On Newman's Lemma and Non-termination

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Abstract

We give novel confluence criteria and examples that can be used by researchers in computer science and software engineering to better recognize situations where the local confluence property of non-terminating abstract rewriting systems is and is not equivalent to the confluence property. We also describe a formalization of the main results in Isabelle proof assistant software.

Keywords [1]

abstract rewriting system, confluence, proof assistant, formal proof

1. Introduction

Newman's lemma [1] is often considered as one of the cornerstones of the rewriting theory [2-6] – a part of foundations of automated reasoning and formal specification and verification of software. This lemma is usually included in computer science and software engineering curricula and can be found in various textbooks. In particular, it can be used in presentations of Knuth-Bendix completion algorithm [7] and Buchberger's algorithm [8]. In modern reformulations, Newman's lemma establishes the equivalence between the (global) confluence and local confluence properties for terminating (Noetherian or strongly normalizing in alternative terminology) abstract rewriting systems (ARS). Although other confluence criteria applicable to wider classes of ARS are available, e.g. the decreasing diagrams method [9] that is known to be complete for the class of ARS with cofinality property [10, 11], Newman's lemma has its own advantages, e.g. abstract treatment of both elements and reductions and no use of auxiliary labelings of elements or reductions.

Taking into account usefulness of Newman's lemma, it is natural to question whether its termination assumption can be relaxed to make the lemma more widely applicable. A formal answer to this question is "yes": the class of all ARS for which confluence and local confluence are equivalent (the union of classes of confluent and not locally confluent ARS) is wider than the class of all terminating ARS (the remaining question is how to make sure that a given, possibly non-terminating system belongs to the former class). However, in the literature on rewriting systems that mention Newman's lemma the termination assumption is quite often presented as a necessary condition for the validity of the lemma's conclusion, e.g.: "*The following examples illustrate that the requirement of noetherianity is necessary to prove confluence from local confluence. …*'' [5, paragraph 1.4.2], "… *For the relations that are not noetherian, much stronger local hypotheses are necessary to yield confluence.*" [2, paragraph 2.2]. In our opinion, such informal statements may unintentionally create an impression that the local confluence property cannot be used in place of the confluence property in any situation where a system under consideration is not terminating (or its termination cannot be established with certainty), although this is not the case.

In this work we give novel results that can help one to better recognize situations where the local confluence property of abstract rewriting systems is and is not equivalent to the confluence property. We also describe formalizations of the main theorems (Theorems 1, 2, 3 from Section 4) in Isabelle proof assistant software [12, 13] using HOL logic (Higher-Order Logic). This work continues the previous works [14] and [15] about confluence proof methods for non-terminating ARS. To the best of our knowledge, proofs of the results given in Section 4 (Main results) below have not been published previously.

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2. Terminology and notation

In this work we will focus on abstract rewriting systems with single reduction relation (without auxiliary labelings of elements or reductions).

We will give novel confluence criteria based on local confluence for special classes of such systems. We will assume that the Axiom of Choice holds and will freely use its corollaries (e.g. Zorn's lemma).

We will use the following (mostly standard) definitions of basic notions that are consistent with the terminology used in many works on the rewriting theory [1-5].

An *abstract rewriting system* (ARS) is a pair $(X, \to)$ of a set (X) and a binary relation $(\to)$ on it called *reduction*. We will denote as $\to^*$ the reflexive transitive closure of $\to$ (on X), and as $\to^+$ the transitive closure of $\to$.

An element $x \in X$ is called *reducible* (in $(X, \to)$), if there exists $y \in X$ such that $x \to y$, and is called *irreducible* (in $(X, \to)$), if it is not reducible in $(X, \to)$.

A *normal form* of an element $x \in X$ (in an *ARS* $(X, \to)$) is an element $y \in X$ such that $x \to^* y$ and y is irreducible in $(X, \to)$.

An ARS $(X, \to)$

- has the *diamond property*, if for all $a, b, c \in X$, if $a \to b$ and $a \to c$, then there exists $d \in X$ such that $b \to d$ and $c \to d$;
- is *confluent*, if for all $a, b, c \in X$, if $a \to^* b$ and $a \to^* c$, then there exists $d \in X$ such that $b \to^* d$ and $c \to^* d$;
- is *locally confluent*, if for all $a, b, c \in X$, if $a \to b$ and $a \to c$, then there exists $d \in X$ such that $b \to^* d$ and $c \to^* d$.

Note that $\to^*$ is a *preorder* (a binary relation that is reflexive and transitive) and the pair $(X, \to^*)$ can be considered as a *preordered set.*

If $(P, \leq)$ is a preordered set, $A \subseteq P$ and $x \in P$, then

- x is a *least* element of A (w.r.t. $\leq$), if $x \in A$ and $x \leq a$ for all $a \in A$;
- x is a *greatest* element of A (w.r.t. $\leq$), if $x \in A$ and $a \leq x$ for all $a \in A$;
- x is a *minimal* element of A (w.r.t. $\leq$), if $x \in A$ and there is no element $y \in A$ such that $y \leq x$ and $\neg (x \leq y)$;
- x is an *upper bound* of A (in $(P, \leq)$), if $a \leq x$ for all $a \in A$;
- x is a *least upper bound* of A (in $(P, \leq)$), if x is a least element of the set of all upper bounds of A in $(P, \leq)$.

In any preordered set $(P, \leq)$ a subset $A \subseteq X$ is

- a *chain*, if every $x, y \in A$, $x \leq y$ or $y \leq x$;
- a *directed set*, if $A \neq \varnothing$ and for every $x, y \in A$ there exists $u \in A$ such that $x \leq u$ and $y \leq u$;
- *closed* [14], if for every nonempty chain C in $(P, \leq)$ and every $x \in P$, if $C \subseteq A$ and x is a least upper bound of C, then x in A ;
- *open* [14], if the set $P \backslash A$ is closed.

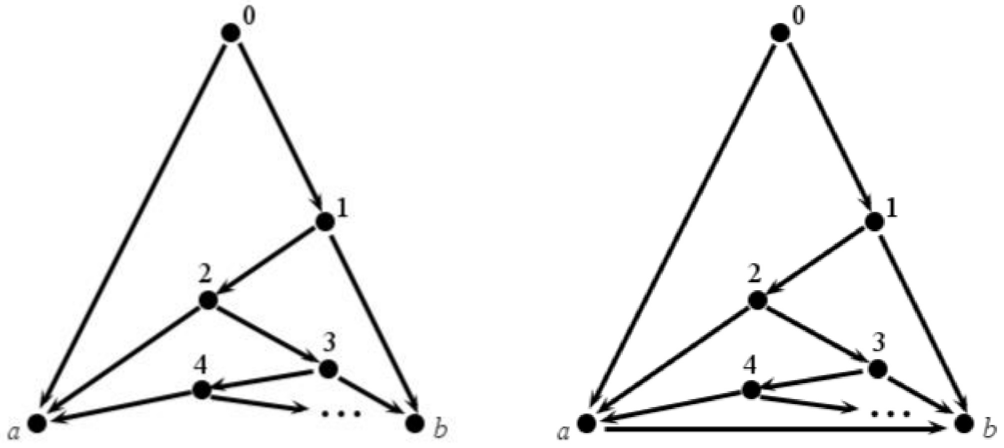
An ARS $(X, \to)$

- is *weakly normalizing*, if each element $x \in X$ has a normal form ;
- is *countable*, if the set X is at most countable, i.e. there exists a total injective function from X to the set of natural numbers ;
- is *terminating*, if there is no infinite sequence $x_1 \to x_2 \to x_3 \to \ldots$, where $x_i \in X$ for all $i=1, 2, \ldots$;
- is *acyclic*, if there is no element $x \in X$ such that $x \to^+ x$;
- is *inductive* [2], if for every infinite sequence $x_1 \to x_2 \to x_3 \to \ldots$ there exists $x \in X$ such that $x_i \to^* x$ for all $i=1, 2, 3, \ldots$ (note that for such a sequence $x_1 \to x_2 \to x_3 \to \ldots$, the set $\{x_1, x_2, x_3, \ldots\}$ is a chain in the preordered set $(X, \to^*)$ and x is an upper bound of this chain in $(X, \to^*)$) ;
- is *strictly inductive* [14-16], if every nonempty chain in the preordered set $(X, \to^*)$ has a least upper bound (note that a strictly inductive ARS is inductive).

3. Motivating example

In accordance with Newman's lemma, any terminating locally confluent ARS is confluent. Any terminating ARS is acyclic, however, generally, an acyclic locally confluent ARS need not be confluent. An example due to Newman [2, Figure 6a] shown in Figure 1(a) below is often used to illustrate the latter fact.

Another widely known example due to Hindley [2, Figure 6b] shows that a finite locally confluent ARS that is not required to be acyclic can be non-confluent.



(a) (b)

Figure 1: Left (a): Newman's counterexample [2. Figure 6a]: inductive, but not strictly inductive. Right (b): an example of a strictly inductive ARS.

Newman's counterexample shown in Figure 1(a) can be formalized as $(X_{\mathrm{NC}}, \to_{\mathrm{NC}})$, where

$$X_{NC} = \{a, b, 0, 1, 2, \dots\},\ a = -1,\ b = -2,$$

and the relation $\to_{\mathrm{NC}}$ is defined as follows:

$$\to_{\mathrm{NC}} = \{ (2n, a) \mid n = 0, 1, 2, \dots \} \cup \{ (2n+1, b) \mid n = 0, 1, 2, \dots \} \cup \{ (n, n+1) \mid n = 0, 1, 2, \dots \}.$$

The ARS $(X_{\mathrm{NC}}, \to_{\mathrm{NC}})$ is countable, acyclic, inductive, locally confluent, but is *not* confluent [2]. The reason of failure of the equivalence between local confluence and confluence conditions for $(X_{\mathrm{NC}}, \to_{\mathrm{NC}})$ is intuitively quite clear: confluence implies that the set of upper bounds of any nonempty subset of X_{NC} is directed in the preordered (actually, partially ordered) set $(X_{\mathrm{NC}}, (\to_{\mathrm{NC}})^*)$, however, the local confluence condition is not sufficient to determine whether this is the case for the set of upper bounds of the infinite chain $\{0, 1, 2, \dots\}$ in $(X_{\mathrm{NC}}, (\to_{\mathrm{NC}})^*)$, in particular, the local confluence condition does not imply the existence of an element x such that $\{ (a, x), (b, x) \} \subseteq (\to_{\mathrm{NC}})^*$.

However, note that in the preordered set $(X_{\mathrm{NC}}, (\to_{\mathrm{NC}})^*)$ the chain $\{1, 2, 3, \dots\}$ has minimal upper bounds (a and b), but has no least upper bound. If the chain $\{1, 2, 3, \dots\}$ had a least upper bound (or no upper bounds), the above mentioned explanation of the possibility of inequivalence of local confluence and confluence would not apply. For example, consider an ARS shown in Figure 1(b) that can be formalized as $(X_{\mathrm{NC}}, \to_{\mathrm{NC}} \cup \{(a, b)\})$. Unlike Newman's counterexample, it is a strictly inductive ARS, i.e. all nonempty chains in the sense of the preorder generated by the reflexive transitive closure of the reduction relation have least upper bounds. Indeed, in the next section we will show rigorously that any acyclic strictly inductive ARS with at most countable set of irreducible elements is confluent if and only if it is locally confluent. We will also show that countability and acyclicity assumptions here are important and cannot be simply omitted from this proposition. Besides, Newman's counterexample implies that strict inductivity cannot be weakened to inductivity in this proposition since $(X_{\mathrm{NC}}, \to_{\mathrm{NC}})$ is acyclic, inductive, has a finite set of irreducible elements, is locally confluent, but is not confluent. These results complement Newman's lemma and clarify the limits of usefulness of the local confluence property as a condition in confluence criteria.

4. Main results

Below we give formulations and proofs of the main results. One can assume that a background theory for understanding them is ZFC.

Theorem 1. *Let $(X, \to)$ be an acyclic strictly inductive ARS such that the set of all its irreducible elements is at most countable.*

Then $(X, \to)$ is confluent if and only if $(X, \to)$ is locally confluent.

Proof.

Any confluent ARS is locally confluent, so it is sufficient to show that if $(X, \to)$ is locally confluent, then $(X, \to)$ is confluent.

Assume that the ARS $(X, \to)$ is locally confluent. Note that since the ARS $(X, \to)$ is acyclic, the relation $\to^*$ on X is antisymmetric: if $x \to^* y$ and $y \to^* x$, then $x = y$ (otherwise, $x \to^+ x$ which contradicts acyclicity). Then $(X, \to^*)$ is not only a preordered set, but is a partially ordered set (poset).

Let us introduce the following notation:

- IR is the set of all irreducible elements of $(X, \to)$;
- $NF(x)$, where $x \in X$, is set of all normal forms of x ;
- ND is the set of all $x \in X$ such that the set $NF(x)$ contains at least two distinct elements.

Let us show that $(X, \to)$ is a weakly normalizing ARS. Let $x \in X$ and $U = \{ y \in X \mid x \to^* y \}$. It is straightforward to show that since $(X, \to)$ is a strictly inductive acyclic ARS, the set U, considered as an induced subposet of $(X, \to^*)$, is a nonempty inductive poset [16] (i.e. a poset where every nonempty chain has an upper bound). Then by Zorn's lemma, U has some maximal element $y_{\max}$ with respect to the order $\to^* \cap (U \times U)$. Such an $y_{\max}$ is also maximal in the poset $(X, \to^*)$, so $y_{\max} \in IR$ (since $(X, \to)$ is acyclic) and $x \to^* y_{\max}$, so $y_{\max} \in NF(x)$. Since $x \in X$ is arbitrary, the ARS $(X, \to)$ is weakly normalizing.

Now let us show that for each $x \in ND$ and $b \in IR$ there exists $y \in ND$ such that $x \to^* y$ and $b \notin NF(y)$. Let us denote $U = \{ y \in X \mid x \to^* y \}$ and $D = \{ y \in U \mid y \to^* b \}$. Denote as $\leq$ the relation $\to^* \cap (U \times U)$. Then $(U, \leq)$ is a (nonempty) induced subposet of $(X, \to^*)$.

Let us fix $x \in ND$ and $b \in IR$ and consider the following 2 cases.

Case 1: $x \to^* b$ does not hold. Then for $y = x$ we have $y \in ND$, $x \to^* y$ and $b \notin NF(y)$.

Case 2: $x \to^* b$ holds. Since $x \in ND$, the set $NF(x) \backslash \{b\}$ is nonempty. Let m be some element of $NF(x) \backslash \{b\}$. Then $x \to^* m$, so $m \in U$. Note that $m \notin D$ since otherwise, $m \to^* b$, and also $m \in IR$, whence $m = b$ which contradicts the fact that $m \in NF(x) \backslash \{b\}$. Thus $m \in U \backslash D$. Let $E = D \cap \{ y \in X \mid y \to^* m \}$. It is straightforward to check that E is a closed set in the poset $(U, \leq)$ (in the sense of Section 2). Moreover, $x \in E$ (because $x \to^* b$ and $x \to^* m$), so $E \neq \varnothing$. Then it is straightforward to check that $(E, \leq \cap (E \times E))$ is a nonempty inductive poset (i.e. a poset where every nonempty chain has an upper bound), so by Zorn's lemma, E has a maximal element $x_{\max}$ with respect to the order $\leq \cap (E \times E)$. Then $x_{\max} \in E$, so $x_{\max} \to^* b$ and $x_{\max} \to^* m$. Then we have $x_{\max} \neq b$ and $x_{\max} \neq m$, because otherwise, $b \to^* m$ or $m \to^* b$ holds, whence $b = m$ since $b \in IR$ and $m \in IR$, which contradicts the fact that $m \in NF(x) \backslash \{b\}$. Then $x_{\max} \to^+ b$ and $x_{\max} \to^+ m$, so there exist $e_1, e_2 \in X$ such that $x_{\max} \to e_1 \to^* b$ and $x_{\max} \to e_2 \to^* m$. Since $(X, \to)$ is a locally confluent ARS, there exists $d \in X$ such that $e_1 \to^* d$ and $e_2 \to^* d$.

Let us show that $e_2 \in ND$.

- Suppose that $e_2 \notin ND$. Note that $e_2 \to^* m$ and $m \in IR$, so $m \in NF(e_2)$. Since $(X, \to)$ is weakly normalizing (as we have shown above) and $e_2 \notin ND$, $NF(e_2)$ is a singleton set, so $NF(e_2) = \{m\}$. Moreover, $e_2 \to^* d$ and $NF(d) \neq \varnothing$, so $m \in NF(d)$. Then $x \to^* x_{\max} \to e_1 \to^* b$, and $e_1 \to^* d \to^* m$, so $e_1 \in E$. Then $x_{\max} \leq e_1$ and $e_1 \in E$. Since $x_{\max}$ is maximal in E with respect to $\leq \cap E \times E$, we have $x_{\max} = e_1$. Then $x_{\max} \to x_{\max}$ and we get a contradiction with the theorem's assumption that $(X, \to)$ is an acyclic ARS.

Thus $e_2 \in ND$. Let us show that $b \notin NF(e_2)$.

- Suppose that $b \in NF(e_2)$. Then $x \to^* x_{\max} \to e_2 \to^* b$ and $e_2 \to^* m$, so $e_2 \in E$. Then $x_{\max} \leq e_2$ and $e_2 \in E$. Since $x_{\max}$ is maximal in E with respect to $\leq \cap E \times E$, we have $x_{\max} = e_2$. Then $x_{\max} \to x_{\max}$ and we get a contradiction with the theorem's assumption that $(X, \to)$ is an acyclic ARS.

Note that $x \to^* x_{\max} \to e_2$, so there exists $y = e_2 \in ND$ such that $x \to^* y$ and $b \notin NF(y)$.

We conclude that indeed, for each $x \in ND$ and $b \in IR$ there exists $y \in ND$ such that $x \to^* y$ and $b \notin NF(y)$.

Now let us show by contradiction that $ND = \varnothing$. Suppose that $ND \neq \varnothing$. Then IR is a nonempty, at most countable set (IR is at most countable by theorem's assumption), so IR is an image of a total function on the set of all non-negative integers $\mathbf{N}_0 = \{ 0, 1, 2, \ldots \}$, so there exists an infinite sequence $f_n \in X$, $n = 0, 1, 2, \ldots$ such that $IR = \{ f_i \mid i \in \mathbf{N}_0 \}$ (note that elements in the sequence f_n, $n = 0, 1, 2, \ldots$ can repeat). As we have shown above, for each $x \in ND$ and $b \in IR$ there exists $y \in ND$ such that $x \to^* y$ and $b \notin NF(y)$. Then for each $x \in ND$ and $i \in \mathbf{N}_0$ there exists $y \in ND$ such that $x \to^* y$ and $f_i \notin NF(y)$. Then since $ND \neq \varnothing$, by the principle of dependent choice there exists an infinite sequence $g_n \in ND$, $n=0,1,2,\ldots$ such that for each $n=0,1,2,\ldots$, $g_n \to^* g_{n+1}$ and $f_n \notin NF(g_{n+1})$. Then the set $\{ g_n \mid n \in \mathbf{N}_0 \}$ is a non-empty chain in the poset $(X, \to^*)$, so it has a least upper bound in $(X, \to^*)$ since $(X, \to)$ is a strictly inductive ARS. Denote as u the least upper bound of $\{ g_n \mid n\in\mathbf{N}_0 \}$ in the poset $(X, \to^*)$. As we have shown above, $(X, \to)$ is a weakly normalizing ARS, so $NF(u) \neq \varnothing$. Let m be some element of $NF(u)$.Then $m \in IR$, so there exists $k\in\mathbf{N}_0$ such that $m = f_k$. Note that $g_{k+1} \to^* u$ (since u is an upper bound of the set $\{ g_n \mid n\in\mathbf{N}_0 \}$) and $u\to^*m$, so $g_{k+1}\to^*m$. Then $m \in NF(g_{k+1})$, since $m \in IR$. On the other hand, $m = f_k \notin NF(g_{k+1})$ by the definition of the sequence g_n , $n = 0, 1, 2, \ldots$, so we have a contradiction: $m\in NF(g_{k+1})$ and $m\notin NF(g_{k+1})$. Thus the assumption $ND \neq \varnothing$ is false and we conclude that $ND = \varnothing$.

Since $ND = \varnothing$, each $x\in X$ has at most one normal form in the ARS $(X, \to)$. As we have shown above, $(X, \to)$ is a weakly normalizing ARS, so each $x\in X$ has exactly one normal form. This straightforwardly implies that the ARS $(X, \to)$ is confluent. □

Note that

- the ARS shown in Figure 1(a) *does not* satisfy the assumptions of Theorem 1: although it is inductive, it is *not strictly* inductive ;
- the ARS shown in Figure 1(b) satisfies the assumptions of Theorem 1.

Corollary 1. *Let* $(X, \to)$ *be an acyclic strictly inductive ARS where each element has at most countable set of normal forms. Then* $(X, \to)$ *is confluent if and only if* $(X, \to)$ *is locally confluent.*

Proof.

Any confluent ARS is locally confluent, so it is sufficient to show that if $(X, \to)$ is locally confluent, then $(X, \to)$ is confluent.

Assume that the ARS $(X, \to)$ is locally confluent. Let $a, b, c \in X$, $a\to^*b$, and $a\to^*c$. Let $U = \{ u \in X \mid a \to^* u \}$. It is straightforward to check that $(U, \to \cap (U\times U))$ is an acyclic strictly inductive locally confluent ARS, where each irreducible element is a normal form of a in $(X, \to)$. Since the set of normal forms of a in $(X, \to)$ is at most countable, the ARS $(U, \to \cap (U\times U))$ is confluent by Theorem 1. Note that $a, b, c \in U$, so there exists $d \in U \subseteq X$ such that (b, d) and (c, d) belong to the reflexive transitive closure of $\to \cap (U\times U)$, so $b\to^*d$ and $c\to^*d$.

Thus we conclude that $(X, \to)$ is confluent. □

Corollary 2. *Let* $(X, \to)$ *be a countable acyclic ARS such that for every infinite reduction sequence* $x_1 \to x_2 \to x_3 \to \ldots$ *the set* $\{x_1, x_2, x_3, \ldots \}$ *has a least upper bound in the poset* $(X,\to^*)$.
Then $(X, \to)$ *is confluent if and only if* $(X, \to)$ *is locally confluent.*

Proof.

Note that since $(X, \to)$ is acyclic, $(X,\to^*)$ is indeed a poset (not just a preordered set).

Let us check that $(X, \to)$ is a strictly inductive ARS. Let C be a nonempty chain in the poset $(X,\to^*)$. Let c_0 be some element of C (it exists since C is nonempty). Let $\to_C$ be a binary relation on C such that for all $x, y \in C$, $x \to_C y$ if and only if $x\to^*y$ (note that $\to^*$ is considered as a binary relation of X, but $\to_C$ is a binary relation on $C \subseteq X$). Note that the ARS $(C, \to_C)$ has the diamond property, because whenever $a, b, c \in C$, $a \to_C b$, and $a \to_C c$, we have $b\to^*c$ or $c\to^*b$ (since C is a chain in the poset $(X,\to^*)$), and in either case there exists $d \in \{c, b\} \subseteq C$ such that $b\to_C d$ and $c\to_C d$. Also, $(C, \to_C)$ is a countable ARS, because $(X, \to)$ is a countable ARS. Since any ARS with diamond property is confluent [5, Remark 1.2.3] and any countable confluent ARS has the cofinality property [10], the ARS $(C, \to_C)$ has the cofinality property. This implies that for c_0 there exists a finite or infinite reduction sequence $c_0 \to^* c_1 \to^* c_2 \to^* \ldots$, where $c_i \in C$ for all i, such that for every $c\in C$, if $c_0 \to^* c$, then $c \to^* c_k$ for some index k. The latter actually implies that for every $c\in C$ there exists k such that $c \to^* c_k$, because C is a chain $(X,\to^*)$. Denote $A = \{ c_1, c_2, c_3, \ldots\}$. Then there exists a nonempty finite or infinite reduction sequence $x_1 \to x_2 \to x_3 \to \ldots$, where $x_i \in X$ for all i, such that $A \subseteq \{ x_1, x_2, x_3, \ldots\}$ and for every $x \in \{ x_1, x_2, x_3, \ldots\}$ there exists $c \in A$ such that $x \to^* c$. Denote $B = \{ x_1, x_2, x_3, \ldots\}$. Then for every $c\in C$ there exists $y \in A \subseteq B$ such that $c \to^* y$.

Note that if the set B is finite, it is a nonempty finite chain in the poset $(X,\to^*)$, so it has a greatest element which is its least upper bound, and if B is an infinite set, then $x_1 \to x_2 \to x_3 \to \ldots$ is an infinite reduction sequence, so B has a least upper bound in the poset $(X,\to^*)$ by assumption. Thus in any case there exists $u \in X$ such that u is a least upper bound of B in $(X,\to^*)$.

Note that u is an upper bound of C in $(X,\to^*)$, because for each $c \in C$ there exists $y \in B$ such that $c \to^* y$ and $y \to^* u$. Moreover, u is a least upper bound of C, because if x is an upper bound of C in

$(X,\to^*)$, then for each $y \in B$ there exists $c \in A \subseteq C$ such that such that $y \to^* c \to^* x$, so x is an upper bound of B in $(X,\to^*)$, whence $u \to^* x$. Thus C has a least upper bound in $(X,\to^*)$.

We conclude that $(X, \to)$ is a strictly inductive ARS. Moreover, it is acyclic by assumption. Since $(X, \to)$ is a countable ARS, the set X is (at most) countable, so the set of irreducible elements in $(X, \to)$ is (at most) countable. Then $(X, \to)$ is confluent if and only if $(X, \to)$ is locally confluent by Theorem 1. □

Hindley's counterexample [2, Figure 6b] shows that the acyclicity assumption *cannot* be simply omitted from the statement of Theorem 1, e.g. the ARS

$$(\{ 0, 1, 2, 3 \}, \{ (1, 0), (2, 3), (1, 2), (2, 1) \})$$

is strictly inductive, has a finite set of irreducible elements, is locally confluent, but is *not* confluent.

The following result shows that the countability assumption also *cannot* be simply omitted from the statement of Theorem 1. The corresponding ARS is illustrated in Figure 2. We call it a *strengthened* Newman's counterexample, because the usual Newman's counterexample (Figure 1(a)) is inductive, but is not *strictly* inductive.

Theorem 2 (*Strengthened Newman's counterexample*).
Let $P = \mathbf{R} \times \mathbf{R} \times (\omega + 1)$, *where* $\mathbf{R}$ *is the set of real numbers and* ω *is the first infinite ordinal*.
Let $\to$ *be a binary relation on* P *such that* $(x, y, n) \to (x', y', m)$ *if and only if one of the following two conditions holds*:

1. $n < \omega$ *and* $m = n + 1$ *and* $| x' - x | < y'/2^m$ *and* $0 < y' < y$;
2. $n < \omega$ *and* $m = \omega$ *and* $x' = x$ *and* $0 = y' < y$.

Then $(P, \to)$ *is an acyclic, strictly inductive, locally confluent ARS that is not confluent*.

Proof.

Since $(x, y, n) \to (x', y', m)$ implies that $n < m$, the relation $(x, y, n) \to^+ (x, y, n)$ does not hold for any $(x, y, n) \in P$, so the ARS $(P, \to)$ is acyclic.

It is straightforward to show that $(x, y, n) \to^+ (x', y', m)$ if and only if one of the following two conditions holds:

(T1) $n < \omega$ and $m < \omega$ and $n < m$ and $| x' - x | < y/2^n - y/2^{m-1} + y'/2^m$ and $0 < y' < y$;

(T2) $n < \omega$ and $m = \omega$ and $| x' - x | < y/2^n$ and $0 = y' < y$.

Then the ARS $(P, \to)$ is not confluent, because if $a = (0, 2, 0)$, $b = (-1, 0, \omega)$, $c = (1, 0, \omega)$, then $a \to^+ b$ and $a \to^+ c$ by (T2), however, by the definition of $\to$, the elements b and c are irreducible, and since they are distinct, the relations $b \to^* d$ and $c \to^* d$ cannot both hold for any $d \in P$.

Let us check that the ARS $(P, \to)$ is locally confluent.
Let $a = (x_1, y_1, n)$, $b = (x_2, y_2, m)$, $c = (x_3, y_3, k) \in P$ be elements such that $a \to b$ and $a \to c$. Then the definition of $\to$ implies that $n < \omega$. Let $d = (x_1, 0, \omega)$.
Consider the following cases.

- If $m < \omega$, then $| x_2 - x_1 | < y_2/2^m$ and $y_2 > 0$, because $(x_1, y_1, n) = a \to b = (x_2, y_2, m)$, whence $b = (x_2, y_2, m) \to^+ (x_1, 0, \omega) = d$ by (T2).
- If $m = \omega$, then $x_2 = x_1$ and $y_2 = 0$, because $(x_1, y_1, n) = a \to b = (x_2, y_2, m)$, whence $b = (x_2, y_2, m) = (x_1, 0, \omega) = d$.

In both cases we have $b \to^* d$. Similarly, consider the following cases.

- If $k < \omega$, then $| x_3 - x_1 | < y_3/2^k$ and $y_3 > 0$, because $(x_1, y_1, n) = a \to c = (x_3, y_3, k)$, whence $c = (x_3, y_3, k) \to^+ (x_1, 0, \omega) = d$ by (T2).
- If $k = \omega$, then $x_3 = x_1$ and $y_3 = 0$, because $(x_1, y_1, n) = a \to c = (x_3, y_3, k)$, whence $c = (x_3, y_3, k) = (x_1, 0, \omega) = d$.

In both cases we have $c \to^* d$. Thus both $b \to^* d$ and $c \to^* d$ hold.

Since a, b, c are arbitrary, we conclude that $(P, \to)$ is a locally confluent ARS.

Finally, let us check that the ARS $(P, \to)$ is strictly inductive.

Let C be a nonempty chain in the poset $(P, \to^*)$ and let

$$N = \{ n < \omega \mid \exists x \exists y \, (x, y, n) \in C \} .$$

Consider the following three cases.

Case 1: there exist $x_1, y_1 \in \mathbf{R}$ such that $(x_1, y_1, \omega) \in C$. Then using (T1), (T2) is straightforward to check that (x_1, y_1, ω) is a greatest element in C with respect to $\rightarrow^*$, so it is a least upper bound of C in $(P, \rightarrow^*)$.

Case 2: the set N is finite and $(x, y, \omega) \notin C$ for all $x, y \in \mathbf{R}$. Then the set N is nonempty (since C is nonempty). Let k_m be the greatest element of N and $x_1, y_1 \in \mathbf{R}$ be elements such that $(x_1, y_1, k_m) \in C$. Then using (T1), (T2) it is straightforward to check that (x_1, y_1, k_m) is a greatest element in C with respect to $\rightarrow^*$, so it is a least upper bound of C in $(P, \rightarrow^*)$.

Case 3: the set N is infinite and $(x, y, \omega) \notin C$ for all $x, y \in \mathbf{R}$. Let

$$Y = \{\, y \in \mathbf{R} \mid \exists\, x\ \exists\, n < \omega\ (x, y, n) \in C \,\}\ .$$

Note that if k_0 is the least element of N and x_0, y_0 are such that $(x_0, y_0, k_0) \in C$, then $y \le y_0$ for all $y \in Y$. This implies that the set Y is empty or bounded from above. Then let y_m be a positive real number such that $y \le y_m$ for all $y \in Y$. This implies that whenever $n < \omega$, $m < \omega$, $(x, y, n) \in C$, and $(x, y, n) \rightarrow^+ (x', y', m)$, the following holds:

$$|\, x' - x \,| < y/2^n - y/2^{m-1} + y'/2^m \le y/2^n - y/2^m = y\,(1/2^n - 1/2^m) \le y_m\,(1/2^n - 1/2^m).$$

Then since C is a chain in $(P, \rightarrow^*)$, the following inequality holds all (x, y, n), $(x', y', m) \in C$ such that $n < \omega$ and $m < \omega$:

$$|\, x' - x \,| \le y_m\, |\, 1/2^n - 1/2^m \,|.$$

Let us fix some bijection $f : \mathbf{N}_0 \rightarrow N$, where $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ is the set of non-negative integers (such a bijection exists since N is infinite and at most countable). Note that the definition of N implies that for each $i \in \mathbf{N}_0$ the set $\{\, x \in \mathbf{R} \mid \exists\, y \in \mathbf{R}\ (x, y, f(i)) \in C \,\}$ is nonempty. Then let us fix some function $g\colon \mathbf{N}_0 \rightarrow \mathbf{R}$ such that for each $i \in \mathbf{N}_0$ there exists $y \in \mathbf{R}$ such that $(g(i), y, f(i)) \in C$.

Let us check that $g(0), g(1), g(2), \ldots$ is a Cauchy sequence of real numbers. Let ε be a positive real number, i.e. $\varepsilon > 0$. Note that the set $\{\, i \in \mathbf{N}_0 \mid f(i) \le 2y_m/\varepsilon \,\}$ is finite, so there is some natural number M such that for all $i \in \mathbf{N}_0$, $f(i) \le 2y_m/\varepsilon$ implies $i < M$. Moreover, whenever $i, j \in \mathbf{N}_0$ and $i > M$, $j > M$, there exist $y, y' \in \mathbf{R}$ such that

$$(g(i), y, f(i)) \in C \text{ and } (g(j), y', f(j)) \in C,$$

where $f(i) < \omega$ and $f(j) < \omega$, so

$$|\, g(j) - g(i) \,| \le y_m\, |\, 1/2^{f(i)} - 1/2^{f(j)} \,| \le y_m \max\{1/2^{f(i)}, 1/2^{f(j)}\} \le y_m\, \varepsilon/(2y_m) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $g(0), g(1), g(2), \ldots$ is a Cauchy sequence of real numbers.

Then the sequence $g(0), g(1), g(2), \ldots$ is convergent in $\mathbf{R}$ and has a limit that we will denote as x_s.

Let us check that $(x_s, 0, \omega)$ is a least upper bound of C.

Firstly, let us check that $(x_s, 0, \omega)$ is an upper bound of C in $(P, \rightarrow^*)$.

Let $(x, y, n) \in C$. The assumption of the Case 3 implies that $n < \omega$. Then $n \in N$. Since N is infinite, there is some $m \in N$ such that $m > n$. Let $x', y' \in \mathbf{R}$ be elements such that $(x', y', m) \in C$. Since C is a chain in $(P, \rightarrow^*)$ and $m > n$, we have $(x, y, n) \rightarrow^+ (x', y', m)$. Then $0 < y' < y$. Note that the set $\{\, i \in \mathbf{N}_0 \mid f(i) \le m \,\}$ is finite, so there is some natural number M such that for all $i \in \mathbf{N}_0$, if $f(i) \le m$ holds, then $i < M$.

Let us check that for all $i \in \mathbf{N}_0$, if $i \ge M$, then $|\, g(i) - x' \,| \le y'/2^m$. Let $i \in \mathbf{N}_0$ and $i \ge M$. Then $f(i) > m$. Since $f(i) \in N$, there exist $y_0 \in \mathbf{R}$ such that $(g(i), y_0, f(i)) \in C$. Since C is a chain in $(P, \rightarrow^*)$ and $f(i) > m$, we have $(x', y', m) \rightarrow^+ (g(i), y_0, f(i))$. Then (T1) implies that $f(i) \ge 1$, $0 \le y_0 \le y'$, and

$$|\, g(i) - x' \,| < y'/2^m - y'/2^{f(i)-1} + y_0/2^{f(i)}.$$

Note that

$$y_0/2^{f(i)} \le 2y'/2^{f(i)} = y'/2^{f(i)-1},$$

so $|\, g(i) - x' \,| \le y'/2^m$. Thus whenever $i \ge M$, the inequality $|\, g(i) - x' \,| \le y'/2^m$ holds. Then

$$x' - y'/2^m \le g(i) \le x' + y'/2^m,$$

holds for all integer $i \ge M$, so the limit x_s of the sequence $g(0), g(1), g(2), \ldots$ satisfies the inequality

$$x' - y'/2^m \le x_s \le x' + y'/2^m\ .$$

Moreover, because $(x, y, n) \rightarrow^+ (x', y', m)$, from (T1) we have:

$$|\, x' - x \,| < y/2^n - y/2^{m-1} + y'/2^m$$

Then since $y' < y$ we have

$$|\, x_s - x \,| \le |\, x_s - x' \,| + |\, x' - x \,| < y'/2^m + y/2^n - y/2^{m-1} + y'/2^m \le y/2^n.$$

Then $|\, x' - x \,| < y/2^n$, and also $y > 0$, so from (T2) we have $(x, y, n) \rightarrow^+ (x_s, 0, \omega)$.

Since $(x, y, n) \in C$ is arbitrary, we conclude that $(x_s, 0, \omega)$ is an upper bound of C.

Now let us check that $(x_s, 0, \omega)$ is least among upper bounds of C in $(P, \to^*)$.

Let $(x_1, y_1, l) \in P$ be an upper bound of C in $(P, \to^*)$. Then $n \leq l$ for all $n \in N$, and since N is infinite, it is not bounded from above by any natural number, so $l = \omega$. Then since C is nonempty, from (T2) it follows that $y_1 = 0$ and for all $(x, y, n) \in C$, $| x_1 - x | < y/2^n$.

Let us check that x_1 is a limit of the sequence $g(0)$, $g(1)$, $g(2)$, … Let ε be a positive real number, i.e. $\varepsilon > 0$. Since the set $\{ i \in \mathbf{N}_0 \mid f(i) \leq 2y_m/\varepsilon \}$ is finite, there is some natural number M such that for all $i \in \mathbf{N}_0$, $f(i) \leq 2y_m/\varepsilon$ implies $i < M$. Moreover, whenever $i \in \mathbf{N}_0$ and $i \geq M$, there exists $y \in \mathbf{R}$ such that $(g(i), y, f(i)) \in C$, where $f(i) < \omega$, so $y \in Y$ and $| x_1 - g(i) | < y/2^{f(i)} \leq y_m\, \varepsilon/(2y_m) < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, x_1 is a limit of the sequence $g(0)$, $g(1)$, $g(2)$, …, and since this sequence converges to x_s, we have $x_1 = x_s$ and $(x_1, y_1, l) = (x_s, 0, \omega)$. Since (x_1, y_1, l) is arbitrary, we conclude that $(x_s, 0, \omega)$ is a least upper bound of C in $(P, \to^*)$.

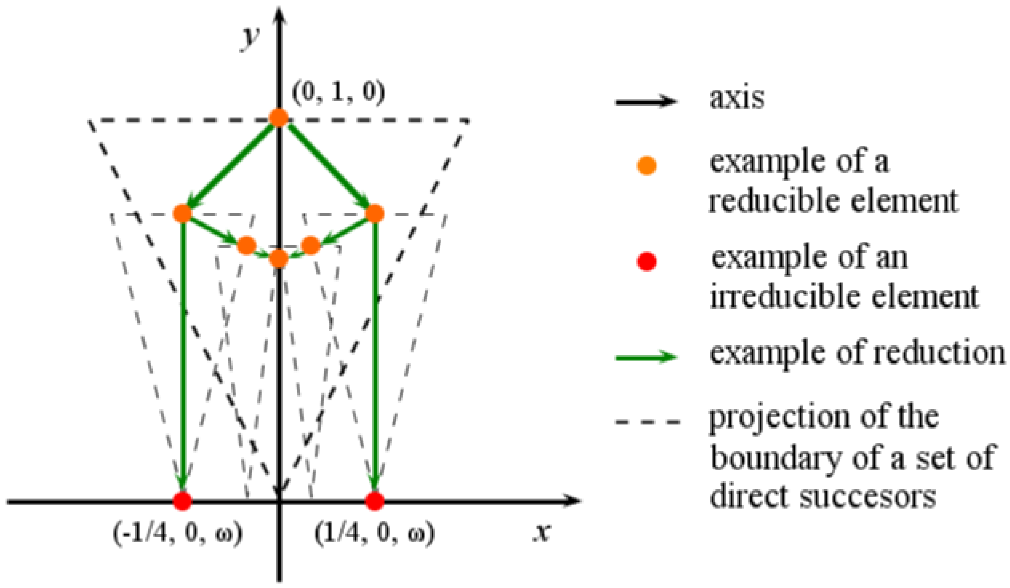


Figure 2: Illustration of the Strengthened Newman's counterexample (Theorem 2) – an example of an acyclic, strictly inductive, locally confluent ARS that is not confluent. Note that only some example elements and reductions are shown. Unlike the case of the usual Newman's counterexample (Figure 1(a)), the set of elements of the Strengthened Newman's counterexample is uncountable, and it is essential that it is uncountable, because Theorem 1 implies that every countable acyclic strictly inductive locally confluent ARS is confluent.

Since C is arbitrary, the ARS $(P, \to)$ is strictly inductive. □

Note that in the ARS $(P, \to)$ given in the statement of Theorem 2, reducible elements have the form (x, y, n), where $y > 0$ and $n < \omega$, and the items (T1), (T2) in the proof of Theorem 2 imply that for any such a reducible element (x, y, n) the set of its normal forms is $\{ (x', 0, \omega) \mid |x' - x| < y/2^n \}$, and the latter is isomorphic to a real interval that is a connected subset of $\mathbf{R}$ in the sense of the standard topology on $\mathbf{R}$ (see an illustration in Figure 3). This observation motivates the result described below (Theorem 3 illustrated in Figure 4).

For any preordered set $(P, \leq)$ and a set $A \subseteq P$, the *downward closure* of A in $(P, \leq)$ is the set

$$\{ y \in P \mid \exists\, a \in A\; y \leq a \}.$$

Given an ARS $(X, \to)$, we will say that a topology T on the set of all irreducible elements of $(X, \to)$ is *downward-compatible with* $(X, \to)$, if for any closed set A in the sense of T, the downward closure of A in the preordered set $(X, \to^*)$ is closed in the sense of Section 2.

Theorem 3. *Let $(X, \to)$ be an acyclic strictly inductive locally confluent ARS and T be a topology on the set of all irreducible elements of $(X, \to)$ that is downward-compatible with $(X, \to)$.*

Then for each $x \in X$, the set of all normal forms of x in $(X, \to)$ is a connected subset of the topological space $(\{x \in X \mid x \text{ is irreducible in } (X, \to) \}, T)$.

Proof.

Let us denote as IR the set of all irreducible elements of $(X, \to)$. For each $x \in X$ denote:

- $U(x) = \{ y \in X \mid x \to^* y \}$;
- $U_{\to^*}(x)$ is the poset $(U(x), \to^* \cap (U(x) \times U(x)))$;
- $NF(x)$ is the set of all normal forms of x in $(X, \to)$.

Since $(X, \to)$ is an acyclic strictly inductive ARS, one can show that $(X, \to)$ is weakly normalizing using the same argument as in the proof of Theorem 1 (i.e. using Zorn's lemma).

Let us fix $x \in X$. Note that $NF(x) \neq \varnothing$. Suppose that $NF(x)$ is not a connected subset of the topological space (IR, T). Then there exist sets $O_1, O_2 \subseteq IR$ that are open in the sense of T and such that $NF(x) \subseteq O_1 \cup O_2$, $O_1 \cap O_2 \cap NF(x) = \varnothing$, $O_1 \cap NF(x) \neq \varnothing$, and $O_2 \cap NF(x) \neq \varnothing$.

Let $A = NF(x) \backslash O_1$ and $B = NF(x) \backslash O_2$. Then $A \cup B = NF(x)$, $A \cap B = \varnothing$, $A \neq \varnothing$, and $B \neq \varnothing$.

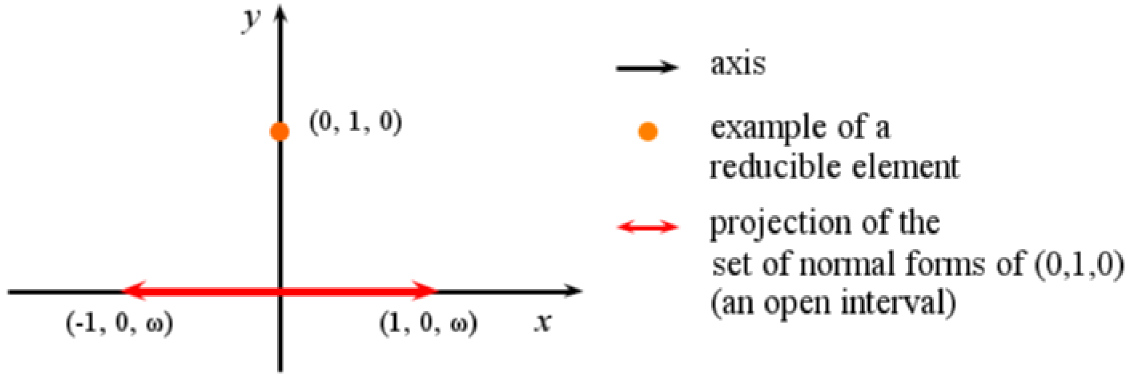


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Figure 3: Illustration of the set of normal forms of a reducible element in the Strengthened Newman's counterexample (Theorem 2).

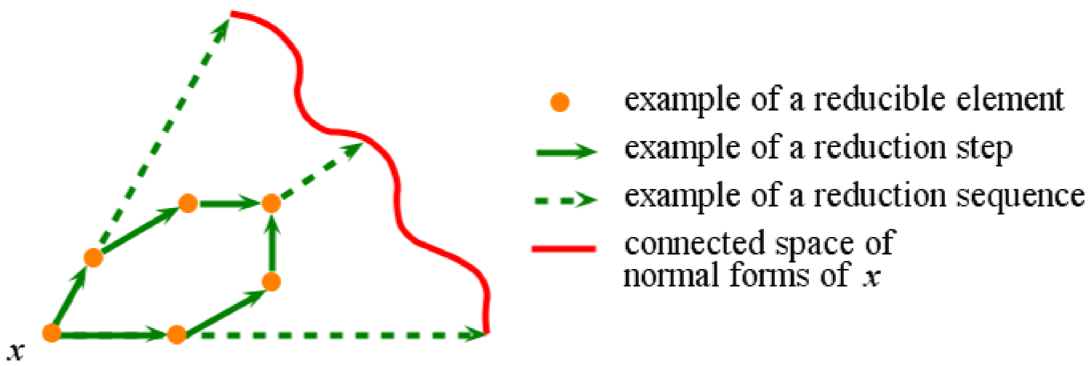

Figure 4: Illustration for the statement of Theorem 3. If an ARS $(X, \to)$ is acyclic, strictly inductive, and locally confluent (but is not necessarily countable or confluent), then for any topology T on the set of irreducible elements of $(X, \to)$ that is downward-compatible with $(X, \to)$ and any $x \in X$, set of normal forms of x, illustrated as a red curve, is a connected subset in the sense of T.

Note that $NF(x) \subseteq U(x)$, so $A, B \subseteq U(x)$. Let D_1 be the downward closure of A in $U_{\to^*}(x)$ and D_2 be the downward closure of B in $U_{\to^*}(x)$.

Since T is downward-compatible with $(X, \to)$, the downward closures of the sets $IR \backslash O_1$, $IR \backslash O_2$ in $(X, \to^*)$ are closed in $(X, \to^*)$ (in the sense of Section 2). Then it is straightforward to check that D_1 and D_2 are closed in $U_{\to^*}(x)$ (in the sense of Section 2).

Since $x \in U(x)$, $A, B \subseteq NF(x)$, and $A, B \neq \varnothing$, we have $x \in D_1$ and $x \in D_2$, so $D_1 \cap D_2 \neq \varnothing$. However, note that $D_1 \cap D_2 \cap NF(x) = \varnothing$, because for each $y \in D_1 \cap D_2 \cap NF(x)$ we have $NF(y) = \{y\}$, $NF(y) \cap A \neq \varnothing$, and $NF(y) \cap B \neq \varnothing$, whence $y \in A \cap B = \varnothing$.

As we have mentioned above, $(X, \to)$ is weakly normalizing, so for each $y \in U(x)$ we have $NF(y) \neq \varnothing$, and also $NF(y) \subseteq NF(x) = A \cup B$, so $NF(y) \cap A \neq \varnothing$ or $NF(y) \cap B \neq \varnothing$, whence $y \in D_1 \cup D_2$. Thus $U(x) \subseteq D_1 \cup D_2$, and also $D_1, D_2 \subseteq U(x)$, so $D_1 \cup D_2 = U(x)$.

Note that $A \cap D_2 = \varnothing$, because whenever $y \in D_2$, there exists $b \in B$ such that $y \to^* b$, so if y is irreducible in $(X, \to)$, then $y = b \notin A$ (because $A \cap B = \varnothing$), and if y is reducible in $(X, \to)$, then $y \notin A$. Similarly, we have $B \cap D_1 = \varnothing$.

Let us define a binary relation $\leq'$ on $U(x)$ such that for all $y_1, y_2 \in U(x)$:

$$y_1 \leq' y_2 \Leftrightarrow (y_1 \to^* y_2 \text{ or } (y_1 \in D_1 \text{ and } y_2 \in A) \text{ or } (y_1 \in D_2 \text{ and } y_2 \in B))$$

Using the facts that D_1, D_2 are closed, $A \cap D_2 = \varnothing$ and $B \cap D_1 = \varnothing$ and strict inductivity of the ARS $(X, \to)$, it is straightforward to show that $\leq'$ is a preorder on $U(x)$, and, moreover, every nonempty chain in the preordered set $(U(x), \leq')$ has a least upper bound.

Let us fix some elements $a_0 \in A$, $b_0 \in B$ (they exist, because $A, B \neq \varnothing$). It is straightforward to check that for all $y \in U(x)$, $y \leq' a_0$ or $y \leq' b_0$.

Let $D = \{ y \in U(x) \mid y \leq' a_0 \text{ and } y \leq' b_0 \}$. Then D is closed in $(U(x), \leq')$ in the sense of Section 2 and $D \neq \varnothing$ since $x \in D$. Then since $(U(x), \leq')$ has a least upper bound for every nonempty chain, using Zorn's lemma (like in Theorem 1) we conclude that D has some maximal element with respect to $\leq'$. Let us denote it as d. Then $d \in U(x)$, $d \leq' a_0$ and $d \leq' b_0$. Since the relation $\leq'$ is defined as a disjunction of 3 conditions ($y_1 \to^* y_2$ or ($y_1 \in D_1$ and $y_2 \in A$) or ($y_1 \in D_2$ and $y_2 \in B$)), the conjunction "$d \leq' a_0$ and $d \leq' b_0$" can be reduced to a disjunction of 9 conditions: "$d \to^* a_0$ and $d \to^* b_0$", "($d \in D_1$ and $a_0 \in A$) and $d \to^* b_0$ ", and so on. It is straightforward to check that each of these cases leads to a contradiction (e.g. if $d \to^* a_0$ and $d \to^* b_0$, then using local confluence of $(X, \to)$ we can obtain a contradiction with maximality of d in D with respect to $\leq'$ like in a similar situation in the proof of Theorem 1). Then the assumption that $NF(x)$ is not a connected subset of (IR, T) is false.

We conclude that the set of all normal forms of x in $(X, \to)$ is a connected subset of (IR, T). □

5. Formalization in proof assistant software

We formalized Theorems 1-3 from Section 4 in Isabelle proof assistant [12, 13] using HOL logic (Higher-Order Logic). The corresponding formal theory entitled *NLNT* extends (and depends on) the previously published theory *GNL.thy* that can be found in the supplementary material for the paper [14].

The statement of a formalized version of Theorem 1 in *NLNT* has the following form (here the symbol σ denotes a reduction relation):

theorem thm_1:
fixes X σ
assumes "*X,σ is ACYCLIC ARS*" **and** "*X,σ is S.I.ARS*"
  and "*{x∈X. x is IRREDUCIBLE in X,σ} is AT MOST COUNTABLE*"
shows "*(X,σ is CONFLUENT ARS) = (X,σ is L.CONFLUENT ARS)*"

The formal notions *ACYCLIC ARS*, *S.I.ARS* (i.e. a strictly inductive ARS), *IRREDUCIBLE*, *CONFLUENT ARS*, *L.CONFLUENT ARS* (i.e. a locally confluent ARS) are defined in the theory *GNL.thy*.

The statement of a formalized version of Theorem 2 in *NLNT* has the following form:

theorem thm_2:
fixes P :: "*(real × real × enat) set*"
 and σ :: "*(real × real × enat) ⇒ (real × real × enat) ⇒ bool*"
assumes "*P = {x::real. True} × {y::real. True} × {n::enat. True}*"
  and "∀ *(x::real) (y::real) (n::enat) (x'::real) (y'::real) (m::enat).*
    σ (x, y, n) (x', y', m) =
     ((n < ∞ ∧ m = n + 1 ∧ | x' – x | < y' / 2^(the_enat m) ∧ 0 < y' ∧ y' < y)
      ∨ *(n < ∞ ∧ m = ∞ ∧ x' = x ∧ 0 = y' ∧ y' < y))*"
shows "*(P,σ is ACYCLIC ARS) ∧ (P,σ is S.I.ARS) ∧ (P,σ is L.CONFLUENT ARS) ∧ (¬ (P,σ is CONFLUENT ARS))*"

The statement of a formalized version of Theorem 3 in *NLNT* has the following form:

definition is_dw_compatible :: "*(('n::topological_space) ⇒ 'a) ⇒ 'a set ⇒ ('a ⇒ 'a ⇒ bool) ⇒ bool*" ("*_ is DW-COMPATIBLE with _,_*")
where
"*(f is DW-COMPATIBLE with X,σ) ≡*
(bij_betw f { n::'n. True } { x ∈X. x is IRREDUCIBLE in X,σ }
*∧ (∀A::('n set). closed A --> ({ y ∈ X. ∃ a ∈A. σ^** y (f a) } is c-CLOSED in X,(σ^**))))*"

theorem thm_3:
fixes X :: "*'a set*" **and** σ :: "*'a ⇒ 'a ⇒ bool*" **and** f :: "*'n::topological_space ⇒ 'a*"
assumes "*X,σ is ACYCLIC ARS*" **and** "*X,σ is S.I.ARS*" **and** "*X,σ is L.CONFLUENT ARS*"
and "*f is DW-COMPATIBLE with X,σ*"
shows " *∀x ∈ X. connected { n::'n. (f n) is N.F. of x in X,σ }*"

The formal notions *c-CLOSED* (a closed subset of a preordered set) and *N.F.* (normal form) are defined in the theory *GNL.thy*.

6. Conclusions

We have proposed novel results (Theorems 1-3 in Section 4) that can help one to better recognize situations where the local confluence property of abstract rewriting systems is and is not equivalent to the confluence property. We have formalized Theorems 1-3 in Isabelle proof assistant software using HOL logic. Note that Theorems 1 and 3 can be generalized to ARS that do not satisfy the acyclicity condition using the notion of *quasi-local confluence* proposed in [14]. Study of other generalizations of the given results is a subject of further work.

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