On generalised Ackermann encodings – the basis issue

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Abstract

In this paper, a generalised version $\mathfrak{A}_\beta$ of the celebrated Ackermann encoding of the hereditarily finite sets, aimed at assigning a real number also to each hereditarily finite hyperset and multiset, is studied. Such a mapping establishes a significant link between real numbers and the theories of such generalised notions of set, so that performing set-theoretic operations can be translated into their number-theoretic equivalent. By appropriately choosing a parameter β, both the Ackermann encoding and the less known map $\mathbb{R}_A$ arise as special cases; a bijective encoding of a subuniverse of hereditarily finite multisets occurs whenever this parameter is chosen among natural numbers, while if it is taken transcendental and within a peculiar interval of the real positive line, then the function is surmised to ensure an injective mapping of both the aforementioned universes.

Keywords

Ackermann encoding, hereditarily finite sets, hypersets, multisets, partition refinement

Introduction

In 1937, Wilhelm Ackermann defined an encoding of the hereditarily finite sets – namely, finite sets whose construction involves only finite sets at any nesting depth – into natural numbers (see [1]). This bijection gives, for each hereditarily finite set, a detailed description of its elements and, recursively, of any set entering its construction. Globally, it induces a well ordering of the universe of hereditarily finite sets via their codes, while for each operation over such sets it provides an exact counterpart over natural numbers. Most remarkably, Ackermann's correspondence enabled also the migration of results about Peano number theory into set theory (cf. [2, Sec. 7.6]); among others, it permits one to prove the essential undecidability of axiomatic theories of sets by an argument *à la* Gödel (see [2]). Regrettably, this bijection has a narrow realm of application, as becomes apparent when one moves on to considering whatever extension of the family of all finite sets to broader ones, e.g., the hereditarily finite *hypersets* and *multisets*.

A hyperset admits cycles in the membership relation, so it violates the so-called *well-foundedness* principle. Despite this, a strongly restrained equality notion – namely, *bisimilarity* (see [3] and [4]) – reconciles liberality with the philosophical Occam's razor criterion. Hypersets find immediate application in several fields. In particular, they can represent finite state automata or, more generally, graphs labelled on edges; showing that two hypersets are bisimilar is analogous to proving the equivalence of two such machines (see, e.g., [5]). If each distinct hyperset is assigned a unique number, up to this equality criterion, then the aim of finding out bisimilarities can hopefully be attained just through a comparison of these codes.

Differently from the previous case, a multiset allows each of its elements to occur with multiplicity higher than one, while the order of those elements is still regarded as irrelevant. Despite their plain application in computer sciences, a formal theory of pure multisets is not uniquely established yet – nonetheless, an attempt can be found in [6] (see also, e.g., [7]). A hereditarily finite set can be regarded as basis for the construction of a hereditarily finite multiset, the latter having as its elements the same

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elements of the former, possibly counted more than once; in this paradigm, an ordering of a multiset is induced by the Ackermann ordering of hereditarily finite sets, and this finds application when writing a *termination function*, which turns out to be much simpler than the one obtained by considering hereditarily finite sets only (see [8]).

The aim of this paper, resulting from a master's degree thesis,[1] is to generalise the Ackermann encoding and its variant $\mathbb{R}_A$ – already introduced and studied in [9], [10], [11] and [12] – into a parametrised family of encoding functions, motivated by missed injectivity over the union of the two aforementioned universes, and aimed at showing the properties which are independent of the specifically chosen one. Some special cases will be analysed in order to determine which parameters yield a well-defined and injective mapping over at least a subuniverse of the joint family of hereditarily finite sets, hypersets, and multisets.

This paper is structured as follows. In Section 1, it introduces the notions of hereditarily finite (hyper-, multi-) sets; Section 2 focuses on the Ackermann encoding and the map $\mathbb{R}_A$. In Section 3, the encoding scheme $\mathfrak{A}_\beta$ is introduced, while the cases $\beta \in \mathbb{N}$ and $e^{-e} \leq \beta \leq e^{1/e}$ are analysed in Sections 4 and 5 resp. – the latter being an interval whose significance emerges from a theorem by Euler. Section 6 is devoted to show two meaningful examples, and Section 7 to point out the conclusions of this study.

1. Hereditarily finite families of sets

Standard set-theoretic notations will be adopted throughout the paper; in particular, $\mathscr{P}(\cdot)$ will represent the *powerset operator*. Recall that, if it exists, the *transitive closure* of a set s is the collection of all its elements, together with their own elements at any nesting depth:

$$\mathrm{trCl}(s) = s \cup \bigcup_{s' \in s} \mathrm{trCl}(s') \,.$$

Notice that its existence is always guaranteed if *well-foundedness*, together with a minimal axiomatic equipment, is assumed on the membership relation (see, e.g., [4] and [13]).

To denote numerical sets, $\mathbb{N}$, $\mathbb{R}$ and so on will be used; the abbreviations $\mathbb{N}^+ := \{n \in \mathbb{N} \mid n > 0\}$, $\mathbb{R}_0^+ := \{z \in \mathbb{R} \mid z \geq 0\}$ and similar are adopted. For classical numerical operations, standard notations will be used; the following generalisation of a notation adopted in [10], namely

$$\beth_\beta(0) = 0, \qquad \beth_\beta(n+1) = \beta^{\beth_\beta(n)} = \underbrace{\beta^{\beta^{\cdot^{\cdot^{\cdot^{\beta}}}}}}_{n},$$

for $\beta \in \mathbb{R}^+$, $n \in \mathbb{N}$ is justified by the frequent use of *iterated exponentials* along this paper.

The following definitions are given by means of the concept of *cumulative hierarchy*, meaning that they are built up by introducing a sequence of *levels* or *layers* such that each one of them is strictly contained in the subsequent one. The first and smallest universe of sets to be introduced is the following.

Definition 1.1 (Hereditarily finite sets)**.**

$$\mathsf{HF}_n = \begin{cases} \emptyset & \text{if } n = 0 \\ \mathscr{P}(\mathsf{HF}_{n-1}) & \text{if } n \in \mathbb{N}^+, \end{cases} \qquad\qquad \mathsf{HF} = \bigcup_{n \in \mathbb{N}} \mathsf{HF}_n$$

defines the cumulative hierarchy of the *hereditarily finite sets* (sometimes referred as HF^0).

Given $h \in \mathsf{HF}$, its *rank* $\mathrm{rk}(h)$ is defined as the least integer r such that $h \in \mathsf{HF}_{r+1}$.

Therefore, hereditarily finite (from now on, often abbreviated as *h.f.*) sets are finite at any nesting depth; observe that this universe is well-founded by construction. The rank of a h.f. set expresses also the maximum depth at which the empty set is nested inside it.

Differently from standard sets, the universe of *multisets* allows each element of any of its members to occur more than once.

[1]Available upon request to S. Boscaratto.

Definition 1.2 (Multisets)**.** Let $O_1, \ldots, O_n$ be n distinct objects and let $m_1, \ldots, m_n \in \mathbb{N}^+$ be positive integers; then the list

$$M = [\underbrace{O_1, \ldots, O_1}_{m_1}, \ldots, \underbrace{O_n, \ldots, O_n}_{m_n}],$$

or equivalently

$$M = \{{}^{m_1}O_1, \ldots, {}^{m_n}O_n\},$$

up to any permutation, defines the *multiset* M containing the objects $O_1, \ldots, O_n$ with *multiplicities* $m_1, \ldots, m_n$ respectively, i.e. containing exactly m_i occurrences of O_i for every $i \in \{1, \ldots, n\}$.[2] The *multiplicity map* of M and its *multiset membership relation* are then defined as

$$\mu_M(O_i) = m_i \quad \iff \quad O_i \ {}^{m_i}\!\in M.$$

Multisets are introduced and often used without much care about formal aspects (see [14]); here, just the case in which their objects are themselves multisets at any nesting depth is taken into account, to be coherent with HF. In this way a cumulative hierarchy of *hereditarily finite multisets* can be defined by introducing an operator which is the multiset analogue to the common powerset. The following definitions and properties are stated and proven in [12].

Definition 1.3 (μ-powerset)**.** Given a multiset X, define

$$\mathscr{P}^\mu(X) = \big\{\{{}^{m_1}x_1, \ldots, {}^{m_n}x_n\} \mid x_1, \ldots, x_n \in X \wedge (\forall i \neq j)(x_i \neq x_j) \\ \wedge\, m_1, \ldots, m_n \in \mathbb{N}^+ \wedge n \in \mathbb{N}\big\}.$$

Definition 1.4 (Hereditarily finite multisets)**.**

$$\mathsf{HF}^\mu_n = \begin{cases} \emptyset & \text{if } n = 0 \\ \mathscr{P}^\mu(\mathsf{HF}^\mu_{n-1}) & \text{if } n \in \mathbb{N}^+, \end{cases} \qquad \mathsf{HF}^\mu = \bigcup_{n \in \mathbb{N}} \mathsf{HF}^\mu_n$$

defines the cumulative hierarchy of the *hereditarily finite multisets.*

Given $H \in \mathsf{HF}^\mu$, its *rank* $\mathrm{rk}(H)$ is the least integer r such that $H \in \mathsf{HF}^\mu_{r+1}$.

Some arithmetical operations such as sum, multiplication by a positive integer, product and exponentiation are defined inside HF^μ, so that polynomials of multisets can be defined, too (see [12]). Moreover, HF^μ is a well-founded universe much as HF, which is in turn naturally embedded into HF^μ, being its subuniverse admitting each distinct element once at any nesting depth.

Differently from the previous cases, the following universe is not defined by means of a cumulative hierarchy but by means of *set systems* describing the transitive closures of its members; the equality criterion between two hypersets generalises from one-to-one correspondence of their elements (*extensionality*, see [13]) to *bisimilarity* (see [3] and the definitions below).

Definition 1.5 (Bisimulation)**.** A dyadic relation $\flat$ on the finite set V of the nodes of an directed graph $\mathcal{M} = (V, E)$ is said to be a *bisimulation* on $\mathcal{M}$ if $u_0 \flat u_1$ always implies that

- for every child v_1 of u_1, u_0 has at least one child v_0 s.t. $v_0 \flat v_1$, and
- for every child v_0 of u_0, u_1 has at least one child v_1 s.t. $v_0 \flat v_1$.

The largest of all bisimulations on $\mathcal{M}$ (relative to inclusion) is the following equivalence relation.[3]

Definition 1.6 (Bisimilarity)**.** The *bisimilarity* of a digraph $\mathcal{M}$ whose set V of nodes is finite is the dyadic relation $\equiv_\mathcal{M}$ over V such that $u \equiv_\mathcal{M} v$ holds between u, v in V if and only if $u \flat v$ holds for some bisimulation $\flat$ on $\mathcal{M}$.

[2] Following [12], multiplicities are unconventionally written as left superscripts in order to avoid confusion with the notation $O^n = \underbrace{O \times \cdots \times O}_{n}$.

[3] See [3, pp. 20–22].

In the wording of [15, pp. 78–80], bisimilarity induces the coarsest partition of V that is stable w.r.t. $\mathcal{M}$. The graphs taken into account here are associated with systems of set equations involving unknowns ς_i (acting as nodes); each equation $\varsigma_i = \{\varsigma_{i,1}, \ldots, \varsigma_{i,m_i}\}$ brings the edges $\langle \varsigma_i, \varsigma_{i,1}\rangle, \ldots, \langle \varsigma_i, \varsigma_{i,m_i}\rangle$ into E.

Definition 1.7 (Hereditarily finite rational hypersets)**.** A *hereditarily finite rational hyperset* is the solution ς_0 (unique, up to bisimilarity) of a finite set system

$$\mathscr{S}(\varsigma_0, \varsigma_1, \ldots, \varsigma_n) = \begin{cases} \varsigma_0 = \{\varsigma_{0,1}, \ldots, \varsigma_{0,m_0}\} \\ \varsigma_1 = \{\varsigma_{1,1}, \ldots, \varsigma_{1,m_1}\} \\ \quad \vdots \\ \varsigma_n = \{\varsigma_{n,1}, \ldots, \varsigma_{n,m_n}\} \end{cases}$$

with $\varsigma_{i,j} \in \{\varsigma_0, \varsigma_1, \ldots, \varsigma_n\}$ for every $i \in \{0, 1, \ldots, n\}$ and $j \in \{1, \ldots, m_i\}$. The class of h.f. rational hypersets is denoted by $\mathsf{HF}^{1/2}$.

Example 1.1. The hyperset $\Omega = \{\{\{\cdots\}\}\}$ is the solution of the equation $\varsigma = \{\varsigma\}$.

Notice that also h.f. well-founded sets can be defined by finite set systems, with the constraint that the elements of ς_i can only be chosen from among $\varsigma_{i+1}, \ldots, \varsigma_n$. Moreover, by allowing multiple occurrences of the same unknown, h.f. multisets can be defined in this way too. *Multihypersets*, i.e., non-well-founded multisets, are neither treated in this paper nor – to the best of authors' knowledge – elsewhere.

2. Encoding sets as natural and real numbers

Definition 2.1 (Ackermann encoding of HF)**.**

$$\mathbb{N}_A(h) = \sum_{h' \in h} 2^{\mathbb{N}_A(h')} \qquad \text{if } h \in \mathsf{HF}$$

recursively defines the *Ackermann encoding* of hereditarily finite sets.

The Ackermann encoding has a number of well-known and interesting properties, which allow an exact match not only between HF and $\mathbb{N}$, but also between the related operations and theories. Some of these properties are shown below.[4]

- $\mathbb{N}_A$ is a bijection between HF and $\mathbb{N}$.
- $h' \in h \in \mathsf{HF}$ if and only if there is a '1' at position $\mathbb{N}_A(h')$ of the binary expansion of $\mathbb{N}_A(h)$.
- $\mathbb{N}_A$ gives a natural, total ordering to HF: this is established as $h \prec h' \Leftrightarrow \mathbb{N}_A(h) < \mathbb{N}_A(h')$.

To extend the domain of an encoding map so as to embrace also h.f. hypersets, in [10] the function $\mathbb{Q}_A$ was introduced. Although it keeps some of the most desirable features of the Ackermann encoding – actually, the restriction of $\mathbb{Q}_A$ to the universe HF equals $\mathbb{N}_A$ –, it is not uniformly extensible to multisets too, and presupposes an ordering of hypersets for which no convenient standard has emerged yet.[5] Another variant, introduced in [9] and adopted also in [10], keeps a stronger formal kinship with $\mathbb{N}_A$.

Definition 2.2 ($\mathbb{R}_A$-code)**.** The $\mathbb{R}_A$*-codes* of hereditarily finite rational hypersets are defined as follows:

$$\mathbb{R}_A(\hbar) = \sum_{\hbar' \in \hbar} 2^{-\mathbb{R}_A(\hbar')} \qquad \text{for } \hbar \in \mathsf{HF}^{1/2}.$$

Notice that this definition allows the codes of h.f. hypersets to be finite, despite not so evidently guaranteeing it. As a very intuitive extension of this encoding, the following definition is introduced.

[4]For a proof see, e.g., [11].
[5][16] shows an attempt towards this direction.

Definition 2.3 ($\mathbb{R}^\mu_A$-code)**.** The $\mathbb{R}^\mu_A$*-codes* of hereditarily finite multisets are defined as follows:

$$\mathbb{R}^\mu_A(H) = \sum_{H' \in H} \mu_H(H') \cdot 2^{-\mathbb{R}^\mu_A(H')} \qquad \text{for } H \in \mathsf{HF}^\mu.$$

Clearly, if the multiplicities are all equal to 1 at any nesting depth, the multiplicity map is unnecessary and the resulting formula is the same as the previous one, thereby showing that $\mathbb{R}^\mu_A\big|_{\mathsf{HF}^{1/2}} = \mathbb{R}_A$.

Observe that this encoding is not injective over the whole HF^μ, indeed

$$\mathbb{R}^\mu_A\big([[\emptyset],[\emptyset]]\big) = 2 \cdot 2^{-1} = 1 = 2^0 = \mathbb{R}^\mu_A\big([\emptyset]\big).$$

Besides, a suitable subuniverse of it is defined in [12] as a domain where injectivity can be conjectured.

Definition 2.4. Let

$$\mathcal{H}_{2,n} = \begin{cases} \big\{H \in \mathsf{HF}^\mu \mid \mu_H(\{\emptyset\}) \le 1\big\} & \text{if } n = 0 \\ \big\{H \in \mathcal{H}_{2,0} \mid H \subseteq \mathcal{H}_{2,n-1}\big\} & \text{if } n \in \mathbb{N}^+. \end{cases}$$

Then define

$$\mathcal{H}_2 = \bigcap_{n=0}^{\infty} \mathcal{H}_{2,n},$$

so that $\mathcal{H}_2$ is the cumulative hierarchy of multisets containing no occurrences of $\{\emptyset\}$ with multiplicity larger than 1 at any nesting depth.[6]

Conjecture 2.1 (Conjecture 4.11, [12])**.** *The encoding map $\mathbb{R}^\mu_A$ is injective over $\mathcal{H}_2$.*

Although this conjecture has some relevant consequences over HF in the first place (see [12]), an example in Section 6 will disprove a preceding conjecture on the injectivity of $\mathbb{R}_A$ over $\mathsf{HF}^{1/2}$; since the violated injectivity is strictly related to the choice of 2^{-1} as basis for the exponentiation, this led to the quest for an appropriate substitute to guarantee this essential property. Moreover, a theorem stating existence and uniqueness of the codes of h.f. rational hypersets (Theorem 4, [11]) turns out to be not fully validated because of an inaccuracy regarding the proof of a preceding lemma (Lemma 4 (vi), [11]). In Section 5 the same lemma will be presented in a generalised version, aimed at going towards the direction of proving this essential feature of an encoding map.

3. A generalised Ackermann map

Consider the following family of encodings of hereditarily finite (hyper-, multi-) sets, including those already known.

Definition 3.1 ($\mathfrak{A}_\beta$-code)**.** Let $\beta \in \mathbb{R}^+ \setminus \{1\}$; then

$$\mathfrak{A}_\beta(h) \overset{\text{def}}{=} \sum_{h' \in h} \mu_h(h') \cdot \beta^{\mathfrak{A}_\beta(h')} \qquad \text{for } h \in \mathsf{HF}^{1/2} \cup \mathsf{HF}^\mu$$

defines the $\mathfrak{A}_\beta$*-codes* of the hereditarily finite (hyper-, multi-) sets.

Remark 1. Although the codes may vary significantly by changing the basis β, two hereditarily finite sets trivially have an established and constant code; they are

$$\emptyset \quad \longmapsto \quad 0, \qquad\qquad \{\emptyset\} \quad \longmapsto \quad 1.$$

As a consequence, the codes of the multisets containing just the empty set but with any multiplicity range over all natural numbers:

$$(\forall \beta \in \mathbb{R}^+ \setminus \{1\})(\forall m \in \mathbb{N})\Big(\mathfrak{A}_\beta\big(\{^m\emptyset\}\big) = m\Big).$$

[6] This notation is an adaption from $\mathfrak{H}_1^\infty$ introduced in [12]: here the subscript '2' stands for the inverse basis of exponentiation. This choice will be justified in Section 5. The motivation behind calling it a cumulative hierarchy can also be found in [12].

Example 3.1. The $\mathfrak{A}_\beta$-code of the *super-singleton* $\{\emptyset\}^n := \{\{\emptyset\}^{n-1}\}$ for $n \in \mathbb{N}^+$ is $\mathfrak{A}_\beta(\{\emptyset\}^n) = \beth_\beta(n)$.

Notice that the definition, as it stands, is not insisting in any way on the injectivity of the codes; indeed, the following two particular cases arise.

Example 3.2. Let $\beta = 2$; then $\mathfrak{A}_2\big|_{\mathsf{HF}} = \mathbb{N}_A$. Since the latter is bijective onto $\mathbb{N}$, the former cannot be injective over its whole domain, which is a much wider family of generalised sets; in particular (see Remark 1):

$$(\forall i \in \mathbb{N}^+)\Big(\mathfrak{A}_\beta(h_i) = \mathfrak{A}_\beta\big(\{^i\emptyset\}\big)\Big).$$

Example 3.3. Let $\beta = 1/2$; then $\mathfrak{A}_{1/2} = \mathbb{R}^\mu_A$. As has been already seen, $\mathbb{R}^\mu_A\big([[\emptyset], [\emptyset]]\big) = 1$.

A remarkable general property is that any basis makes the codes of h.f. sets grow beyond any bound.

Proposition 3.1. *Let $\beta \in \mathbb{R}^+ \setminus \{1\}$; then, the set of $\mathfrak{A}_\beta$-codes of* HF *is superiorly unbounded.*

Proof. Fix $n \in \mathbb{N}$. For any odd number $j \in \mathbb{N}$, consider h_j as the j-th h.f. set with respect to the Ackermann ordering, therefore $\emptyset \in h_j$; then

$$\mathfrak{A}_\beta(h_j) = \sum_{h' \in h_j} \beta^{\mathfrak{A}_\beta(h')} = \sum_{h' \in h_j} \mathfrak{A}_\beta\big(\{h'\}\big) \geq 1.$$

Case $\beta < 1$. Observe that, in this case,

$$\mathfrak{A}_\beta\big(\{h_j\}\big) = \beta^{\mathfrak{A}_\beta(h_j)} \leq \beta.$$

Besides,

$$\mathfrak{A}_\beta\big(\{\{h_j\}\}\big) = \beta^{\mathfrak{A}_\beta(\{h_j\})} \geq \beta^\beta > \beta.$$

Let $k = 2n\lceil\beta^{-1}\rceil$ and consider the h.f. set $h = \{\{h_{k'}\} \mid k' \leq k\}$. Thus,

$$\mathfrak{A}_\beta(h) = \sum_{k'=0}^{k} \mathfrak{A}_\beta\big(\{\{h_{k'}\}\}\big) \geq \sum_{\substack{k'=0 \\ k' \text{ odd}}}^{k} \mathfrak{A}_\beta\big(\{\{h_{k'}\}\}\big) > \beta \cdot \frac{k}{2} = n.$$

Case $\beta > 1$. Let $j > 1$; then,

$$\mathfrak{A}_\beta\big(\{h_j\}\big) = \beta^{\mathfrak{A}_\beta(h_j)} > \beta.$$

Consider again $k = 2n\lceil\beta^{-1}\rceil$, and $h = \{h_{k'} \mid k' \leq k\}$. As in the previous case,

$$\mathfrak{A}_\beta(h) = \sum_{k'=0}^{k} \mathfrak{A}_\beta\big(\{h_{k'}\}\big) \geq \sum_{\substack{k'=0 \\ k' \text{ odd}}}^{k} \mathfrak{A}_\beta\big(\{h_{k'}\}\big) > \beta \cdot \frac{k}{2} = n.$$

□

4. Natural bases

Generalising the Ackermann map from h.f. sets to natural numbers, consider a basis $\beta \in \mathbb{N}^+$, $\beta \geq 2$; even when $\beta > 2$, the Ackermann ordering is clearly kept over HF, but there may be a gap between two consecutive codes; e.g., if $\beta = 3$, their codes are $0, 1, 3, 4, 27, \ldots$.

A rather intuitive feature that any of these encodings share with Ackermann's is that the expression of the codes as base-β numbers shows the membership relation as the presence of a '1' at the position given by the Ackermann ordering; therefore, codes of h.f. sets are sequences of just 0s and 1s in that representation. Notably, the missing codes can be filled with multisets with multiplicities at most $\beta - 1$ at any nesting depth; in this way, $\mathfrak{A}_\beta$ over a proper subfamily of HF^μ turns out to be bijective, and the code of a multiset as expressed as a base-β number defines completely its members with their appropriate multiplicities.

To achieve a reasonable definition of the subuniverse of HF^μ that can be encoded with a natural basis, consider an alternative version of powerset which is compatible with multisets.

Definition 4.1 (m-powerset)**.** Given a multiset X and $m \in \mathbb{N}^+$, define

$$\mathscr{P}^{(m)}(X) = \big\{\{^{m_1}x_1, \ldots, {}^{m_n}x_n\} \mid x_1, \ldots, x_n \in X \wedge (\forall i \neq j)(x_i \neq x_j) \\ \wedge\, m_1, \ldots, m_n \in \mathbb{N}^+ \wedge (\forall i)(m_i \leq m) \wedge n \in \mathbb{N}\big\}.$$

By using the classical powerset operation and the multiplication of a multiset by an integer (see [12]), $\mathscr{P}^{(m)}(X) = \mathscr{P}(mX)$ may serve as an equivalent definition: this follows from the upper bound this generalised powerset imposes to the multiplicity. Notice that the two operations coincide when $m = 1$. On the other hand, since Definition 1.3 of μ-powerset admits arbitrarily large multiplicities, the multiset resulting from its application can be reproduced by the union of all the multisets generated from m-powersets: $\mathscr{P}^{\mu}(X) = \bigcup_{m \in \mathbb{N}^+} \mathscr{P}^{(m)}(X)$.

With this new operator, the h.f. multisets that can be properly encoded by $\mathfrak{A}_m$ with $m \in \mathbb{N}^+ \setminus \{1\}$ can reasonably be delimited.

Definition 4.2 (Hereditarily finite m-multisets)**.** Let $m \in \mathbb{N}^+ \setminus \{1\}$. Then,

$$\mathsf{HF}^{(m)}_n = \begin{cases} \emptyset & \text{if } n = 0 \\ \mathscr{P}^{(m-1)}(\mathsf{HF}^{(m)}_{n-1}) & \text{if } n \in \mathbb{N}^+, \end{cases} \qquad \mathsf{HF}^{(m)} = \bigcup_{n \in \mathbb{N}} \mathsf{HF}^{(m)}_n$$

defines the cumulative hierarchy of the *hereditarily finite m-multisets.*

The simplest case occurs when $m = 2$, since $\mathsf{HF} = \mathsf{HF}^{(2)}$ as subfamilies of h.f. multisets. More generally, it is guaranteed that the elements of each $h \in \mathsf{HF}^{(m)}$ have multiplicities at most $m-1$ at any nesting depth. Each $\mathsf{HF}^{(m)}$ ranges over every layer HF^{μ}_n and $\mathsf{HF}^{\mu} = \bigcup_{m \in \mathbb{N}^+ \setminus \{1\}} \mathsf{HF}^{(m)}$, in complete analogy with the relationship between $\mathscr{P}^{\mu}$ and $\mathscr{P}^{(m)}$. Moreover, the rank of an h.f. multiset inside $\mathsf{HF}^{(m)}$ is the same as inside HF^{μ}:

$$\mathsf{HF}^{(m)}_n = \mathsf{HF}^{(m)} \cap \mathsf{HF}^{\mu}_n.$$

Given the above definitions, the following statement holds true.

Theorem 4.1. *Let $m \in \mathbb{N}^+ \setminus \{1\}$. Then the encoding map*

$$\mathfrak{A}_m\big|_{\mathsf{HF}^{(m)}}: \quad \mathsf{HF}^{(m)} \longrightarrow \mathbb{N}$$

is bijective.

Proof. Given any $h \in \mathsf{HF}^{(m)}$, its code is well-defined by the recursive construction of the map $\mathfrak{A}_m$. On the other hand, any $n \in \mathbb{N}$ can be expressed as a sum of powers of m:

$$n \quad = \quad a_0 + a_1 m + a_2 m^2 + \cdots + a_k m^k\,, \quad \text{with} \quad a_i \in \{0, \ldots, m-1\} \;\text{ and }\; k \in \mathbb{N}.$$

Then the proof follows by induction on the degree k, given the base case

$$a_0 = \mathfrak{A}_m\big(\underbrace{[\emptyset, \ldots, \emptyset]}_{a_0}\big).$$

□

This meaningful property ensures a total ordering of each $\mathsf{HF}^{(m)}$; a total ordering of the whole HF^{μ} would be a limit case of all such encodings, but it cannot be implemented without introducing transfinite ordinals. Moreover, since $\mathsf{HF}^{(m)}$ is naturally embedded into $\mathsf{HF}^{(m+1)}$, the ordering given by $\mathfrak{A}_m$ to the former is kept by the ordering given by $\mathfrak{A}_{m+1}$ to the latter.

Despite these promising results about $\mathfrak{A}_m$ over $\mathsf{HF}^{(m)}$, no $\beta = m$ would remain acceptable over the whole families of h.f. sets and multisets, due to violated injectivity: with the given definition, every $\mathfrak{A}_m$-code of a set in $\mathsf{HF} \setminus \{\emptyset, \{\emptyset\}\}$ is the code of at least one multiset in $\mathsf{HF}^{\mu} \setminus \mathsf{HF}$.

Moreover, recalling what this paper is aimed at, the most significant reason to abandon any encoding attempt with an integer basis is that there is no convergence on the codes of hypersets.

$\mathfrak{A}_3(h)$	$(\mathfrak{A}_3(h))_3$	$\sum_{h'\in h}\mu_h(h')\cdot 3^{\mathfrak{A}_3(h')}$	Multiset	Corr. set
0	0	0	$\emptyset$	$\emptyset$
1	1	3^0	$[\emptyset]$	$\{\emptyset\}$
2	2	$2\cdot 3^0$	$[\emptyset,\emptyset]$	$\{\emptyset\}$
3	10	3^1	$[[\emptyset]]$	$\{\{\emptyset\}\}$
4	11	3^1+3^0	$[[\emptyset],\emptyset]$	$\{\{\emptyset\},\emptyset\}$
5	12	$3^1+2\cdot 3^0$	$[[\emptyset],\emptyset,\emptyset]$	$\{\{\emptyset\},\emptyset\}$
6	20	$2\cdot 3^1$	$[[\emptyset],[\emptyset]]$	$\{\{\emptyset\}\}$
7	21	$2\cdot 3^1+3^0$	$[[\emptyset],[\emptyset],\emptyset]$	$\{\{\emptyset\},\emptyset\}$
8	22	$2\cdot 3^1+2\cdot 3^0$	$[[\emptyset],[\emptyset],\emptyset,\emptyset]$	$\{\{\emptyset\},\emptyset\}$
9	100	3^2	$[[\emptyset,\emptyset]]$	$\{\{\emptyset\}\}$
10	101	3^2+3^0	$[[\emptyset,\emptyset],\emptyset]$	$\{\{\emptyset\},\emptyset\}$
11	102	$3^2+2\cdot 3^0$	$[[\emptyset,\emptyset],\emptyset,\emptyset]$	$\{\{\emptyset\},\emptyset\}$
12	110	3^2+3^1	$[[\emptyset,\emptyset],[\emptyset]]$	$\{\{\emptyset\}\}$
13	111	$3^2+3^1+3^0$	$[[\emptyset,\emptyset],[\emptyset],\emptyset]$	$\{\{\emptyset\},\emptyset\}$
14	112	$3^2+3^1+2\cdot 3^0$	$[[\emptyset,\emptyset],[\emptyset],\emptyset,\emptyset]$	$\{\{\emptyset\},\emptyset\}$
15	120	$3^2+2\cdot 3^1$	$[[\emptyset,\emptyset],[\emptyset],[\emptyset]]$	$\{\{\emptyset\}\}$
16	121	$3^2+2\cdot 3^1+3^0$	$[[\emptyset,\emptyset],[\emptyset],[\emptyset],\emptyset]$	$\{\{\emptyset\},\emptyset\}$
17	122	$3^2+2\cdot 3^1+2\cdot 3^0$	$[[\emptyset,\emptyset],[\emptyset],[\emptyset],\emptyset,\emptyset]$	$\{\{\emptyset\},\emptyset\}$
18	200	$2\cdot 3^2$	$[[\emptyset,\emptyset],[\emptyset,\emptyset]]$	$\{\{\emptyset\}\}$
19	201	$2\cdot 3^2+3^0$	$[[\emptyset,\emptyset],[\emptyset,\emptyset],\emptyset]$	$\{\{\emptyset\},\emptyset\}$

Table 1
The first 20 multisets of $\mathsf{HF}^{(3)}$ with respect to the ordering given by the $\mathfrak{A}_3$ encoding, together with their corresponding h.f. sets (multiple occurrences are merged together), their $\mathfrak{A}_3$-codes as base-10 and base-3 numbers, and as a sum of powers of 3 to underline the strict correspondence between digits and multiplicities. Notice that the codes 0,1,3,4 etc. of multisets also belonging to HF are sequences of just 0s and 1s in base 3.

5. Other bases

5.1. Over hypersets

A most remarkable result concerning iterated exponentials, first discovered by L. Euler in 1777 and subsequently re-discovered and proven in several papers, is shown below.[7] Here, its statement is taken with slight changes from [18].

Theorem 5.1. *The function $x = f(z) = \lim_{n\to\infty} \beth_z(n) = z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}$ converges when $e^{-e} \le z \le e^{1/e}$ and diverges for all other positive z outside this interval. On this interval f is the partial inverse of g [namely, $g(x) = x^{1/x}$], that is,*

$$g\big(f(z)\big) = z \qquad \text{if } e^{-e} \le z \le e^{1/e},$$
$$f\big(g(x)\big) = x \qquad \text{if } e^{-1} \le x \le e$$

[$e = 2.71828\ldots$ is the Euler number*]. In particular, four nontrivial modes of convergence and divergence occur.*

Case 1*: $z > 1$. The sequence of hyperpowers increases monotonically: $\beth_z(1) < \beth_z(2) < \beth_z(3) < \cdots$.*

Subcase 1c*: $1 < z \le e^{1/e}$. The sequence is bounded by e, and so $f(z)$ converges.*

[7] [17]. A brief history of this theorem can be found in, e.g., [18] and [19].

Subcase 1d*:* $e^{1/e} < z$. *The sequence increases without bound, and so* $f(z)$ *diverges.*

Case 2*:* $z < 1$. *The sequence of hyperpowers oscillates:* $\daleth_z(2n) < \daleth_z(2n-1)$ *for* $n > 1$ *and, moreover, the two subsequences* $\daleth_z(2) < \daleth_z(4) < \daleth_z(6) < \cdots$ *and* $\cdots < \daleth_z(5) < \daleth_z(3) < \daleth_z(1)$ *each converge.*

Subcase 2c*:* $e^{-e} \le z < 1$. *The preceding two subsequences of odd and even hyperpowers converge to the same value, and so* $f(z)$ *converges.*

Subcase 2d*:* $z < e^{-e}$. *The preceding two subsequences each converge separately to different values, and so* $f(z)$ *diverges.*

Notice that, if the infinitely iterated exponential function f converges, then its limit is a solution of the exponential equation $x = z^x$; the function $k(x,z) = z^x - x$ has a unique zero if $0 < z \le 1$, two zeros if $1 < z \le e^{1/e}$ – actually, for $z = e^{1/e}$ there is a unique zero with multiplicity 2, namely $x = e$ – and no zeros if $z > e^{1/e}$. In the second case, the limit of the function is the lower of such zeros (see, e.g., [19]).

Definition 5.1. Let $\beta \in \mathbb{R}^+$, $\beta \le e^{1/e}$; then $\boldsymbol{\Omega}_{\beta^{-1}}$ is defined as the lower solution of the equation $x = \beta^x$, i.e.

$$\boldsymbol{\Omega}_{\beta^{-1}} = \min_{x\in\mathbb{R}^+} \{x \mid x = \beta^x\}.$$

Corollary 5.1. *Let* $\beta \in \mathbb{R}^+$, $s_n = \daleth_\beta(n) = \mathfrak{A}_\beta(\{\emptyset\}^n)$.

Case 1*:* $e^{-e} \le \beta < 1$. *The following hold true.*

$$0 = s_0 < s_2 < \cdots < s_{2i} < \cdots < \boldsymbol{\Omega}_{\beta^{-1}} < \cdots < s_{2i+1} < \cdots < s_3 < s_1 = 1,$$
$$\lim_{i\to\infty} s_{2i} = \boldsymbol{\Omega}_{\beta^{-1}} = \lim_{i\to\infty} s_{2i+1}.$$

Case 2*:* $1 < \beta \le e^{1/e}$. *The following hold true.*

$$0 = s_0 < 1 = s_1 < \beta = s_2 < \cdots < s_i < \cdots < \boldsymbol{\Omega}_{\beta^{-1}},$$
$$\lim_{i\to\infty} s_i = \boldsymbol{\Omega}_{\beta^{-1}}.$$

For a further generalisation of $\mathfrak{A}_\beta$-codes to $\mathsf{HF}^{1/2}$, which follows naturally from the previous corollary on the code of Ω, consider the following extension of the concept of *code system* (Definition 4, [11]).

Definition 5.2 ($\mathfrak{A}_\beta$-code systems)**.** Consider the set system

$$\mathscr{S}(\varsigma_0, \varsigma_1, \ldots, \varsigma_n) = \begin{cases} \varsigma_0 = [\varsigma_{0,1}, \ldots, \varsigma_{0,m_0}] \\ \varsigma_1 = [\varsigma_{1,1}, \ldots, \varsigma_{1,m_1}] \\ \quad\vdots \\ \varsigma_n = [\varsigma_{n,1}, \ldots, \varsigma_{n,m_n}] \end{cases}$$

and define its *index map*

$$I_{\mathscr{S}}\colon \quad \bigcup_{i=0}^{n} \{\langle i,j\rangle \mid 1 \le j \le m_i\} \longrightarrow \{0,1,\ldots,n\},$$

that associates the index of the unknown $\varsigma_{i,j}$ to its corresponding index in the list $\varsigma_0, \varsigma_1, \ldots, \varsigma_n$, namely $\varsigma_{i,j} = \varsigma_{I_{\mathscr{S}}(i,j)}$. Given $\beta \in \mathbb{R}^+ \setminus \{1\}$, $e^{-e} \le \beta \le e^{1/e}$, the $\mathfrak{A}_\beta$*-code system* of $\mathscr{S}(\varsigma_0, \varsigma_1, \ldots, \varsigma_n)$ in the real unknowns $x_0, x_1, \ldots, x_n$ is

$$\mathscr{C}_\beta(x_0, x_1, \ldots, x_n) = \begin{cases} x_0 = \beta^{x_{0,1}} + \cdots + \beta^{x_{0,m_0}} \\ x_1 = \beta^{x_{1,1}} + \cdots + \beta^{x_{1,m_1}} \\ \quad\vdots \\ x_n = \beta^{x_{n,1}} + \cdots + \beta^{x_{n,m_n}}, \end{cases}$$

where $x_{i,j}$ is a shorthand for $x_{I_{\mathscr{S}}(i,j)}$ for $i \in \{0,1,\ldots,n\}$ and $j \in \{1,\ldots,m_i\}$.

Definition 5.3 (Normal set systems). A set system $\mathscr{S}(\varsigma_0, \varsigma_1, \ldots, \varsigma_n)$ is said to be *normal* if there exist $n+1$ pairwise non-bisimilar hypersets $\hbar_0, \hbar_1 \ldots, \hbar_n \in \mathsf{HF}^{1/2}$ that satisfy all the equations in $\mathscr{S}(\varsigma_0, \varsigma_1, \ldots, \varsigma_n)$ once assigned to $\varsigma_0, \varsigma_1, \ldots, \varsigma_n$.

The following definitions are taken from [11] to show how rational hypersets can be arbitrarily approximated by sequences of sets or multisets.

Definition 5.4 (Multiset approximating sequences). Consider a normal set system $\mathscr{S}(\varsigma_0, \varsigma_1, \ldots, \varsigma_n)$, and let $I_{\mathscr{S}}$ be its index map. The *multiset approximating sequence* for the solution of $\mathscr{S}(\varsigma_0, \varsigma_1, \ldots, \varsigma_n)$ is the sequence $(\langle H_i^j \mid 0 \leq i \leq n\rangle)_{j\in\mathbb{N}}$ of the $(n+1)$-tuples of well-founded hereditarily finite multisets defined by

$$\langle H_i^j \mid 0 \leq i \leq n\rangle = \begin{cases} \langle \emptyset \mid 0 \leq i \leq n\rangle & \text{if } j = 0 \\ \big\langle [H_{i,1}^j, \ldots, H_{i,m_i}^j] \mid 0 \leq i \leq n\big\rangle & \text{if } j > 0, \end{cases}$$

where $H_{i,u}^{j-1}$ is a shorthand for $H_{I_{\mathscr{S}}(i,u)}^{j-1}$, for every $i \in \{0, 1, \ldots, n\}$ and every $u \in \{1, \ldots, m_i\}$.

Given a normal set system $\mathscr{S}(\varsigma_0, \varsigma_1, \ldots, \varsigma_n)$, two distinct unknowns ς_i and $\varsigma_{i'}$, with $i, i' \in \{0, 1, \ldots, n\}$, are said to be *distinguished* at step $k > 0$ by its multiset approximating sequence if $H_i^k \neq H_{i'}^k$.[8] Further, H_i^j is referred as the j-th *multiset approximation value* of ς_i.

Some significant properties of set and multiset approximating sequences are shown in [11]. Another definition is here adapted from [11], with the aim of mirroring the multiset approximating sequence for the solution of a normal set system to a numerical approximating sequence for the solution of the corresponding $\mathfrak{A}_\beta$-code. The interval initially considered is delimited by e^{-e} and $e^{1/e}$ to guarantee at least the convergence on the code of Ω by Corollary 5.1.

Definition 5.5 ($\mathfrak{A}_\beta$-code increment sequences). Assume $\beta \in \mathbb{R}^+ \setminus \{1\}$, $e^{-e} \leq \beta \leq e^{1/e}$ and consider the set system $\mathscr{S} = (\varsigma_0, \varsigma_1, \ldots, \varsigma_n)$ with index map $I_{\mathscr{S}}$ and multiset approximating sequence $(\langle H_i^j \mid 0 \leq i \leq n\rangle)_{j\in\mathbb{N}}$. The $\mathfrak{A}_\beta$*-code increment sequence* $(\langle \delta_i^j \mid 0 \leq i \leq n\rangle)_{j\in\mathbb{N}}$ for the system $\mathscr{S}(\varsigma_0, \varsigma_1, \ldots, \varsigma_n)$ is defined as

$$\delta_i^j = \mathfrak{A}_\beta(H_i^{j+1}) - \mathfrak{A}_\beta(H_i^j)$$

for every $i \in \{0, 1, \ldots, n\}$ and $j \in \mathbb{N}$.

Some meaningful properties are stated below.

Lemma 5.1. *Given* $\beta \in \mathbb{R}^+$, $\beta < 1$,

$$(\forall x, y \in \mathbb{R})\big(|y| \leq |x| \wedge xy \leq 0 \Rightarrow |\beta^y - 1| \leq \big|\beta^{-x} - 1\big|\big).$$

Proof. Let $x, y \in \mathbb{R}$ such that $xy \leq 0$, and suppose $|y| \leq |x|$. Then, since $\beta < 1$,

$$1 \leq \beta^{-|y|} \leq \beta^{-|x|}.$$

Suppose $y \leq 0 \leq x$, then

$$1 \leq \beta^y \leq \beta^{-x} \quad \Rightarrow \quad 0 \leq \beta^y - 1 \leq \beta^{-x} - 1 \quad \Rightarrow \quad |\beta^y - 1| \leq |\beta^{-x} - 1|.$$

Otherwise, i.e. $x \leq 0 \leq y$,

$$1 \leq \beta^{-y} \leq \beta^x \quad \Rightarrow \quad \beta^{-x} \leq \beta^y \leq 1 \quad \Rightarrow \quad \beta^{-x} - 1 \leq \beta^y - 1 \leq 0 \quad \Rightarrow \quad 0 \leq |\beta^y - 1| \leq |\beta^{-x} - 1|.$$

□

[8] If two unknowns are distinguished at a certain step, then they are distinguished at every subsequent step; see Lemma 2 (a), [11].

Lemma 5.2. *Given $\beta \in \mathbb{R}^+ \setminus \{1\}$, $e^{-e} \leq \beta \leq e^{1/e}$, let $\mathscr{S}(\varsigma_0, \varsigma_1, \ldots, \varsigma_n)$ be a normal set system with index map $I_{\mathscr{S}}$, multiset approximating sequence $(\langle H_i^j \mid 0 \leq i \leq n\rangle)_{j\in\mathbb{N}}$ and $\mathfrak{A}_\beta$-code increment sequence $(\langle \delta_i^j \mid 0 \leq i \leq n\rangle)_{j\in\mathbb{N}}$. Then, for every $i \in \{0, 1, \ldots, n\}$ and $j \in \mathbb{N}$, the following facts hold true.*

$$\mathfrak{A}_\beta(H_i^{j+1}) = \sum_{k=0}^{j} \delta_i^k \tag{1}$$

$$\delta_i^0 = m_i \tag{2}$$

$$\delta_i^{j+1} = \sum_{u=1}^{m_i} \beta^{\mathfrak{A}_\beta(H_{i,u}^j)}\left(\beta^{\delta_{i,u}^j} - 1\right) \qquad \text{where } H_{i,u}^j = H^j_{I_{\mathscr{S}}(i,u)} \tag{3}$$

Moreover, if $\beta < 1$,

$$\delta_i^{2j+1} \leq 0 \leq \delta_i^{2j}, \tag{4}$$

$$|\delta_i^{j+1}| \leq |\delta_i^j|, \tag{5}$$

while, if $\beta > 1$,

$$0 \leq \delta_i^j, \tag{6}$$

$$(\exists k)\left(\delta_i^{k+1} \geq \delta_i^k\right) \Rightarrow \lim_{j\to\infty} \delta_i^j > 0. \tag{7}$$

Proof. Fix an index $i \in \{0, 1, \ldots, n\}$; along the proof, the following shorthand notation will be adopted.

$$H_{i,u}^j = H^j_{I_{\mathscr{S}}(i,u)}, \qquad \delta_{i,u}^j = \delta^j_{I_{\mathscr{S}}(i,u)}, \qquad m_{i,u} = m_{I_{\mathscr{S}}(i,u)}.$$

Claim (1). It is inductively proven by using the definition of $\mathfrak{A}_\beta$-code increment sequence:

$$\mathfrak{A}_\beta(H_i^1) = \delta_i^0 + \mathfrak{A}_\beta(H_i^0) = \delta_i^0$$

$$\mathfrak{A}_\beta(H_i^{j+1}) = \delta_i^j + \mathfrak{A}_\beta(H_i^j) = \delta_i^j + \sum_{k=0}^{j-1} \delta_i^k = \sum_{k=0}^{j} \delta_i^k.$$

Claim (2). This is another trivial consequence of the given definitions. Indeed, the codes of the first step of multiset approximating sequence are the sum of as many 1s as the elements of the considered hyperset, the empty set being the step 0 of whatever sequence.

Claim (3). By expanding

$$\delta_i^{j+1} = \sum_{u=1}^{m_i} \beta^{\mathfrak{A}_\beta(H_{i,u}^{j+1})} - \sum_{u=1}^{m_i} \beta^{\mathfrak{A}_\beta(H_{i,u}^j)} = \sum_{u=1}^{m_i}\left(\beta^{\mathfrak{A}_\beta(H_{i,u}^{j+1})} - \beta^{\mathfrak{A}_\beta(H_{i,u}^j)}\right) =$$

$$= \sum_{u=1}^{m_i}\left(\beta^{\mathfrak{A}_\beta(H_{i,u}^j)+\delta_{i,u}^j} - \beta^{\mathfrak{A}_\beta(H_{i,u}^j)}\right) = \sum_{u=1}^{m_i} \beta^{\mathfrak{A}_\beta(H_{i,u}^j)} \cdot \left(\beta^{\delta_{i,u}^j} - 1\right).$$

the result is proven.

The next two claims hold just in the case $\beta < 1$; since this is also the case of $\mathfrak{A}_{1/2} = \mathbb{R}_A$, they conclude the part treated also by Lemma 4, [11] in that specific case.

Claim (4). It follows by (2) and (3), since $\delta_i^0 \geq 0$ for every $i \in \{0, 1, \ldots, n\}$, so that δ_i^1 is a non-positive sum, because $\beta^{\delta_i^0} - 1 \leq 0$. Therefore, by induction, the $\mathfrak{A}_\beta$-code increment sequence assumes alternatively non-negative and non-positive values, for even and odd indices respectively.

Claim (5). The key step in the proof of this claim is the inequality

$$\beta^{\delta_i^j} \cdot \left|\beta^{\delta_i^{j+1}} - 1\right| \leq \left|\beta^{\delta_i^j} - 1\right|. \tag{8}$$

which will now be proven by induction on $j \in \mathbb{N}$, for every $i \in \{0, 1, \ldots, n\}$. Since

$$\left|\delta_i^1\right| = \sum_{u=1}^{m_i} \beta^{\mathfrak{A}_\beta(H_{i,u}^0)} \cdot \left|\beta^{\delta_{i,u}^0} - 1\right| = \sum_{u=1}^{m_i} \left|\beta^{\delta_{i,u}^0} - 1\right| = \sum_{u=1}^{m_i} (1 - \beta^{m_{i,u}}) \leq m_i = \left|\delta_i^0\right|, \tag{9}$$

and thus $0 \leq -\delta_i^1 \leq \delta_i^0$ by claim (4), the base case of (8) is proven by

$$0 \geq \delta_i^1 \geq -\delta_i^0 \quad \Rightarrow \quad 1 \leq \beta^{\delta_i^1} \leq \beta^{-\delta_i^0} \quad \Rightarrow \quad 0 \leq \beta^{\delta_i^1} - 1 \leq \beta^{-\delta_i^0} - 1 \quad \Rightarrow$$
$$\Rightarrow \quad 0 \leq \beta^{\delta_i^0} \cdot \left(\beta^{\delta_i^1} - 1\right) \leq 1 - \beta^{\delta_i^0} \quad \Rightarrow \quad \beta^{\delta_i^0} \cdot \left|\beta^{\delta_i^1} - 1\right| \leq \left|\beta^{\delta_i^0} - 1\right|.$$

Then, to move on from the induction hypothesis (8) to the next value of j, we argue as follows:

$$\left|\delta_i^{j+2}\right| = \sum_{u=1}^{m_i} \beta^{\mathfrak{A}_\beta(H_{i,u}^{j+1})} \cdot \left|\beta^{\delta_{i,u}^{j+1}} - 1\right| = \sum_{u=1}^{m_i} \beta^{\mathfrak{A}_\beta(H_{i,u}^{j}) + \delta_{i,u}^j} \cdot \left|\beta^{\delta_{i,u}^{j+1}} - 1\right| =$$
$$= \sum_{u=1}^{m_i} \beta^{\mathfrak{A}_\beta(H_{i,u}^{j})} \cdot \beta^{\delta_{i,u}^j} \cdot \left|\beta^{\delta_{i,u}^{j+1}} - 1\right| \leq \sum_{u=1}^{m_i} \beta^{\mathfrak{A}_\beta(H_{i,u}^{j})} \cdot \left|\beta^{\delta_{i,u}^{j}} - 1\right| = \left|\delta_i^{j+1}\right|.$$

Observe that (4) and $|\delta_i^{j+2}| \leq |\delta_i^{j+1}|$ for some $i \in \{0, 1, \ldots, n\}$ make δ_i^{j+1} and δ_i^{j+2} suitable values for x and y of Lemma 5.1, so that

$$\left|\beta^{\delta_i^{j+2}} - 1\right| \leq \left|\beta^{-\delta_i^{j+1}} - 1\right|$$

holds, and thus

$$\beta^{\delta_i^{j+1}} \cdot \left|\beta^{\delta_i^{j+2}} - 1\right| \leq \left|\beta^{\delta_i^{j+1}} - 1\right|.$$

This completes the proof of the claim (8). Claim (5) then follows for $j > 0$ as an immediate by-product of its proof, while in the case $j = 0$, it amounts to (9).

Consider next the case $\beta > 1$.

Claim (6). The $\mathfrak{A}_\beta$-code increment sequence is positive, since

$$\delta_i^{j+1} = \sum_{u=1}^{m_i} \beta^{\mathfrak{A}_\beta(H_{i,u}^{j})} \cdot \left(\beta^{\delta_{i,u}^{j}} - 1\right) \geq \sum_{u=1}^{m_i} \beta^{\mathfrak{A}_\beta(H_{i,u}^{j})} \geq 0,$$

being $\delta_i^0 = m_i \geq 0$ non-negative in the first place.

Claim (7). Assume $\delta_i^{k+1} \geq \delta_i^k$; it follows

$$\begin{aligned}
\delta_i^{k+2} - \delta_i^{k+1} &= \sum_{u=1}^{m_i} \beta^{\mathfrak{A}_\beta(H_{i,u}^{k+1})} \cdot \left(\beta^{\delta_{i,u}^{k+1}} - 1\right) - \sum_{u=1}^{m_i} \beta^{\mathfrak{A}_\beta(H_{i,u}^{k})} \cdot \left(\beta^{\delta_{i,u}^{k}} - 1\right) \\
&= \sum_{u=1}^{m_i} \beta^{\mathfrak{A}_\beta(H_{i,u}^{k})} \cdot \beta^{\delta_{i,u}^k} \cdot \left(\beta^{\delta_{i,u}^{k+1}} - 1\right) - \sum_{u=1}^{m_i} \beta^{\mathfrak{A}_\beta(H_{i,u}^{k})} \cdot \left(\beta^{\delta_{i,u}^{k}} - 1\right) \\
&= \sum_{u=1}^{m_i} \beta^{\mathfrak{A}_\beta(H_{i,u}^{k})} \cdot \left(\beta^{\delta_{i,u}^k}\left(\beta^{\delta_{i,u}^{k+1}} - 1\right) - \left(\beta^{\delta_{i,u}^{k}} - 1\right)\right) \\
&= \sum_{u=1}^{m_i} \beta^{\mathfrak{A}_\beta(H_{i,u}^{k})} \cdot \left(\beta^{\delta_{i,u}^k + \delta_{i,u}^{k+1}} - 2\beta^{\delta_{i,u}^{k}} + 1\right).
\end{aligned}$$

Since $\beta^{\mathfrak{A}_\beta(H_{i,u}^{k})} \geq 0$, focus now on the sum between parentheses.

$$\beta^{\delta_{i,u}^k + \delta_{i,u}^{k+1}} - 2\beta^{\delta_{i,u}^{k}} + 1 \geq \beta^{2\delta_{i,u}^k} - 2\beta^{\delta_{i,u}^{k}} + 1 = \left(\beta^{\delta_{i,u}^{k}} - 1\right)^2 \geq 0$$

Therefore, the $\mathfrak{A}_\beta$-code increment sequence is either stable or increasing from the k-th step on. □

Remark 2. Observe that, despite not being proven over $\mathsf{HF}^{1/2}$ yet, convergence on the codes of h.f. well-founded sets and multisets is guaranteed by the convergence of the multiset approximating sequence within a number of steps equal to their rank, so that the $\mathfrak{A}_\beta$-code increment sequence is constantly 0 for all the subsequent steps.

If $e^{-e} \leq \beta < 1$, the convergence on the $\mathfrak{A}_\beta$-code of Ω appears to imply at least the convergence on the $\mathfrak{A}_\beta$-codes of the other rational hypersets $\hbar$ such that $\hbar \in \hbar$. Otherwise, if $1 < \beta \leq e^{1/e}$ the convergence is not guaranteed, since whenever $\delta_i^{k+1} \geq \delta_i^k$ the code sequence $\langle \mathfrak{A}_\beta(H_i^j) \rangle_{j\in\mathbb{N}}$ diverges due to statement (7) of the preceding lemma. An initial result is presented below, showing a minimum requirement to get $\delta_i^1 < \delta_i^0$.

Proposition 5.1. *Given $\beta > 1$, consider the set system $\mathscr{S}(\varsigma_0, \varsigma_1, \dots, \varsigma_n)$ with index map $I_\mathscr{S}$; let $i \in \{0, 1, \dots, n\}$ be such that $m_i > 0$ and define $m_{i,v} = \max_{u\in\{1,\dots,m_i\}}\{m_{i,u}\}$. If $m_{i,v} < \log_\beta 2$, then $\delta_i^1 < \delta_i^0$.*

Proof. Recalling that $\delta_i^0 = m_i = \mathfrak{A}_\beta(H_i^1)$, $m_{i,v} < \log_\beta 2$ implies

$$\begin{aligned}\delta_i^1 = \sum_{u=1}^{m_i} \beta^{\mathfrak{A}_\beta(H_{i,u}^0)}(\beta^{\delta_{i,u}^0} - 1) = \sum_{u=1}^{m_i}(\beta^{\delta_{i,u}^0} - 1) = \sum_{u=1}^{m_i}(\beta^{m_{i,u}} - 1) \\ \leq m_i(\beta^{m_{i,v}} - 1) < m_i(2-1) = m_i = \delta_i^0.\end{aligned}$$

□

Observe that the above constraint appears to be quite restrictive when dealing with the convergence of the multiset approximating sequence's $\mathfrak{A}_\beta$-codes, also because it has to be strengthened at every subsequent step. Although a further analysis needs to be done on the convergence on $\mathfrak{A}_\beta$-codes of $\mathsf{HF}^{1/2}$, the existence and uniqueness of $\mathfrak{A}_\beta$-codes of all rational hypersets for a basis chosen between e^{-e} and 1 can be conjectured.[9]

Conjecture 5.1. *Consider $\beta \in \mathbb{R}$, $e^{-e} \leq \beta < 1$, and $\hbar \in \mathsf{HF}^{1/2}$. Then, there exists and is unique its $\mathfrak{A}_\beta$-code.*

$$(\forall \hbar \in \mathsf{HF}^{1/2})(\forall \beta \in \mathbb{R})\big(e^{-e} \leq \beta < 1 \Rightarrow \exists! \mathfrak{A}_\beta(\hbar) \in \mathbb{R}^+\big).$$

The challenging question of injectivity is still open; it will follow from the injectivity of the map over HF^μ, by the method of multiset approximating sequence.

5.2. Over multisets

For what concerns the application of the map $\mathfrak{A}_\beta$ to h.f. multisets, observe that in all the cases $\beta = 1/m$ with $m \in \mathbb{N}^+ \setminus \{1\}$, analogues of the properties valid for $\mathbb{R}_A^\mu$ can be found too. Injectivity is violated at the very first levels of the hierarchy, since $\mathfrak{A}_{1/m}\big(\{^m\{\emptyset\}\}\big) = m \cdot 1/m = 1$, so that to require it an analog of $\mathcal{H}_2$ has to be introduced as domain for this map.

Definition 5.6. Let $m \in \mathbb{N}^+ \setminus \{1\}$ and

$$\mathcal{H}_{m,n} \overset{\text{def}}{=} \begin{cases} \{H \in \mathsf{HF}^\mu \mid \mu_H(\{\emptyset\}) \leq m-1\} & \text{if } n = 0 \\ \{H \in \mathcal{H}_{m,0} \mid H \subseteq \mathcal{H}_{m,n-1}\} & \text{if } n \in \mathbb{N}^+. \end{cases}$$

Then define

$$\mathcal{H}_m \overset{\text{def}}{=} \bigcap_{n=0}^{\infty} \mathcal{H}_{m,n},$$

thus $\mathcal{H}_m$ is the collection of multisets containing no occurrences of $\{\emptyset\}$ with multiplicity larger than $m-1$ at any nesting depth.

[9]The case $\beta > 1$ shall be excluded in view of counterexamples such as Example 6.2 below.

Further, by developing tools analogous to the ones of [12], the following can be stated.

Conjecture 5.2. *Given $m \in \mathbb{N}^+ \setminus \{1\}$, the coding map $\mathfrak{A}_{1/m}$ is injective over the collection $\mathcal{H}_m$.*

Theorem 5.2. *Under Conjecture 5.2, every hereditarily finite set of rank at least 4 has a transcendental $\mathfrak{A}_{1/m}$-code.*

Proof. It is a rearrangement of the proof of Theorem 4.12, [12], where the *reduction operator* is generalised to

$$\rho_m(H) = \left(H \setminus \left\{{}^{m\left\lfloor \frac{k}{m} \right\rfloor}\{\emptyset\}\right\}\right) + \left\{{}^{\left\lfloor \frac{k}{m} \right\rfloor}\emptyset\right\},$$

so that it replaces every m-uple of $\{\emptyset\}$ in H with a single occurrence of $\emptyset$. □

Since multisets introduce multiple occurrences of their elements, for every algebraic basis $e^{-e} \leq \beta < 1$ there are issues similar to the ones already encountered for $\beta = 1/m$ with m natural, the latter being a subcase of the former. Indeed, since by definition an algebraic number is the root of a polynomial with integer coefficients, an algebraic basis β satisfies

$$P(\beta) = 0 \quad \text{where} \quad P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k \in \mathbb{Z}[x].$$

Therefore, in these cases it would be necessary to introduce a proper subfamily of HF^μ to exclude the possibility that the same polynomial is reproduced by the code of any multiset in it. The case of interest is the one in which $a_0 < 0$, $a_i \geq 0$ for $i \in \{1, \ldots, k\}$, so that the $\mathfrak{A}_\beta$-codes of $\{{}^{-a_0}\emptyset\}$ and $\{{}^{a_1}\{\emptyset\}, {}^{a_2}\{{}^2\emptyset\}, \ldots, {}^{a_k}\{{}^k\emptyset\}\}$ would coincide.

These observations lead to focus on transcendental real numbers within e^{-e} and 1. The most obvious choice appears to be $e^{-1} = 1/e \sim 0.36788\ldots$, since e is one of the most studied transcendental mathematical constants and is used for both the definitions of the natural logarithm and the *product logarithm* (see, e.g., [20]).

Despite being the only possible choice to encode completely both HF^μ and $\mathsf{HF}^{1/2}$, e^{-1} and any other transcendental basis suffer of a lack of knowledge about their behaviour when involved in (iterated) exponentiation. Nonetheless, having already excluded all the algebraic numbers from count, the following conjecture is stated as a motivation for future research.

Conjecture 5.3. *The $\mathfrak{A}_{e^{-1}}$ encoding of h.f. multisets and hypersets is injective over the whole universe $\mathsf{HF}^{1/2} \cup \mathsf{HF}^\mu$.*

6. Other results

Example 6.1 (Non-injectivity of $\mathbb{R}_A$). Consider $\beta = 1/2$; then, $\mathfrak{A}_{1/2}(\hbar) = \mathbb{R}_A(\hbar) = 1$, where

$$\hbar = \{\{\{\{\ldots\}, \{\emptyset\}\}, \{\emptyset\}\}, \{\emptyset\}\}$$

is the solution of the set system

$$\mathscr{S}(\varsigma_0, \varsigma_1, \varsigma_2) = \begin{cases} \varsigma_0 = \{\varsigma_0, \varsigma_1\} \\ \varsigma_1 = \{\varsigma_2\} \\ \varsigma_2 = \{\}. \end{cases}$$

It suffices to observe that the corresponding $\mathfrak{A}_{1/2}$-code is the solution of $x = 2^{-x} + 2^{-1}$, which is trivially 1. Moreover, notice that by iteratively putting $\hbar_0 = \hbar$ and $\hbar_n = \{\hbar_n, \hbar_{n-1}\}$ for $n \in \mathbb{N}^+$, countably many non-bisimilar hypersets with $\mathfrak{A}_{1/2}$-code 1 are obtained.

Example 6.2 (Convergence and divergence at the two extreme cases)**.** Consider the hyperset

$$\hbar = \{\{\{\{\dots\},\emptyset\},\emptyset\},\emptyset\},$$

solution of

$$\mathscr{S}(\varsigma_0,\varsigma_1) = \begin{cases} \varsigma_0 = \{\varsigma_0,\varsigma_1\} \\ \varsigma_1 = \{\}. \end{cases}$$

By considering its multiset approximating sequence and the corresponding $\mathfrak{A}_\beta$-code approximating sequence for the bases $\beta = e^{-e}$ and $\beta = e^{1/e}$, it turns out that the former guarantees convergence to a finite real number, while the latter does not. This is easily explained by using an analytical approach: indeed, observe that the $\mathfrak{A}_\beta$-code of $\hbar$ is the root of the equation $x = \beta^x + 1$, so that its existence depends on the behaviour of the function $g(x) = (x-1)^{1/x}$. To find out its extrema, consider

$$\frac{d}{dx}g(x) = \frac{(x-1)^{1/x-1}\big(x-(x-1)\ln(x-1)\big)}{x^2}.$$

that is zero at the point

$$x = e^{W(1/e)+1} + 1 \sim 4.59112\dots \quad \text{s.t.} \quad z = g(x) = (e^{W(1/e)+1})^{1/(e^{W(1/e)+1}+1)} \sim 1.32110\dots$$

where $W(\cdot)$ is the principal branch of the product logarithm. This last value, representing the maximum value of $g(x)$, is the greatest that can be assigned to β to ensure the convergence on the $\mathfrak{A}_\beta$-code of $\hbar$.

7. Conclusions and open problems

A parameterised encoding scheme has been defined, in such a way so as to embrace both the celebrated Ackermann encoding and the new and less known $\mathbb{R}_A^\mu$ map, the latter being a version of the former that permits an extension of its domain from hereditarily finite well-founded sets to the universes of h.f. ill-founded sets (hypersets) and h.f. (well-founded) multisets. This generalised Ackermann encoding, defined as

$$\mathfrak{A}_\beta(h) = \sum_{h' \in h} \mu_h(h')\beta^{\mathfrak{A}_\beta(h')}$$

– where $\mu_h(h')$ expresses the multiplicity of h' inside h – depends on a real, positive and non-1 basis β, whose choice turns out to be significant to obtain both an everywhere convergent and injective map, at least on a subuniverse.

Selecting such a basis β among natural numbers has shown how one can define a subfamily of h.f. multisets over which the map is bijective. This mapping then gives a total ordering of this subfamily, and given the code of a multiset in it one can extrapolate which multisets belong to it with its corresponding multiplicity. This result appears to be promising in algorithmics to encode multisets whose maximum multiplicity is known, at any nesting depth. Despite this, all these encodings cannot apply to h.f. hypersets, since all their codes would be missing.

Given this first example, the focus has shifted to an interval – including 1 – on which a theorem by Euler ensures at least the convergence on the code of the hyperset $\Omega = \{\Omega\}$. Following the example of a previous paper concerning $\mathbb{R}_A$, a way to approximate hypersets and their codes has been shown (with a basis within this interval); some properties of these approximating sequences have been proven, concluding that it is plausible that any of such bases can encode properly the universe of h.f. hypersets and multisets if lower than 1.

Once the most promising interval where to look for β was found, the algebraic bases have been excluded, due to issues regarding injectivity over h.f. multisets. This suggested that one should adopt a transcendental basis, e.g., the inverse Euler number e^{-1}, for future studies; the main difficulty in this case is the lack of knowledge on the behaviour of iterated exponentials with a transcendental basis.

Despite the results obtained about this encoding scheme, several problems are still left open. The challenge of proving existence and uniqueness of codes of all the h.f. rational hypersets is still there, even when choosing a basis within the aforementioned interval and lower than 1. Moreover, determining the range in which a $\beta > 1$ must lie in to ensure existence of the code of a h.f. hyperset might be a way to introduce a non-arbitrary concept of rank for the universe of such aggregates. Nonetheless, this would follow from a deep study of nested exponential equations in two variables, which are made possible just with a significant knowledge of real analysis.

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