficient matrix.

In block-based CS scheme, $\boldsymbol{X}$ is split into a lot of blocks with dimension of $\sqrt{n}\times\sqrt{n}$. It can be represented by

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 & \boldsymbol{X}_2 & \cdots & \boldsymbol{X}_{\sqrt{L}} \\ \boldsymbol{X}_{\sqrt{L}+1} & \boldsymbol{X}_{\sqrt{L}+2} & \cdots & \boldsymbol{X}_{2\sqrt{L}} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{X}_{L-\sqrt{L}+1} & \boldsymbol{X}_{L-\sqrt{L}+2} & \cdots & \boldsymbol{X}_L \end{bmatrix}, \tag{5}$$

where $\boldsymbol{X}_i$ is i-th block of $\boldsymbol{X}$ and $L = N/n$.

Let $\boldsymbol{x}_i \in R^{n\times n}$ be the vectorized signal of $\boldsymbol{X}_i$, then we can be obtain the measurement vector by

$$\boldsymbol{y}_i = \boldsymbol{\Phi}_B \boldsymbol{x}_i, \tag{6}$$

where $\boldsymbol{\Phi}_B \in R^{m\times n}$ is a measurement matrix.

Let $K_i = \|\boldsymbol{x}_i\|_0$, then, the block sparsity level vector of $\boldsymbol{X}$ can be denoted by $\boldsymbol{K} = [K_1, K_2, ..., K_L]$. Let $K_{\max} = \|\boldsymbol{K}\|_\infty$ be the maximum block sparsity level. If $\boldsymbol{\Phi}_B$ satisfies RIP with order $K_{\max}$, we can reconstruct all blocks of $\boldsymbol{X}$, and then reconstruct the original image $\boldsymbol{D}$.

3. A NEW ERROR PERFORMANCE

3.1. Error performance bound of CS for 1D signals

For a signal $\boldsymbol{s} \in R^n$, the best k-term approximation is defined by

$$\sigma_k(\boldsymbol{s})_1 := \min_{z\in\Sigma_k} \|\boldsymbol{s} - \boldsymbol{z}\|_1. \tag{7}$$

For compressible signals, the CS reconstruction via solving (3) is nearly as good as that using the best k-term approximation of $\boldsymbol{s}$.

Lemma 1 [19]: Suppose that $\boldsymbol{\Phi}_B$ is a measurement matrix obeying RIP with order $2k$, and $\delta_{2k} \le \sqrt{2} - 1$. Then, for a signal $\boldsymbol{s}$, the solution $\hat{s}$ to (3) obeys

$$\|\boldsymbol{s} - \hat{s}\|_2 \le c_0 \frac{\sigma_k(\boldsymbol{s})_1}{\sqrt{k}}. \tag{8}$$

for some constant c_0.

Obviously, the above lemma can also be generalized to 2D signals, a representative result can be found in [15]. Lemma 1 and its generalized result for 2D signals show the fact that the CS reconstruction error via solving (3) is bound by the best k-term approximation error. But, what is the bound for the best k -term approximation error? Whether the best k -term approximation error can be bound by an explicit function of k ? Since if this kind of boundary exists, the CS reconstruction error for 2D signals can be explicitly bound by a function of the maximum block sparsity level. In next section, this kind of boundary will be established firstly, and then a new error performance of BCS for 2D signals is proposed.

3.2. A new error performance bound for 2D signals

In this section, we carefully research the reconstruction error bound of BCS for 2D signals. Before we give the main theorem of this paper, we provide two lemmas that will be used in the proof.

Lemma 2. Suppose that $\boldsymbol{s}\in R^n$ is a k -sparse signal, and $\bar{k}$ is a positive integer, then we have

$$\sigma_{\bar{k}}(\boldsymbol{s})_1 := \min_{z\in\Sigma_{\bar{k}}} \|\boldsymbol{s}-\boldsymbol{z}\|_1 \le \frac{1}{k}\left|k-\bar{k}\right|\cdot\|\boldsymbol{s}\|_1 . \tag{9}$$

Proof: In order to prove Lemma 2, we will require the following lemma.

Lemma 2.1. Suppose that $\boldsymbol{u}\in R^n$ is a signal whose entries are arranged in amplitude descending order, i.e., $|\boldsymbol{u}(1)|\ge|\boldsymbol{u}(2)|\ge\cdots\ge|\boldsymbol{u}(n)|$. When $l\ge 2$, we have

$$\|\boldsymbol{u}\|_1 \ge \frac{n}{n-l+1}\sum_{i=l}^{n}|\boldsymbol{u}(i)| . \tag{10}$$

Proof of Lemma 2.1: When l=2, the above inequality can be rewritten as

$$\|\boldsymbol{u}\|_1 \ge \frac{n}{(n-1)}\sum_{i=2}^{n}|\boldsymbol{u}(i)| . \tag{11}$$

Obviously, the inequality (11) is equivalent to

$$(n-1)\|\boldsymbol{u}\|_1 - n\sum_{i=2}^{n}|\boldsymbol{u}(i)| \ge 0 . \tag{12}$$

Because

$$\begin{aligned}
&(n-1)\|\boldsymbol{u}\|_1 - n\sum_{i=2}^{n}|\boldsymbol{u}(i)| \\
&= (n-1)\left(|\boldsymbol{u}(1)|+\sum_{i=2}^{n}|\boldsymbol{u}(i)|\right) - n\sum_{i=2}^{n}|\boldsymbol{u}(i)| \\
&= (n-1)|\boldsymbol{u}(1)| - \sum_{i=2}^{n}|\boldsymbol{u}(i)| = \sum_{i=2}^{n}\left(|\boldsymbol{u}(1)|-|\boldsymbol{u}(i)|\right) \ge 0 ,
\end{aligned} \tag{13}$$

which gives inequality (10) with $l=2$.

Now, we generalize inequality (10) to $l\ge 2$,

$$\begin{aligned}
\|\boldsymbol{u}\|_1 &\ge \frac{n}{n-1}\sum_{i=2}^{n}|\boldsymbol{u}(i)| \ge \frac{n}{n-2}\sum_{i=3}^{n}|\boldsymbol{u}(i)| \\
&\ge\cdots\ge \frac{n}{n-l+1}\sum_{i=l}^{n}|\boldsymbol{u}(i)|,
\end{aligned} \tag{14}$$

which complete the proof.

Proof of Lemma 2: Firstly, we prove inequality (9) in the case of $\bar{k}\le k$.

We define the support set of $\boldsymbol{s}$ as $F=\mathrm{supp}(\boldsymbol{s})$, i.e., the set of i for which $\boldsymbol{s}_i\ne 0$. Let $\boldsymbol{s}_F\in R^k$ be a signal vector which only reserve the entries of $\boldsymbol{s}$ in the support set. Let $\boldsymbol{s}'_F\in R^k$ be a signal vector obtained by rearranging $\boldsymbol{s}_F$ in amplitude descending order, i.e., $|\boldsymbol{s}'_F(1)|\ge|\boldsymbol{s}'_F(2)|\ge\cdots\ge|\boldsymbol{s}'_F(k)|$. Then, we have

$$\|\boldsymbol{s}\|_1 = \|\boldsymbol{s}_F\|_1 = \|\boldsymbol{s}'_F\|_1 . \tag{15}$$

where the first equality holds because $\|\boldsymbol{s}\|_1 = \sum_{i\in F}|\boldsymbol{s}(i)| + \sum_{i\notin F}|\boldsymbol{s}(i)| = \sum_{i\in F}|\boldsymbol{s}(i)| + \sum_{i\notin F}0 = \|\boldsymbol{s}_F\|_1$ and the second equality uses the fact that $\boldsymbol{s}_F'$ is just a signal vector obtained by rearranging the entries order of $\boldsymbol{s}_F$.

Let $\boldsymbol{s}'$ be the best $\bar{k}$-term approximation of $\boldsymbol{s}$, then we have

$$\sigma_{\bar{k}}(\boldsymbol{s})_1 = \|\boldsymbol{s}-\boldsymbol{s}'\|_1 = \sum_{i=\bar{k}+1}^{k}\left|\boldsymbol{s}_F'(i)\right| \le \frac{k-\bar{k}}{k}\|\boldsymbol{s}_F'\|_1 = \frac{|k-\bar{k}|}{k}\|\boldsymbol{s}\|_1, \qquad (16)$$

where the inequality uses the result of Lemma 2.1.

Now we consider the case of $\bar{k} > k$. We have

$$\sigma_{\bar{k}}(\boldsymbol{s})_1 := \min_{z\in\Sigma_{\bar{k}}}\|\boldsymbol{s}-\boldsymbol{z}\|_1 = 0 \le \frac{1}{k}|k-\bar{k}|\cdot\|\boldsymbol{s}\|_1, \qquad (17)$$

which gives (9) with $\bar{k} > k$.

Lemma 3. Suppose that $\boldsymbol{s}\in R^n$ is a k-sparse signal, then we have

$$\|\boldsymbol{s}\|_1^2 \le k\|\boldsymbol{s}\|_2^2. \qquad (18)$$

Proof: Firstly, we consider arbitrary signal vector $\boldsymbol{v}=[v_1,v_2,...,v_n]\in R^n$. For any signal $\boldsymbol{v}$, we have

$$\|\boldsymbol{v}\|_1^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}|\boldsymbol{v}_i|\cdot|\boldsymbol{v}_j|$$

$$\le \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|\boldsymbol{v}_i|^2+|\boldsymbol{v}_j|^2}{2} = n\|\boldsymbol{v}\|_2^2, \qquad (19)$$

where the equality condition holds if and only if $|\boldsymbol{v}_1| = |\boldsymbol{v}_2| = \cdots = |\boldsymbol{v}_n|$.

Of course, sparse signal $\boldsymbol{s}\in R^n$ also satisfies (19). However, we can derive a tighter bound for sparse signal by taking the sparse characteristic of $\boldsymbol{s}$ into consideration. Let $\boldsymbol{s}_F\in R^k$ be a signal vector which only reserves the nonzero entries of $\boldsymbol{s}$, as used in Lemma 3.

According to (19), we have

$$\|\boldsymbol{s}_F\|_1^2 \le k\|\boldsymbol{s}_F\|_2^2. \qquad (20)$$

Combining (15) with (20), we obtain

$$\|\boldsymbol{s}\|_1^2 \le k\|\boldsymbol{s}\|_2^2. \qquad (21)$$

which complete the proof.

Leveraging the above lemmas, we can establish the main result of this paper.

Theorem 1. Suppose that $\boldsymbol{\Phi}_B$ is a measurement matrix obeying RIP with order $2\bar{K}$, $\delta_{2\bar{K}} \le \sqrt{2}-1$ and $\bar{K} \le 2K_{\max}K_{\min}/(K_{\max}+K_{\min})$, where $K_{\min}$ is the minimum block sparsity level. Assume that each block of $\boldsymbol{x} = \left[\boldsymbol{x}_1^{\mathrm{T}},\boldsymbol{x}_2^{\mathrm{T}},...,\boldsymbol{x}_L^{\mathrm{T}}\right]^{\mathrm{T}} \in R^{N\times1}$ is sampled by $\boldsymbol{\Phi}_B$ via (6), where $\boldsymbol{x}_i \in R^{n\times1}$ is i-th block of $\boldsymbol{x}$. Let $\hat{\boldsymbol{x}}_i$ be the reconstruction signal of $\boldsymbol{x}_i$ via solving (3), then for $\boldsymbol{x}$, the CS recovery error $\mu = \|\boldsymbol{x}-\hat{\boldsymbol{x}}\|_2/\|\boldsymbol{x}\|_2$ obeys

$$\mu \le c\frac{K_{\max}-\bar{K}}{\bar{K}}, \qquad (22)$$

where $\hat{\boldsymbol{x}} = \left[\hat{\boldsymbol{x}}_1^{\mathrm{T}},\hat{\boldsymbol{x}}_2^{\mathrm{T}},...,\hat{\boldsymbol{x}}_L^{\mathrm{T}}\right]^{\mathrm{T}} \in R^{N\times1}$ is the reconstruction signal of $\boldsymbol{x}$ and c is a finite constant.

Proof: According Lemma 1 and Lemma 2, we have

$$\|\boldsymbol{x}_i - \hat{\boldsymbol{x}}_i\|_2 \le c_i \frac{\sigma_{\bar{K}}(\boldsymbol{x}_i)_1}{\sqrt{\bar{K}}} \le c \frac{|K_i - \bar{K}|}{K_i\sqrt{\bar{K}}} \|\boldsymbol{x}_i\|_1 . \tag{23}$$

where c_i is a finite constant and $c = \max_i c_i$.

We bound the square of reconstruction error term by

$$\begin{aligned}
\mu^2 &= \frac{\|\boldsymbol{x}_1 - \hat{\boldsymbol{x}}_1\|_2^2 + \|\boldsymbol{x}_2 - \hat{\boldsymbol{x}}_2\|_2^2 + \cdots + \|\boldsymbol{x}_L - \hat{\boldsymbol{x}}_L\|_2^2}{\|\boldsymbol{x}\|_2^2} \\
&\le \frac{1}{\|\boldsymbol{x}\|_2^2} \sum_{i=1}^{L} \left(c^2 \frac{(1-\bar{K}/K_i)^2}{\bar{K}} \|\boldsymbol{x}_i\|_1^2 \right) \\
&\le c^2 \frac{(1-\bar{K}/K_{\max})^2}{\bar{K}} \frac{1}{\|\boldsymbol{x}\|_2^2} \sum_{i=1}^{L} \|\boldsymbol{x}_i\|_1^2 \\
&\le c^2 \frac{(1-\bar{K}/K_{\max})^2}{\bar{K}} \frac{1}{\|\boldsymbol{x}\|_2^2} \sum_{i=1}^{L} K_i \|\boldsymbol{x}_i\|_2^2 \\
&\le c^2 \frac{(1-\bar{K}/K_{\max})^2 K_{\max}}{\bar{K}} \frac{1}{\|\boldsymbol{x}\|_2^2} \sum_{i=1}^{L} \|\boldsymbol{x}_i\|_2^2 \\
&= c^2 \left(\frac{K_{\max} - \bar{K}}{\sqrt{\bar{K} K_{\max}}}\right)^2 \\
&\le c^2 \frac{K_{\max} - \bar{K}}{\bar{K}} .
\end{aligned} \tag{24}$$

where the first inequality uses the result of (23), the second inequality uses the inequality $\bar{K} \le 2K_{\max}K_{\min}/(K_{\max} + K_{\min})$, the third inequality uses the result of Lemma 3, the fourth inequality uses the inequality $K_i \le K_{\max}$ and the last inequality use the fact $\bar{K} \le K_{\max}$.

Taking the square-root on both sides of (24) gives the inequality (22).

In conclusion, Theorem 1 shows that the reconstruction error bound of BCS depends on the maximum block sparsity level of the sparse signal. In the best case, when $K_{\max} = \bar{K}$, which means that the nonzero elements of the 2D signal is distributed among the blocks evenly, the 2D sparse signal can be recovered perfectly by solving (3).

The main application of Theorem 1 is the permutation-based BCS scheme for CS-based image compression applications. According to Theorem 1, better recovery quality can be achieved if the signal has smaller maximum block sparsity level. Therefore, we can improve sampling efficiency by using permutation strategies prior to sampling. In practice, a lot of permutation strategies [11-18] have been proposed. The simulation results of [11-18] have shown that the reconstructed-images quality can be improved significantly if we can reduce the maximum block sparsity level by using permutation strategies prior to sampling. The successful application of permutation-based BCS scheme in CS-based image compression field can be regarded as a favorable evidence for Theorem 1.

4. Conclusions

In this paper, we analyze the error performance bound of BCS. It is revealed that better reconstruction performance can be achieved if the 2D signal has smaller maximum block sparsity level. We also show its potential applications in image compression field.

5. ACKNOWLEDGMENT

This work was supported by the Scientific and Technological Research Program of Chongqing Municipal Education Commission under Grant KJZD-K201801901. Lin Jiang (Jianglin4901@qq.com) is the corresponding author of this paper.

6. REFERENCES

[1] D. L. Donoho, "Compressed sensing," *IEEE Trans. on Information Theory*, vol. 52, no. 4, pp. 1289-1306, 2006.

[2] R. G. Baraniuk, "Compressive sensing," *IEEE Signal Processing Magazine*, vol. 24, no. 4, pp. 118-121, 2007.

[3] B. Zhang, D. Xiao, and Y. Xiang, "Robust coding of encrypted images via 2D compressed sensing," *IEEE Trans. on Multimedia*, vol. 23, pp. 2656-2671, 2021.

[4] B. Zhang, D. Xiao, Y. Li, et al., "2D compressed sensing using nonlocal low-rank prior reconstruction for cipher-image coding," *IEEE Signal Processing Letters*, no. 29, pp. 2033-2037, 2022.

[5] L. Y. Zhang, K. W. Wong, Y. Zhang, et al., Bi-level protected compressive sampling, *IEEE Trans. on Multimedia*, vol. 18, no. 9, pp. 1720-1732 , 2016.

[6] T. T. Thuy, J. Peetakul, D. K. Chi, et al., "Bi-directional intra prediction based measurement coding for compressive sensing images," *in Proc. of IEEE International Workshop on Multimedia Signal Processing*, 2020.

[7] L. Wang, X. Wu, and G. Shi, "Binned progressive quantization for compressive sensing," *IEEE Trans. on Image Processing*, vol. 21, no. 6, pp. 2980–2990, 2012.

[8] G. Hu, D. Xiao, Y. Wang, et al., "An image coding scheme using parallel compressive sensing for simultaneous compression-encryption applications," *Journal of Visual Communication and Image Representation*, vol. 44, pp. 116-127, 2017.

[9] J. Zhang, D. Zhao, and F. Jiang, "Spatially directional predictive coding for block-based compressive sensing of natural images," *in Proc. of IEEE International Conference on Image Processing*, 2013, pp. 1021–1025.

[10] R. Wan, J. Zhou, B. Huang, et al., "Adjacent pixels based measurement coding system for compressively sensed images," *IEEE Trans. on Multimedia*, vol. 24, pp.3558–3569, 2022.

[11] Z. Gao, C. Xiong, L. Ding, et al., "Image representation using block compressive sensing for compression applications," *Journal of Visual Communication and Image Representation*, vol. 24, no. 7, pp. 885-894, 2013.

[12] B. Zhang, Y. L. Liu, X. J. Jing, et al., "Interweaving permutation meets block compressed sensing," *Chinese Journal of Electronics*, vol. 27, no. 5, pp. 1056-1062, 2018.

[13] B. Zhang, Y. Liu, J. Zhuang, et al., "Matrix permutation meets block compressed sensing," *Journal of Visual Communication and Image Representation*, vol. 60, no. 4, pp. 69-78, 2019.

[14] Y. S. Zhang, J. Zhou, and F. Chen, "Embedding cryptographic features in compressive sensing," *Neurocomputing*, vol. 205, pp. 472-480, 2016.

[15] F. Hao, A. V. Sergiy, J. Hai, et al., "Permutation meets parallel compressed sensing: how to relax restricted isometry property for 2D sparse signals," *IEEE Trans. on Signal Processing*, vol. 62, no. 1, pp. 196-210, 2014.

[16] Z. Gao, C. Xiong, C. Zhou, and H. Wang, "Compressive sampling with coefficients random permutations for image compression," *in Proc. of International Conference on Multimedia and Signal Processing*, pp. 321-324, 2011.

[17] B. Zhang, Y. Liu, J. Zhuang, et al., "A novel block compressed sensing based on matrix permutation," *in Proc. of the 30th International Conference on Visual Communications and Image Processing*, pp. 1-4, 2016.

[18] Y. Q. Cao, W. G. Gong, B. Zhang, et al., "Optimal permutation based block compressed sensing for image compression applications," *IEICE Trans. on Information and Systems*, vol. 101, no. 1, pp. 215-224, 2018.

[19] A. Cohen, W. Dahmen, and R. DeVore, "Compressed sensing and best k-term approximation," *Journal of the American Mathematical Society*, vol. 22, no. 1, pp. 211-231, 2009.