Continued Hereditarily Finite Set-Approximations*

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Abstract

We study an encoding $\mathbb{R}_A$ that assigns a real number to each hereditarily finite set, in a broad sense. In particular, we investigate whether the map $\mathbb{R}_A$ can be used to produce codes that approximate any positive real number α to arbitrary precision, in a way that is related to continued fractions. This is an interesting question because it connects the theory of hereditarily finite sets to the theory of real numbers and continued fractions, which have important applications in number theory, analysis, and other fields.

Keywords

Ackermann codes, hereditarily finite sets, continued fractions, set-approximations

1. Introduction

We consider the following mapping of (hereditarily finite) sets into real numbers (see [1])

$$\mathbb{R}_A(x) = \sum_{y \in x} 2^{-\mathbb{R}_A(y)},$$

defined in complete analogy with the celebrated function

$$\mathbb{N}_A(x) = \sum_{y \in x} 2^{\mathbb{N}_A(y)}$$

proposed by W. Ackermann in 1937 as a recursive encoding of hereditarily finite sets by natural numbers (see [2]).

The encoding $\mathbb{R}_A(\cdot)$ can be used to map its domain, which is formed by the union of the following *universes*, into real numbers:

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1. HF: the well-founded *hereditarily finite sets* (h.f. sets, for short);
2. HF^{μ}: the hereditarily finite *multisets* (see [3]);
3. HF^{1}: the hereditarily finite *circular* sets (*hypersets*,[1] from now on; see [4, 5]);
4. $\mathsf{HF}^{1/2}$: the hereditarily finite *rational* hypersets, that is the sub-universe of HF^1 consisting of those hypersets whose *transitive closures* are finite.

The following inclusions hold:

$$\mathsf{HF} \subsetneq \mathsf{HF}^{1/2} \subsetneq \mathsf{HF}^{1} \quad \wedge \quad \mathsf{HF} \subsetneq \mathsf{HF}^{\mu}.$$

In what follows, for any set $\hbar$ that belongs to one of the aforementioned universes, we will refer to the real number $\mathbb{R}_A(\hbar)$ as the *code* of $\hbar$.[2]

In [6], it has been proven that each element of $\mathsf{HF}^{1/2}$ has a uniquely defined code. In the following, we contend that sets belonging to any of the universes listed above have unique codes as well.

Furthermore, we will demonstrate that every real number can be approximated arbitrarily closely by a (possibly infinite) sequence of codes of well-founded hereditarily finite sets. Building on this result, we will then show that every positive real number can be expressed as the code of a single element in HF^1. The proof of this fact relies on introducing a set-theoretic version of the process used to define the continued fraction uniquely associated with any given positive real number.

Our construction introduces the concept of *set-approximation*, which is a sequence of sets that may not be unique and may have different ways of being obtained, but whose codes eventually converge to any given non-negative real number $\alpha \in \mathbb{R}_0^+$. However, we will show that there exists a unique *first* approximation, which is obtained by using codes of sets in HF selected according to a minimality criterion (minimum $\mathbb{N}_A$-code). This notion of optimality will serve as a set-theoretic counterpart to the concept of first approximation introduced in the study of continued fractions.

The mapping of universes of hereditarily finite sets into finitely-branching directed graphs provides a connection between the material presented here and graph theory.

2. Basics

Let $\mathbb{N}$ and $\mathbb{N}^+$ be the set of natural numbers and positive integers, respectively, and let $\mathscr{P}(\cdot)$ denote the *powerset* operator.

Definition 1 (Hereditarily finite sets). $\mathsf{HF} = \bigcup_{n\in\mathbb{N}} \mathsf{HF}_n$ is the collection of all *hereditarily finite* sets, where

$$\begin{cases} \mathsf{HF}_0 = \emptyset, \\ \mathsf{HF}_{n+1} = \mathscr{P}(\mathsf{HF}_n), \quad \text{for } n \in \mathbb{N}. \end{cases}$$

[1] A term introduced by Barwise and Etchemendy in [4].

[2] We will use $\hbar$ instead of plain h to stress that $\hbar$ may represent a hyperset.

The Ackermann code $\mathbb{N}_A$ recalled above, where

$$\mathbb{N}_A(x) = \sum_{y \in x} 2^{\mathbb{N}_A(y)}$$

for every hereditarily finite set x, induces a natural ordering

$$\mathsf{h}_0, \mathsf{h}_1, \mathsf{h}_2, \mathsf{h}_3, \ldots$$

of the hereditarily finite sets, known as the *Ackermann order*, where $\mathbb{N}_A(\mathsf{h}_i) = i$ for every $i \in \mathbb{N}$.

Consider next the following map $\mathbb{R}_A$ over HF, obtained from $\mathbb{N}_A$ by simply placing a minus sign in front of each exponent in the definition of $\mathbb{N}_A$:

$$\mathbb{R}_A(x) = \sum_{y \in x} 2^{-\mathbb{R}_A(y)}. \tag{1}$$

From (1), it readily follows that all (valid) $\mathbb{R}_A$-codes are nonnegative. For instance, we have:

$$\mathbb{R}_A(\emptyset) = 0, \qquad \mathbb{R}_A(\{\emptyset\}) = 1, \qquad \mathbb{R}_A(\{\emptyset\}^2) = \frac{1}{2},$$
$$\mathbb{R}_A(\{\emptyset\}^3) = \frac{1}{\sqrt{2}}, \qquad \mathbb{R}_A(\{\emptyset\}^4) = 2^{-\frac{1}{\sqrt{2}}}, \qquad \mathbb{R}_A(\{\emptyset\}^5) = 2^{-2^{-\frac{1}{\sqrt{2}}}}, \quad \text{etc.,}$$

where $\{\emptyset\}^0 = \emptyset$ and, recursively, $\{\emptyset\}^{n+1} = \big\{\{\emptyset\}^n\big\}$.

In the following, it will be convenient to introduce a graph-theoretic point of view on hereditarily finite sets (see [7]). Such view is based on the introduction of the so-called membership graph (see Section 2.3.1). Just two notions will be instrumental in order to introduce membership graphs: the notion of *transitive closure* of a set and the notion of *bisimulation*, which we briefly recall here.

Definition 2. The *transitive closure* of a set $\hbar$ is defined, recursively or otherwise, as the collection

$$\mathsf{TrCl}(\hbar) \quad = \quad \hbar \cup \textstyle\bigcup_{x \in \hbar} \mathsf{TrCl}(x).$$

Definition 3. A *bisimulation* on a given graph $G = (V, E)$ is a relation $B \subseteq V \times V$ such that, for all $u_0, u_1 \in V$ for which $\langle u_0, u_1 \rangle \in B$ holds, the following two conditions hold as well:

- $\forall v_1[\langle u_1, v_1 \rangle \in E \rightarrow \exists v_0(\langle u_0, v_0 \rangle \in E \wedge \langle v_0, v_1 \rangle \in B)]$;
- $\forall v_0[\langle u_0, v_0 \rangle \in E \rightarrow \exists v_1(\langle u_1, v_1 \rangle \in E \wedge \langle v_0, v_1 \rangle \in B)]$.

It can be shown that, given a graph G, there always exists a bisimulation on G that includes all others. When the graph G is understood, the symbol $\cong$ will be used to denote such a maximal bisimulation, which is an equivalence relation — named *bisimilarity* — on the set V of nodes.

2.1. Continued Fractions

Continued fractions (see [8]) are introduced, among many other reasons, as a simple and elegant means to denote real numbers. In the most basic setting, where elements of the continued fractions are simply natural numbers, the idea is the following. Given a positive real number $\alpha \in \mathbb{R}^+$, either $\alpha \in \mathbb{N}$, in which case we are done, or $0 < \alpha - \lfloor \alpha \rfloor < 1$. In the latter case, we can express α as $\lfloor \alpha \rfloor + \frac{1}{r}$, for some $r > 1$. By iterating the above process, we ultimately obtain a (possibly infinite) continued fraction

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{1 + \cdots}}} \ ,$$

where the a_i are natural numbers, which can be conveniently represented as $[a_0; a_1, a_2, \ldots]$.

Based on the above initial steps, a rich theory has been developed. In particular, it can be easily proved that every $\alpha \in \mathbb{R}^+$ can be arbitrarily approximated by a (possibly infinite) sequence $\left\langle \frac{p_k}{q_k} \right\rangle_k$ of rational numbers, satisfying the following recursive relations: for any $k \geqslant 2$,

$$\begin{cases} p_k = a_k p_{k-1} + p_{k-2} \ , \\ q_k = a_k q_{k-1} + q_{k-2} \ . \end{cases}$$

Continued approximations built using the above ideas turn out to be optimal in the sense described in [8]:

> Let us agree that a fraction a/b (for $b > 0$) is a *best approximation* of a real number α if every other rational fraction with the same or smaller denominator differs from α by a greater amount, in other words, if the inequalities $0 < d \leqslant b$, and $a/b \not\equiv c/d$ imply that:
>
> $$\left| \alpha - \frac{c}{d} \right| > \left| \alpha - \frac{a}{b} \right| .$$

2.2. The Universe HF^μ of Hereditarily Finite Multisets

To define the collection HF^μ of hereditarily finite multisets, we use the finitary *μ-power-set operator* $\mathscr{P}^\mu$ (see [3]). Given a multiset X, we put

$$\mathscr{P}^\mu(X) = \left\{ \{^{m_1}x_1, \ldots, {}^{m_n}x_n\} \mid x_1, \ldots, x_n \in X \text{ (distinct)} \, , m_1, \ldots, m_n \in \mathbb{N}^+, n \in \mathbb{N} \right\}.$$

Thus, each element e of $\mathscr{P}^\mu(X)$ has the form $\left\{^{m_1}x_1, \ldots, {}^{m_n}x_n\right\}$, where $x_1, \ldots, x_n$ are finitely many distinct members of X and $m_1, \ldots, m_n \in \mathbb{N}^+$ are their multiplicities in e.

Then, the *cumulative hierarchy* HF^μ *of the hereditarily finite multisets* is defined as

$$\mathsf{HF}^\mu = \bigcup_{n \in \mathbb{N}} \mathsf{HF}^\mu_n,$$

where $\mathsf{HF}^\mu_0 = \emptyset$ and, recursively, $\mathsf{HF}^\mu_{n+1} = \mathscr{P}^\mu(\mathsf{HF}^\mu_n)$ for $n \in \mathbb{N}$.

The map $\mathbb{R}_A$ can be extended in a very natural manner to a map $\mathbb{R}_A^{\mu}$ over the collection HF^{μ} of the h.f. multisets, by putting recursively, for every multiset $H \in \mathsf{HF}^{\mu}$,

$$\mathbb{R}_A^{\mu}(H) = \sum_{K \in H} \mu_H(K) \cdot 2^{-\mathbb{R}_A^{\mu}(K)}.$$

2.3. Set Systems

Both well-founded and circular sets can be presented as (unique) solutions to systems of set-theoretic equations such as the ones introduced by the following definition (taken from [6], see also [5]). A collection $\hbar_1, \ldots, \hbar_n$ of sets solving a given set system in the variables $x_1, \ldots, x_n$ is downward closed with respect to membership and is fully described by the equations appearing in the system. As a matter of fact, the solution set turns out to be a complete listing of the sets in $\mathsf{TrCl}\,(\{\hbar_1, \ldots, \hbar_n\})$.

Definition 4 (Set systems). A *set system* $\mathscr{S}(x_1, \ldots, x_n)$ in the distinct set unknowns $x_1, \ldots, x_n$ is a collection of set-theoretic equations of the form

$$\begin{cases} x_1 = \{x_{1,1}, \ldots, x_{1,m_1}\} \\ \quad \vdots \\ x_n = \{x_{n,1}, \ldots, x_{n,m_n}\}, \end{cases} \tag{2}$$

with $m_i \geqslant 0$ for $i \in \{1, \ldots, n\}$, where each unknown $x_{i,u}$, for $i \in \{1, \ldots, n\}$ and $u \in \{1, \ldots, m_i\}$, also occurs among the unknowns $x_1, \ldots, x_n$.[3]

The directed graph $G_{\mathscr{S}} = (V_{\mathscr{S}}, E_{\mathscr{S}})$ associated with the system $\mathscr{S} = \mathscr{S}(x_1, \ldots, x_n)$, with

$$\begin{aligned} V_{\mathscr{S}} &= \{x_1, \ldots, x_n\}, \\ E_{\mathscr{S}} &= \big\{\langle x_i, x_{i,u}\rangle \,:\, i \in \{1, \ldots, n\},\ u \in \{1, \ldots, m_i\}\big\}, \end{aligned}$$

is the *membership graph* of $\mathscr{S}$.

Remark 2.1. Note that we are not insisting that the membership graph be acyclic; if we did, we would be considering only conventional, well-founded sets; this is because we want our notions to adjust to all intricacies inherent in Aczel's notion of 'non-well-founded' (albeit finite) sets, cf. [5].

Definition 5. A set system $\mathscr{S}(x_1, \ldots, x_n)$ is *normal* if there exist n pairwise distinct (i.e., non-bisimilar[4]) hypersets $\hbar_1, \ldots, \hbar_n \in \mathsf{HF}^{1/2}$ such that the assignment $x_i \mapsto \hbar_i$ satisfies all the set equations of $\mathscr{S}(x_1, \ldots, x_n)$.

For every $h = \{h_1, \ldots, h_n\} \in \mathsf{HF}$ with n members, the code $\mathbb{R}_A(h)$ can be expressed as the sum

$$\mathbb{R}_A(\{h_1\}) + \cdots + \mathbb{R}_A(\{h_n\}).^5$$

[3]When $m_i = 0$, the expression $\{x_{i,1}, \ldots, x_{i,m_i}\}$ reduces to $\{\}$, designating the empty set.
[4]Bisimilarity (see lines below Definition 3) is now referred to the graph $G_{\mathscr{S}}$.
[5]*Disjoint additivity* property.

$$\begin{cases} x_1 = \{x_2, x_3\} \\ x_2 = \{\,\} \\ x_3 = \{x_4, x_5\} \\ x_4 = \{x_2\} \\ x_5 = \{x_6, x_7\} \\ x_6 = \{x_4\} \\ \quad \vdots \end{cases}$$

Figure 1: A set system for $\hbar_1 \in \mathsf{HF}^1 \setminus \mathsf{HF}^{1/2}$.

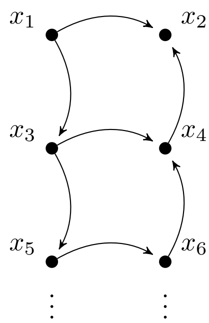

Figure 2: The membership graph for $\hbar_1$.

Since $\mathbb{R}_A(\{h_i\}) = 2^{-\mathbb{R}_A(h_i)} \leqslant 1$, we can therefore conclude that $\mathbb{R}_A(h) \leqslant |h|$. As a matter of fact, in [6] it is proved that if h is the (unique) solution to a system $\mathscr{S}$ of set-theoretic equations involving one variable for each of the elements in $\mathsf{TrCl}\,(h)$, $\mathbb{R}_A(h)$ is the point of convergence of a sequence of codes of *multisets* approximating h. Such multiset code-values, whose definition is given so as to generalise naturally the definition given for sets, start with 0 and $|h|$ and oscillate alternatively below and above $\mathbb{R}_A(h)$, eventually converging to it.

2.3.1. The Universes HF^1 and $\mathsf{HF}^{1/2}$ of Hereditarily Finite Hypersets

The collection HF^1 of hereditarily finite hypersets can be seen as consisting of the unique solutions to *infinite* set systems defined as in the previous subsection. Equivalently, HF^1 can be seen as the smallest collection of finite hypersets, all of whose elements are finite. Albeit each element of HF^1 has finitely many members, this is not necessarily true for its transitive closure.

$\mathsf{HF}^{1/2}$ is the sub-collection of the hypersets in HF^1 whose transitive closures are finite.

Example 1. In Figure 1 is an infinite set system describing a set $\mathsf{TrCl}\,(\{\hbar_1\})$ with $\hbar_1 \in \mathsf{HF}^1 \setminus \mathsf{HF}^{1/2}$ and in Figure 2 its membership graph.

In analogy with what has been done for set systems, it is convenient to *"depict"* also a hyperset $\hbar$ by its *membership graph* $G_\hbar = (V_\hbar, E_\hbar)$, namely a directed graph whose nodes are the hypersets in $\mathsf{TrCl}\,(\{\hbar\})$ and whose arcs represent the membership relation among them:

$$\langle \hbar'', \hbar' \rangle \in E_\hbar \quad \text{if and only if} \quad \hbar' \leftarrow \hbar'' \quad \text{if and only if} \quad \hbar' \in \hbar''.$$

Graphs bisimilar to $G_\hbar$ (with, possibly, more than $|\mathsf{TrCl}\,(\{\hbar\})|$ nodes) will depict (possibly redundantly) the same hyperset $\hbar$: $G_\hbar$ will be the (minimal) representative of its $\cong$-equivalence class. All graphs G bisimilar to $G_\hbar$ will also be dubbed *membership* graphs. It turns out that every node in each such graph G is reachable from a node bisimilar to $\hbar$, which is called the *point* of the (*pointed*) graph G.

We can identify HF^1 as the quotient by bisimulation of the collection $\mathcal{G}$ of directed and pointed graphs, all of whose nodes have a finite-size in-neighborhood.

The *unfolding* of a graph in $\mathcal{G}$ is a finitely branching tree and can be seen as an "approximation" of the graph. More formally:

Definition 6. For all $i \in \mathbb{N}$ and $G \in \mathcal{G}$, the i-th unfolding $\mathbf{u}_i(G)$ of G is the finitely branching tree whose nodes are paths of length less than or equal to i starting from the point in G, and whose arcs correspond to a one-step extension of paths of length less than i.

Given $\hbar \in \mathsf{HF}^1$, the *infinite* unfolding $\mathbf{u}(G_\hbar)$ of $G_\hbar$ is defined to be $\mathbf{u}(G_\hbar) = \lim_{i\to\infty} \mathbf{u}_i(G_\hbar)$.[6]

Since a finite tree is essentially a hereditarily finite multi-set (see [3]), the above definitions allow us to extend the notion of $\mathbb{R}_A$-code to HF^1, via the generalisation $\mathbb{R}_A^\mu$ of $\mathbb{R}_A$ to multi-sets.

Lemma 2.2. *For any $\hbar \in \mathsf{HF}^1$, there exists $\alpha \in \mathbb{R}_0^+$ such that:*

$$\alpha = \lim_{i\to\infty} \mathbb{R}_A^\mu(\mathbf{u}_i(G_\hbar)).$$

Proof. The proof of this fact is a generalisation to infinite sets of Lemma 4 and Theorem 4 in [6]. ☐

The above lemma allows us to extend the mapping $\mathbb{R}_A$ to hypersets, according to the following definition:

Definition 7. For any $\hbar \in \mathsf{HF}^1$, we put

$$\mathbb{R}_A(\hbar) = \lim_{i\to\infty} \mathbb{R}_A^\mu(\mathbf{u}_i(G_\hbar)).$$

3. Continued Approximations of Codes

As observed in [6], the codes of elements of HF can get arbitrarily large and arbitrarily small. This is proved in the following lemma.

Lemma 3.1. *For every $\alpha \in \mathbb{R}^+$ there exist (nonempty) $h, h' \in \mathsf{HF}$ such that $\mathbb{R}_A(h) > \alpha$ and $0 < \mathbb{R}_A(h') < \alpha$.*

Proof. Notice that for any odd natural number j, we have $\emptyset \in \mathsf{h}_j$. Thus, we have $\mathbb{R}_A(\mathsf{h}_j) \geqslant \mathbb{R}_A(\{\emptyset\}) = 2^0 = 1$, $\mathbb{R}_A(\{\mathsf{h}_j\}) = 2^{-\mathbb{R}_A(\mathsf{h}_j)} \leqslant 2^{-1} = \frac{1}{2}$, and $\mathbb{R}_A(\{\{\mathsf{h}_j\}\}) = 2^{-\mathbb{R}_A(\{\mathsf{h}_j\})} \geqslant 2^{-1/2} > \frac{1}{2}$.

Given $\alpha \in \mathbb{R}^+$, let $k = \lceil 4\alpha \rceil$ and consider the $(k+1)$-element hereditarily finite set $h = \big\{\{\mathsf{h}_j\} \ : \ j \leqslant k\big\}$. Then, we have:

$$\mathbb{R}_A(h) = \sum_{j=0}^{k} \mathbb{R}_A\big(\,\{\{\mathsf{h}_j\}\}\,\big) \geqslant \sum_{\substack{j=0 \\ j \text{ is odd}}}^{k} \mathbb{R}_A\big(\,\{\{\mathsf{h}_j\}\}\,\big) > \frac{1}{2}\cdot\left\lceil\frac{k}{2}\right\rceil = \frac{1}{2}\cdot\left\lceil\frac{\lceil 4\alpha\rceil}{2}\right\rceil \geqslant \frac{1}{2}\cdot\frac{\lceil 4\alpha\rceil}{2} \geqslant \alpha\,.$$

Next, to prove that for all $\alpha \in \mathbb{R}^+$ there exists $h' \in \mathsf{HF}$ such that $0 < \mathbb{R}_A(h') < \alpha$, it suffices to pick any $h \in \mathsf{HF}$ such that $\mathbb{R}_A(h) > \frac{1}{\alpha}$ and set $h' = \{h\}$. Indeed, by recalling that the inequality $x^x > \frac{1}{2}$ holds for all $x > 0$, we have:

$$0 < \mathbb{R}_A(h') = \mathbb{R}_A(\{h\}) = 2^{-\mathbb{R}_A(h)} < 2^{-\frac{1}{\alpha}} < \alpha.$$ ☐

[6]The definition of infinite unfolding is well-given, since $\mathbf{u}_i(G)$ is a subtree of $\mathbf{u}_{i+1}(G)$ for all $G \in \mathcal{G}$ and $i \in \mathbb{N}$.

An immediate consequence of the preceding lemma is the following:

Corollary 3.2. *For every $\alpha \in \mathbb{R}^+$ there exist infinitely many $h, h' \in \mathsf{HF}$ such that $\mathbb{R}_A(h) > \alpha$ and $\mathbb{R}_A(h') < \alpha$.*

The above lemma allows us to conclude that we can arbitrarily approximate any real number in the following sense.

Definition 8. Given $\alpha \in \mathbb{R}_0^+$ $(= \mathbb{R}^+ \cup \{0\})$, a sequence $\langle h_{\alpha,i} \rangle_{i \in \mathbb{N}}$ of hereditarily finite sets is said to *approximate* α or to be an α*-approximation*, if

$$\sum_{i \in \mathbb{N}} \mathbb{R}_A(h_{\alpha,i}) = \alpha.$$

Even though Lemma 3.1 easily implies, for every $\alpha \in \mathbb{R}^+$, the existence of α-approximating sequences (in which repetitions are allowed), the above definition, clearly, by no means identifies a unique such a sequence.

In the following, we will address the question of giving a sensible notion of *first* approximation and proving its uniqueness.

Remark 3.3. We will prove below that introducing (first) approximating sequences — whose elements are codes of elements in HF — is a way to capture (by approximations) the uncountably many real numbers by means of the countably many (codes of) elements of HF.

Since the cardinality of HF^1 is larger than ω and since our notion of $\mathbb{R}_A$-code extends to hypersets, the above remark suggests the following question, which will be addressed in Section 3.3:

Question (HF^1**-approximations).**
Given $\alpha \in \mathbb{R}$, is there any element $\hbar(\alpha) \in \mathsf{HF}^1$ such that $\mathbb{R}_A(\hbar(\alpha)) = \alpha$?

3.1. On the Existence and Uniqueness of *First* Approximations

Given $\alpha \in \mathbb{R}_0^+$, consider any sequence $\langle h_{\alpha,i} \rangle_{i \in \mathbb{N}}$ that approximates α. In the most interesting case, that is when α can be approximated only by sequences that are not eventually constant, the values $\mathbb{R}_A(h_{\alpha,i})$ get arbitrarily small. Since for any $i \in \mathbb{N}$ there are only finitely many hereditarily finite sets whose Ackermann number $\mathbb{N}_A$ is smaller than $\mathbb{N}_A(h_{\alpha,i})$, this implies that $\mathbb{N}_A(h_{\alpha,i})$ (as well as the rank $\mathsf{rk}(h_{\alpha,i})$) must get arbitrarily large.

The above considerations motivate, for any $\alpha \in \mathbb{R}_0^+$, the following definition, setting the stage for uniquely identifying a first approximating sequence.

Definition 9. Given $\alpha \in \mathbb{R}^+$, the *least approximation* of α, denoted h_α, is the (nonempty) h.f. set whose $\mathbb{N}_A$-code is minimum among all the sets h in HF such that $0 < \mathbb{R}_A(h) \leqslant \alpha$.

We also let $\emptyset$ be the least approximation of 0.

In view of Lemma 3.1, the above definition is well-given. The same idea used in it, which allowed us to identify the single *first* approximation for $\alpha \in \mathbb{R}_0^+$, can iteratively be exploited to characterize a *sequence* of least approximations.

Definition 10. Given $\alpha \in \mathbb{R}_0^+$, the *first*[7] *set-theoretic approximating sequence* $\langle h_{\alpha,i} \rangle_{i \in \mathbb{N}}$ of α is

[7]In the conclusions we will justify our choice of calling the approximation defined here as "first" instead of "best".

recursively defined as follows, for $i \in \mathbb{N}$:

1. if $\mathbb{R}_A(\bigcup_{j=0}^{i-1} h_{\alpha,j}) = \alpha$, then $h_{\alpha,i} = \emptyset$;

2. otherwise $h_{\alpha,i}$ is the set in HF whose $\mathbb{N}_A$-code is minimum among the sets in HF such that
$$h \not\subseteq \bigcup_{j=0}^{i-1} h_{\alpha,j} \qquad \text{and} \qquad \mathbb{R}_A(h \cup \bigcup_{j=0}^{i-1} h_{\alpha,j}) \leqslant \alpha\,. \tag{3}$$

We will refer to the (possibly infinite) set $\bigcup_{i\in\mathbb{N}} h_{\alpha,i}$ as the *cumulus* of the first set-theoretic approximating sequence $\langle h_{\alpha,i}\rangle_{i\in\mathbb{N}}$ of α.

Remark 3.4. The preceding definition is well-given. Indeed, if $\mathbb{R}_A(\bigcup_{j=0}^{i-1} h_{\alpha,j}) < \alpha$ holds for some $i \in \mathbb{N}$, then Corollary 3.2 yields the existence of infinitely many sets h in HF such that $\mathbb{R}_A(h) \leqslant \alpha - \mathbb{R}_A(\bigcup_{j=0}^{i-1} h_{\alpha,j})$. Hence, there exist infinitely many sets h in HF such that (3) holds.

In the rest of the section, we let $\alpha \in \mathbb{R}_0^+$ and $\langle h_{\alpha,i}\rangle_{i\in\mathbb{N}}$ be the *first set-theoretic approximating sequence* of α.

Denote by h_α^i the hereditarily finite set $\bigcup_{j\leqslant i} h_{\alpha,j}$ and call the sequence of $\langle h_\alpha^i\rangle_{i\in\mathbb{N}}$ the *cumulative* version of the set-theoretic approximating sequence $\langle h_{\alpha,i}\rangle_{i\in\mathbb{N}}$. The cumulative sequence $\langle h_\alpha^i\rangle_{i\in\mathbb{N}}$ is $\subseteq$-monotone, that is:
$$h_\alpha^0 \subseteq h_\alpha^1 \subseteq \cdots \subseteq h_\alpha^i \subseteq \cdots\,. \tag{4}$$
Plainly, its set-theoretic limit $h_\alpha^\infty = \bigcup_{i\in\mathbb{N}} h_\alpha^i$ coincides with the cumulus of the sequence $\langle h_{\alpha,i}\rangle_{i\in\mathbb{N}}$ from which it derives.

As yet, we cannot exclude that $h_{\alpha,i} = h_{\alpha,j}$ holds for some pair i, j of distinct pedices. Moreover, even if we had $h_{\alpha,i} \neq h_{\alpha,j}$ when $i \neq j$, it could still be the case that $h_{\alpha,i} \cap h_{\alpha,j} \neq \emptyset$ with $i \neq j$. These possibilities are ruled out by the following lemma.

Lemma 3.5. *Given a first set-theoretic approximating sequence $\langle h_{\alpha,i}\rangle_{i\in\mathbb{N}}$ of some $\alpha \in \mathbb{R}_0^+$, it holds that:*

i) $h_{\alpha,i} \cap h_{\alpha,j} = \emptyset$, *for all distinct* $i, j \in \mathbb{N}$;

ii) $h_{\alpha,i}$ *is the least approximation of* $\alpha - \mathbb{R}_A(h_\alpha^{i-1})$ *when* $i > 0$;

iii) $h_\alpha^i \cap h_{\alpha,i+1} = \emptyset$, *for all* $i \in \mathbb{N}$;

iv) either

- *for all* $i \in \mathbb{N}$ *it holds that* $h_\alpha^i \subsetneq h_\alpha^{i+1}$, *or*
- *there exists* $\bar{i}$ *such that* $h_\alpha^i \subsetneq h_\alpha^{i+1}$, *for all* $i < \bar{i}$, *and* $h_\alpha^i = h_\alpha^{\bar{i}}$, *for all* $i \geqslant \bar{i}$;

v) $\mathbb{R}_A(h_\alpha^i) = \mathbb{R}_A\left(\biguplus_{j\leqslant i} h_{\alpha,j}\right) = \sum_{j\leqslant i} \mathbb{R}_A(h_{\alpha,j})$.

Proof. To see i), arguing by contradiction assume that $h_{\alpha,i} \cap h_{\alpha,j} \neq \emptyset$, for some $j < i$. Then $h_{\alpha,i}$ and $h_{\alpha,j}$ are nonempty and $h_{\alpha,i} \not\subseteq h_{\alpha,j}$ holds by point 2 of Definition 10. Hence, by setting $h'_{\alpha,i} = h_{\alpha,i} \setminus h_{\alpha,j}$, we would have $h'_{\alpha,i} \not\subseteq \bigcup_{j \leqslant i-1} h_{\alpha,j}$, $\mathbb{R}_A(h'_{\alpha,i} \cup \bigcup_{j \leqslant i-1} h_{\alpha,j}) = \mathbb{R}_A(h_{\alpha,i} \cup \bigcup_{j \leqslant i-1} h_{\alpha,j}) \leqslant \alpha$, and $\mathbb{N}_A(h'_{\alpha,i}) < \mathbb{N}_A(h_{\alpha,i})$, contradicting the minimality of the $\mathbb{N}_A$-code of $h_{\alpha,i}$.

As for ii), notice that from i) it follows that $\mathbb{R}_A(\bigcup_{j \leqslant i} h_{\alpha,j}) = \mathbb{R}_A(h_{\alpha,i}) + \mathbb{R}_A(h^{i-1}_\alpha)$. Hence the claim follows Definitions 9 and 10.

Finally, points iii), iv), and v) easily follow from i). □

Theorem 3.6. *Given $\alpha \in \mathbb{R}^+_0$, the first set-theoretic approximating sequence $\langle h_{\alpha,i} \rangle_{i \in \mathbb{N}}$ is unique.*

Proof. Arguing by contradiction, let $\langle h_{\alpha,i} \rangle_{i \in \mathbb{N}}$ and $\langle h'_{\alpha,i} \rangle_{i \in \mathbb{N}}$ be two *distinct* first approximating sequences of the same real number $\alpha \in \mathbb{R}^+_0$. We can immediately rule out the case $\alpha = 0$, since from Definition 10 the first approximating sequence of 0 is the constant null-set sequence $\langle \emptyset, \emptyset, \emptyset, \ldots \rangle$. Hence, let $\alpha > 0$, and let $i \in \mathbb{N}$ be the first index such that $h_{\alpha,i} \neq h'_{\alpha,i}$. Thus, $\bigcup_{j \leqslant i-1} h_{\alpha,j} = \bigcup_{j \leqslant i-1} h'_{\alpha,j}$ holds, so that by points 1 and 2 of Definition 10 it readily follows that $h_{\alpha,i} = h'_{\alpha,i}$, a contradiction. □

A useful technical fact is stated in the following lemma.

Lemma 3.7. *Let $\langle h_{\alpha,i} \rangle_{i \in \mathbb{N}}$ be the first set-theoretic approximating sequence of a given $\alpha \in \mathbb{R}^+_0$, and let $\langle h^i_\alpha \rangle_{i \in \mathbb{N}}$ be its cumulative version. Also, let $h \in \mathsf{HF}$ and $k \in \mathbb{N}$ be such that*

$$h \not\subseteq h^k_\alpha \qquad \textit{and} \qquad \mathbb{R}_A(h^k_\alpha \cup h) \leqslant \alpha.$$

Then it holds that

$$\mathbb{R}_A\big(h^{k+\mathbb{N}_A(h)}_\alpha\big) \geqslant \mathbb{R}_A(h^k_\alpha \cup h). \tag{5}$$

Proof. If, for contradiction, (5) were false, we would have $h \not\subseteq h^{k+\mathbb{N}_A(h)}_\alpha$. Hence, the $\mathbb{N}_A$-codes of the following $\mathbb{N}_A(h)$ sets

$$h_{\alpha,k+1}, \quad h_{\alpha,k+2}, \quad \ldots, \quad h_{\alpha,k+\mathbb{N}_A(h)}$$

in the sequence $\langle h_{\alpha,i} \rangle_{i \in \mathbb{N}}$ would be pairwise distinct, non-null, and strictly less than $\mathbb{N}_A(h)$. However, this is impossible, as there are at most $\mathbb{N}_A(h) - 1$ pairwise distinct h.f. sets with $\mathbb{N}_A$-code greater than 0 and less than $\mathbb{N}_A(h)$. □

3.2. Convergence of First Approximating Sequences

Consider the limit set $h^\infty_\alpha = \bigcup_{i \in \mathbb{N}} h^i_\alpha$ of $\langle h_{\alpha,i} \rangle_{i \in \mathbb{N}}$ (and of $\langle h^i_\alpha \rangle_{i \in \mathbb{N}}$). We extend the notion of $\mathbb{R}_A$-code to the (possibly infinite) set h^∞_α by putting

$$\mathbb{R}_A(h^\infty_\alpha) = \sum_{i \in \mathbb{N}} \mathbb{R}_A(h_{\alpha,i}) = \lim_{i \to \infty} \mathbb{R}_A(h^i_\alpha). \tag{6}$$

In other words, the code of the limit set h^∞_α is defined as the limit of the codes of the components of $\langle h^i_\alpha \rangle_{i \in \mathbb{N}}$, thus generalising property v) of Lemma 3.5. Clearly, the interesting case arises when h^∞_α is infinite, and the natural question to ask is whether we can prove that $\mathbb{R}_A(h^\infty_\alpha) = \alpha$. This will be our task in the current section.

The following lists of sets will be very useful for the purpose:

- super-singletons s_i, defined as:

$$s_0 = \{\emptyset\}^0 = \emptyset \quad \text{and} \quad s_i = \{\emptyset\}^i = \{s_{i-1}\}, \text{ for } i \in \mathbb{N}^+;$$

- sets of n super-singletons s_i^n, defined as:

$$s_i^n = \{s_i, s_{i+1}, \ldots, s_{i+n-1}\};$$

- sets of m sets of n super-singletons $s_i^{n,m}$, defined as:

$$s_i^{n,m} = \{s_i^n, s_{i+1}^n, \ldots, s_{i+m-1}^n\}.$$

Next, let Ω be the (unique) real solution to the equation $x = 2^{-x}$, which turns out to have the approximate value 0.6411857...; then, the following approximation results hold:

i) the limit of the codes of super-singletons is Ω;

ii) the limit of the codes of sets of n super-singletons is $n\Omega$;

iii) the limit of the codes of sets of m sets of n super-singletons is $m\Omega^n$.

The first limit follows from the following result, whose proof can be carried out along the same lines of the proof of Theorem 4 in [6]:

Lemma 3.8. *The sequences $\langle \mathbb{R}_A(s_{2j}) \rangle_{j\in\mathbb{N}}$ and $\langle \mathbb{R}_A(s_{2j+1}) \rangle_{j\in\mathbb{N}}$ are strictly increasing and strictly decreasing, respectively, and they both converge to Ω. Hence, $\lim_{i\to\infty} \mathbb{R}_A(s_i) = \Omega$.*

The limits ii) and iii) are proved in the following two lemmas.

Lemma 3.9. $\lim_{i\to\infty} \mathbb{R}_A(s_i^n) = n\Omega$, *for all* $n \in \mathbb{N}$.

Proof. If $n \in \mathbb{N}^+$, then for all $i \in \mathbb{N}$ we have

$$\begin{aligned} \mathbb{R}_A(s_i^n) &= \mathbb{R}_A(\{s_i, s_{i+1}, \ldots, s_{i+n-1}\}) \\ &= \mathbb{R}_A(\{s_i\}) + \mathbb{R}_A(\{s_{i+1}\}) + \cdots + \mathbb{R}_A(\{s_{i+n-1}\}) = \sum_{k=1}^{n} \mathbb{R}_A(s_{i+k}). \end{aligned}$$

Hence,

$$\lim_{i\to\infty} \mathbb{R}_A(s_i^n) = \sum_{k=1}^{n} \lim_{i\to\infty} \mathbb{R}_A(s_{i+k}) = \sum_{k=1}^{n} \Omega = n\Omega.$$

On the other hand, if $n = 0$ then $s_j^0 = \emptyset$, for all $j \in \mathbb{N}$, and therefore

$$\lim_{i\to\infty} \mathbb{R}_A(s_i^0) = \lim_{i\to\infty} \mathbb{R}_A(\emptyset) = 0.$$

Hence, the thesis follows for all $n \in \mathbb{N}$. □

Lemma 3.10. $\lim_{i\to\infty} \mathbb{R}_A(s_i^{n,m}) = m\Omega^n$, *for all* $n, m \in \mathbb{N}$.

Proof. If $m \in \mathbb{N}^+$, then for all i and n in $\mathbb{N}$ we have

$$\begin{aligned}\mathbb{R}_A(s_i^{n,m}) &= \mathbb{R}_A(\{s_i^n, s_{i+1}^n, \dots, s_{i+m-1}^n\}) \\ &= \mathbb{R}_A(\{s_i^n\}) + \mathbb{R}_A(\{s_{i+1}^n\}) + \cdots + \mathbb{R}_A(\{s_{i+m-1}^n\}) = \sum_{k=0}^{m-1} 2^{-\mathbb{R}_A(s_{i+k}^n)}.\end{aligned}$$

Hence, by Lemma 3.9 and recalling that $\Omega = 2^{-\Omega}$, we have:

$$\lim_{i\to\infty} \mathbb{R}_A(s_i^{n,m}) = \sum_{k=0}^{m-1} \lim_{i\to\infty} 2^{-\mathbb{R}_A(s_{i+k}^n)} = \sum_{k=0}^{m-1} 2^{-n\Omega} = m\Omega^n.$$

On the other hand, if $m = 0$ then $s_j^{n,0} = \emptyset$, for all j and n in $\mathbb{N}$, and therefore

$$\lim_{i\to\infty} \mathbb{R}_A(s_i^{n,0}) = \lim_{i\to\infty} \mathbb{R}_A(\emptyset) = 0.$$

Hence, the thesis follows for all $m, n \in \mathbb{N}$. □

Next, after recalling when a set of positive reals is dense in $\mathbb{R}^+$, we prove that the set $\{m\Omega^n \mid n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}^+$.

Definition 11. A set $A \subseteq \mathbb{R}^+$ is *dense in* $\mathbb{R}^+$, if for all $b \in \mathbb{R}^+$ and $\varepsilon \in \mathbb{R}^+$ there exists $a \in A$ such that $|a - b| < \varepsilon$.

Lemma 3.11. *For every* $n_0 \in \mathbb{N}$*, the set* $A_{n_0} = \{m\Omega^{n_0+n} \mid n, m \in \mathbb{N}\}$ *is dense in* $\mathbb{R}^+$*.*

Proof. Let $n_0 \in \mathbb{N}$. To see that A_{n_0} is dense in $\mathbb{R}^+$, let α and ε be any positive reals. Then, there exists $n \in \mathbb{N}$ such that:

$$0 \leqslant \alpha - \frac{\lfloor \alpha\Omega^{-n_0-n} \rfloor}{\Omega^{-n_0-n}} < \varepsilon.$$

In fact, it is enough to take $n \in \mathbb{N}$ such that $\Omega^{n_0+n} \leqslant \varepsilon$ (we recall that $\Omega \approx 0.6411857 < 1$), and since $0 \leqslant \alpha\Omega^{-n_0-n} - \lfloor \alpha\Omega^{-n_0-n} \rfloor < 1$, we have

$$0 \leqslant \frac{\alpha\Omega^{-n_0-n} - \lfloor \alpha\Omega^{-n_0-n} \rfloor}{\Omega^{-n_0-n}} < \Omega^{n_0+n} \leqslant \varepsilon.$$

Hence, putting $m = \lfloor \alpha\Omega^{-n_0-n} \rfloor$, we have $0 \leqslant \alpha - m\Omega^{n_0+n} < \varepsilon$, proving that the set A_{n_0} is dense in $\mathbb{R}^+$. □

Lemma 3.12. *The* $\mathbb{R}_A$*-code of the cumulus* $h_\alpha^\infty = \bigcup_{i\in\mathbb{N}} h_\alpha^i = \bigcup_{i\in\mathbb{N}} h_{\alpha,i}$ *of the first set-theoretic approximating sequence* $\langle h_{\alpha,i} \rangle_{i\in\mathbb{N}}$ *of a given* $\alpha \in \mathbb{R}_0^+$ *is equal to* α*, i.e.,*

$$\mathbb{R}_A(h_\alpha^\infty) = \alpha.$$

Proof. In view of (6), it is enough to show that $\lim_{i\to\infty} \mathbb{R}_A(h_\alpha^i) = \alpha$.

If $\mathbb{R}_A(h_\alpha^{i_0}) = \alpha$ for some $i_0 \in \mathbb{N}$, then $h_\alpha^i = h_\alpha^{i_0}$ for all $i \geqslant i_0$, and so $\lim_{i\to\infty} \mathbb{R}_A(h_\alpha^i) = \alpha$ holds trivially.

Thus, let us assume that we have $\mathbb{R}_A(h_\alpha^i) < \alpha$ for all $i \in \mathbb{N}$.

We intend to show the existence of a function $c\colon \mathbb{N} \to \mathbb{N}$ such that the following inequalities

$$c(k) > k \qquad \text{and} \qquad \frac{\mathbb{R}_A(h^k_\alpha) + \alpha}{2} < \mathbb{R}_A(h^{c(k)}_\alpha) < \alpha \tag{7}$$

are verified for all $k \in \mathbb{N}$.

Thus, let k be any natural number in $\mathbb{N}$ and consider the set h^k_α, and let $n_0 = \mathsf{rk}(h^k_\alpha)$ and $\alpha' = \alpha - \mathbb{R}_A(h^k_\alpha)$. Then, from Lemma (3.11) there exist $n \geqslant n_0$ and $m \in \mathbb{N}^+$ such that

$$\frac{5\alpha'}{8} < m\Omega^n < \frac{7\alpha'}{8}.$$

In addition, from Lemma (3.10) there exists $i \in \mathbb{N}$ such that

$$|\mathbb{R}_A(s^{m,n}_i) - m\Omega^n| < \frac{\alpha'}{8}.$$

Hence, $\frac{\alpha'}{2} < \mathbb{R}_A(s^{m,n}_i) < \alpha'$, and since $n \geqslant \mathsf{rk}(h^k_\alpha)$ we have $s^{m,n}_i \cap h^k_\alpha = \emptyset$ too. Thus, $\mathbb{R}_A(h^k_\alpha \cup s^{m,n}_i) = \mathbb{R}_A(h^k_\alpha) + \mathbb{R}_A(s^{m,n}_i)$ and so

$$\frac{\mathbb{R}_A(h^k_\alpha) + \alpha}{2} = \mathbb{R}_A(h^k_\alpha) + \frac{\alpha'}{2} < \mathbb{R}_A(h^k_\alpha \cup s^{m,n}_i) < \mathbb{R}_A(h^k_\alpha) + \alpha' = \alpha.$$

Therefore, by Lemma (3.7), we have

$$\frac{\mathbb{R}_A(h^k_\alpha) + \alpha}{2} < \mathbb{R}_A(h^k_\alpha \cup s^{m,n}_i) \leqslant \mathbb{R}_A\left(h^{k+\mathbb{N}_A(s^{m,n}_i)}_\alpha\right) < \alpha.$$

By letting $c(k) = k + \mathbb{N}_A(s^{m,n}_i)$, it is immediate to check that the sought-after inequalities (7) hold.

Using the function $c\colon \mathbb{N} \to \mathbb{N}$ just defined, we can set forth a sequence $\langle n_i \rangle_{i\in\mathbb{N}}$ obeying the following recurrence:

$$\begin{cases} n_0 = 0 \\ n_{i+1} = c(n_i)\,, \quad \text{for } i \in \mathbb{N}. \end{cases}$$

Next, let us consider the subsequence $\langle h^{n_i}_\alpha \rangle_{i\in\mathbb{N}}$ of the cumulative sequence $\langle h^i_\alpha \rangle_{i\in\mathbb{N}}$. As is plain, $\lim_{i\to\infty} \mathbb{R}_A(h^{n_i}_\alpha) = \lim_{i\to\infty} \mathbb{R}_A(h^i_\alpha)$, hence to complete the proof it suffices to show that $\lim_{i\to\infty} \mathbb{R}_A(h^{n_i}_\alpha) = \alpha$.

Letting $r_i = \mathbb{R}_A(h^{n_i}_\alpha)$ for $i \in \mathbb{N}$, we have:

$$\begin{cases} r_0 = \mathbb{R}_A(h^{n_0}_\alpha) < \alpha \\ \dfrac{r_i + \alpha}{2} < r_{i+1} < \alpha, \quad \text{for } i \in \mathbb{N}. \end{cases} \tag{8}$$

Since $r_i < \alpha$, we have

$$r_i = \frac{r_i + r_i}{2} < \frac{r_i + \alpha}{2} < r_{i+1}, \quad \text{for } i \in \mathbb{N}.$$

Thus, the sequence $\langle r_i \rangle_{i\in\mathbb{N}}$ is strictly increasing and bounded above by α, and so it converges. Letting $\mathbf{r}$ be its limit, by (8) we have

$$\mathbf{r} \leqslant \frac{\mathbf{r}+\alpha}{2} \leqslant \mathbf{r},$$

which yields $\mathbf{r} = \alpha$.

In conclusion, we have

$$\mathbb{R}_A(h_\alpha^\infty) = \lim_{i\to\infty} \mathbb{R}_A(h_\alpha^i) = \lim_{i\to\infty} \mathbb{R}_A(h_\alpha^{n_i}) = \lim_{i\to\infty} r_i = \mathbf{r} = \alpha,$$

proving the lemma. □

3.3. On the Existence of HF^1-Approximations

We now state our final result and highlight its proof:

any $\alpha \in \mathbb{R}_0^+$ is the code of some set in HF^1.

Proposition 3.13. *For every $\alpha \in \mathbb{R}_0^+$, there exists $\hbar \in \mathsf{HF}^1$ such that*

$$\mathbb{R}_A(\hbar) = \alpha.$$

Proof sketch. Given $\alpha \in \mathbb{R}_0^+$, we provide a (possibly infinite) procedure to build a set $\hbar_\alpha \in \mathsf{HF}^1$ such that $\mathbb{R}_A(\hbar_\alpha) = \alpha$.

If $\alpha = 0$, then $\mathbb{R}_A(\emptyset) = 0$, and we are done. Otherwise, if $\alpha \in \mathbb{R}^+$, let $\langle h_{\alpha,i} \rangle_{i\in\mathbb{N}}$ be the first set-theoretic approximation for α, $\langle h_\alpha^i \rangle_{i\in\mathbb{N}}$ be its cumulative variant.

We define two (possibly finite) sequences

$$\alpha_0,\ \alpha_1,\ \alpha_2,\ \ldots \qquad \text{and} \qquad p_0,\ p_1,\ p_2,\ \ldots \tag{9}$$

of positive reals and related indices, respectively, by putting:

- $\alpha_0 = \alpha$,

and, for $i \in \mathbb{N}$,

- $\alpha_{i+1} = -\log(\alpha_i - \mathbb{R}_A(h_{\alpha_i}^{p_i}))$,

  where p_i is the least integer in $\mathbb{N}$ such that:

  - $\alpha_i - \mathbb{R}_A(h_{\alpha_i}^{p_i}) \leqslant 1$ and
  - for all $j \leqslant p_i$, it holds that $\alpha_i - \mathbb{R}_A(h_{\alpha_i}^{p_i}) \neq \mathbb{R}_A(h_{\alpha_i,j})$,

  provided that $\alpha_i - \mathbb{R}_A(h_{\alpha_i}^{p_i}) \neq 0$; otherwise α_{i+1} is not defined and the sequences are terminated.[8]

[8] Plainly, $\langle h_{\alpha_i}^j \rangle_{j\in\mathbb{N}}$ is the cumulative set-theoretic approximating sequence for α_i.

When the sequences (9) are finite, say equal to $\langle \alpha_0,\ \alpha_1,\ \ldots,\ \alpha_k \rangle$ and to $\langle p_0,\ p_1,\ \ldots,\ p_k \rangle$ for some $k \in \mathbb{N}$, respectively, we put

$$\begin{cases} H_k = h^{p_k}_{\alpha_k}, \\ H_i = h^{p_i}_{\alpha_i} \cup \{H_{i+1}\}, \quad \text{for } i < k. \end{cases}$$

It can be shown that

$$H_j \in \mathsf{HF} \qquad \text{and} \qquad \mathbb{R}_A(H_j) = \alpha_j,$$

for $j = 0, 1, \ldots, k$. Hence,

$$\mathbb{R}_A(H_0) = \alpha_0 = \alpha.$$

In the case in which the sequences (9) are infinite, we need to step into the universe HF^1. Let us consider the infinite set system

$$x_i = h^{p_i}_{\alpha_i} \cup \{x_{i+1}\}, \qquad \text{for } i \in \mathbb{N}. \tag{10}$$

Letting $\langle \hbar_i \rangle_{i \in \mathbb{N}}$ be the solution in HF^1 of (10), we have

$$\mathbb{R}_A(\hbar_i) = \alpha_i, \qquad \text{for } i \in \mathbb{N},$$

and, in particular,

$$\mathbb{R}_A(\hbar_0) = \alpha_0 = \alpha.$$

This is a consequence of the following facts:

- We can build an infinite sequence $\mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_i \subset \cdots$ of finitely-branching finite trees, where $\mathcal{T}_i$ can be seen as the picture of the hereditarily finite set $\hbar_i$, truncated by replacing $\hbar_{i+1}$ by $\emptyset$ in the definition of $\hbar_i$.

  The limit for i that goes to infinity of $\mathcal{T}_i$ is a picture $\mathcal{T}$ of $\hbar_0$ (including pictures of $\hbar_i$, for $i > 0$).

- Every finite system of set-theoretic equations induced by $\mathcal{T}_i$, for $i \geqslant 0$, introduces also hereditarily finite sets, $\hbar^j_k$, for $0 < k \leqslant j \leqslant i$, that are increasingly more complete set-theoretic approximations of $\hbar_k$, for $k \leqslant i$.

- The codes $\mathbb{R}_A(\hbar^j_i)$, for $i \leqslant j$, approximate α_i. That is $\lim_{j \to \infty} \mathbb{R}_A(\hbar^j_i) = \alpha_i$. □

4. Conclusions

In this paper we investigated a sort of set-theoretic counterpart of the apparatus (as called by Khinchin in [8]) of continued fractions. Instrumental in establishing a parallel between the continued fraction approach to denote real numbers is the notion of the code $\mathbb{R}_A$, which assigns a real number to every hyperset in HF^1.

The two, rather initial, results we proved are the possibility of capturing every positive real α as the code of an infinite set H of hereditarily sets, and — building on the previous result — the fact that there exists $\hbar \in \mathsf{HF}^1$ such that $\mathbb{R}_A(\hbar) = \alpha$.

The approximation defined here as "first" is, in fact, coarser than necessary — and hence not "best". E.g., consider the case of $h \in \mathsf{HF}$, $\mathbb{N}_A(h)$ even, and $\alpha = \mathbb{R}_A(h) > 1$. According to Definition 10, $h_{\alpha,0} = \{\emptyset\}$ will be a subset of the first HF^1-approximation of α: this prevents the possibility of producing h as a (correct) set-theoretic approximation of α, since $\emptyset \notin h$ follows from the fact that $\mathbb{N}_A(h)$ is even.

There are still many intriguing questions that are yet to be answered, including the following:

Open Question (Recurrence).
Can we find a recursive relation providing the hereditarily finite set $h^{p_i}_{\alpha_i}$ in terms of previous hereditarily finite sets $h^{p_{i-1}}_{\alpha_{i-1}}, h^{p_{i-2}}_{\alpha_{i-2}}$, in case H_i is infinite?

The sketched proof of Proposition 3.13 bears a strong similarity with the proof of the construction of a (regular) continued fraction for a given real number. A positive answer to the above question would provide a recurrence relation that could be seen as the set-theoretic counterpart of what Khinchin calls "the rule for the formation of the convergents" (mentioned in Section 2.1, see also Theorem 1 in [8]):

$$p_k = a_k p_{k-1} + p_{k-2},$$
$$q_k = a_k q_{k-1} + q_{k-2}.$$

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