On the Importance of the Deadlock Trap Property for Monotonic Liveness

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Abstract. In Petri net systems, liveness is an important property capturing the idea of no transition (action) becoming non-fireable (unattainable). Additionally, in some situations it is particularly interesting to check if the net system is (marking) monotonically live, i.e., it remains live for any marking greater than the initial one. In this paper, we discuss structural conditions preserving liveness under arbitrary marking increase. It is proved that the deadlock trap property (DTP) is a necessary condition for liveness monotonicity of ordinary nets, and necessary and sufficient for some subclasses. We illustrate also how the result can be used to study liveness monotonicity for non-ordinary nets using a simulation preserving the firing language. Finally, we apply these conditions to several case studies of biomolecular networks.

1 Motivation

Petri nets are a natural choice to represent biomolecular networks. Various types of Petri nets may be useful – qualitative, deterministically timed, stochastic, continuous or hybrid ones, depending on the available information and the kind of properties to be analysed. Accordingly, the integrative framework demonstrated by several case studies in [GHR$^+$08], [HGD08], [HDG10] applies a family of related Petri net models, sharing structure, but differing in their kind of kinetic information.

A key notion of the promoted strategy of biomodel engineering is the *level concept*, which has been introduced in the Petri net framework in [GHL07]. Here, a token stands for a specific amount of mass, defined by the total mass divided by the number of levels. Thus, increasing the token number to represent a certain amount of mass means to increase the resolution of accuracy.

This procedure silently assumes some kind of behaviour preservation while the marking is increased (typically multiplied by a factor) to represent a finer granularity of the mass flowing through the network. However, as it is well-known in Petri net theory, liveness is not monotonic with respect to (w.r.t.) the initial marking for general Petri nets. Thus, there is no reason to generally assume that

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there is no significant change in the possible behaviour by marking increase. Contrary, under liveness monotonicity w.r.t. the initial marking we can expect continuization (fluidization) to be reasonable. However, only a particular kind of monotonicity seems to be needed for continuization: *homothetic liveness*, i.e., liveness preservation while multiplying the initial marking by k [RTS99], [SR02].

At structural level, (monotonic) liveness can be considered using transformation (reduction) rules [Ber86], [Sil85], [Mur89], [Sta90], the classical analysis for ordinary nets based on the Deadlock Trap Property (DTP) [Mur89], [Sta90], or the results of *Rank Theorems*, which are directly applicable to non-ordinary nets [TS96], [RTS98]. In this paper, we concentrate on the DTP, which will initially be used for ordinary net models, and later extended to non-ordinary ones.

This paper is organized as follows. We start off with recalling relevant notions and results of Petri net theory. Afterwards we introduce the considered subject by looking briefly at two examples, before turning to our main result yielding a necessary condition for monotonic liveness. We demonstrate the usefulness of our results for the analysis of biomolecular networks by a variety of case studies. We conclude with an outlook on open issues.

2 Preliminaries

We assume basic knowledge of the standard notions of place/transition Petri nets, see e.g. [DHP$^+$93], [HGD08], [DA10]. To be self-contained we recall the fundamental notions relevant for our paper.

Definition 1 (Petri net, syntax).
A Petri net *is a tuple* $\mathcal{N} = \langle P, T, \boldsymbol{Pre}, \boldsymbol{Post}\rangle$*, and a* Petri net system *is a tuple* $\Sigma = \langle \mathcal{N}, \boldsymbol{m}_0\rangle$*, where*

- P *and* T *are finite, non-empty, and disjoint sets.* P *is the set of* places*.* T *is the set of* transitions*.*
- $\boldsymbol{Pre}, \boldsymbol{Post} \in \mathbb{N}^{|P|\times|T|}$ *are the pre- and post-matrices, where* $|\cdot|$ *is the cardinality of a set, i.e., its number of elements. For a place* $p_i \in P$ *and a transition* $t_j \in T$*,* $Pre(p_i, t_j)$ *is the weight of the arc connecting* p_i *to* t_j *(*0 *if there is no arc), while* $Post(p_i, t_j)$ *is the weight of the arc connecting* t_j *to* p_i*.*
- $\boldsymbol{m}_0 \in \mathbb{N}^{|P|}_{\geq 0}$ *gives the* initial marking*.*
- $\boldsymbol{m}(p)$ *yields the number of tokens on place* p *in the marking* $\boldsymbol{m}$*. A place* p *with* $\boldsymbol{m}(p) = 0$ *is called* empty (unmarked) *in* $\boldsymbol{m}$*, otherwise it is called* marked *(non-empty). A set of places is called empty if all its places are empty, otherwise marked.*
- *The* preset *and* postset *of a node* $x \in P \cup T$ *are denoted by* ${}^\bullet x$ *and* $x^\bullet$*. They represent the input and output transitions of a place* x*, or the input and output places of a transition* x*. More specifically, if* $t_j \in T$*,* ${}^\bullet t_j = \{p_i \in P | Pre(p_i, t_j) > 0\}$ *and* $t_j{}^\bullet = \{p_i \in P | Post(p_i, t_j) > 0\}$*. Similarly, if* $p_i \in P$*,* ${}^\bullet p_i = \{t_j \in T | Post(p_i, t_j) > 0\}$ *and* $p_i{}^\bullet = \{t_j \in T | Pre(p_i, t_j) > 0\}$*.*
  We extend both notions to a set of nodes $X \subseteq P \cup T$ *and define the set of all prenodes* ${}^\bullet X := \bigcup_{x\in X} {}^\bullet x$*, and the set of all postnodes* $X^\bullet := \bigcup_{x\in X} x^\bullet$*.*

- *A node* $x \in P \cup T$ *is called* source node*, if* $^\bullet x = \emptyset$*, and* sink node *if* $x^\bullet = \emptyset$*. A* boundary node *is either a sink or a source node (but not both, because we assume a connected net).*

Definition 2 (Petri net, behaviour). *Let* $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$ *be a net system.*

- *A transition* t *is* enabled *at marking* $\boldsymbol{m}$*, written as* $\boldsymbol{m}[t\rangle$*, if* $\forall p \in {}^\bullet t : \boldsymbol{m}(p) \geq Pre(p,t)$*, else* disabled*.*
- *A transition* t*, enabled in* $\boldsymbol{m}$*, may* fire *(occur), leading to a new marking* $\boldsymbol{m'}$*, written as* $\boldsymbol{m}[t\rangle\boldsymbol{m'}$*, with* $\forall p \in P : \boldsymbol{m'}(p) = \boldsymbol{m}(p) - Pre(p,t) + Post(p,t)$*.*
- *The set of all markings* reachable *from a marking* $\boldsymbol{m}_0$*, written as* $[\boldsymbol{m}_0\rangle$*, is the smallest set such that* $\boldsymbol{m}_0 \in [\boldsymbol{m}_0\rangle$*,* $\boldsymbol{m} \in [\boldsymbol{m}_0\rangle \wedge \boldsymbol{m}[t\rangle\boldsymbol{m'} \Rightarrow \boldsymbol{m'} \in [\boldsymbol{m}_0\rangle$*.*
- *The* reachability graph (RG) *is a directed graph with* $[\boldsymbol{m}_0\rangle$ *as set of nodes, and the labelled arcs denote the reachability relation* $\boldsymbol{m}[t\rangle\boldsymbol{m'}$*.*

Definition 3 (Behavioural properties). *Let* $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$ *be a net system.*

- *A place* p *is* k-bounded *(bounded for short) if there is a positive integer number* k*, serving as an upper bound for the number of tokens on this place in all reachable markings of the Petri net:* $\exists\, k \in \mathbb{N}_0 : \forall \boldsymbol{m} \in [\boldsymbol{m}_0\rangle : \boldsymbol{m}(p) \leq k$ *.*
- *A Petri net system is* k-bounded *(bounded for short) if all its places are* k-bounded*.*
- *A transition* t *is* dead at marking $\boldsymbol{m}$ *if it is not enabled in any marking* $\boldsymbol{m'}$ *reachable from* $\boldsymbol{m}$*:* $\nexists\, \boldsymbol{m'} \in [\boldsymbol{m}\rangle : \boldsymbol{m'}[t\rangle$*.*
- *A transition* t *is* live *if it is not dead in any marking reachable from* $\boldsymbol{m}_0$*.*
- *A marking* $\boldsymbol{m}$ *is* dead *if there is no transition which is enabled in* $\boldsymbol{m}$*.*
- *A Petri net system is* deadlock-free (weakly live) *if there are no reachable dead markings.*
- *A Petri net system is* live (strongly live) *if each transition is live.*

Definition 4 (Net structures). *Let* $\mathcal{N} = \langle P, T, \boldsymbol{Pre}, \boldsymbol{Post} \rangle$ *be a Petri net.* $\mathcal{N}$ *is*

- Homogeneous (HOM) *if* $\forall p \in P : t, t' \in p^\bullet \Rightarrow Pre(p,t) = Pre(p,t')$*;*
- Ordinary (ORD) *if* $\forall p \in P$ *and* $\forall t \in T$*,* $Pre(p,t) \leq 1$ *and* $Post(p,t) \leq 1$*;*
- Extended Simple (ES) *(sometimes also called* asymmetric choice*) if it is ORD and* $\forall\, p,q \in P : p^\bullet \cap q^\bullet = \emptyset \vee p^\bullet \subseteq q^\bullet \vee q^\bullet \subseteq p^\bullet$*;*
- Extended Free Choice (EFC) *if it is ORD and* $\forall\, p,q \in P : p^\bullet \cap q^\bullet = \emptyset \vee p^\bullet = q^\bullet$*.*

Definition 5 (DTP). *Let* $\mathcal{N} = \langle P, T, \boldsymbol{Pre}, \boldsymbol{Post} \rangle$ *be a Petri net.*

- *A* siphon *(structural deadlock, co-trap) is a non-empty set of places* $D \subseteq P$ *with* $^\bullet D \subseteq D^\bullet$ *.*
- *A* trap *is a non-empty set of places* $Q \subseteq P$ *with* $Q^\bullet \subseteq {}^\bullet Q$*.*
- *A* minimal siphon (trap) *is a siphon (trap) not including a siphon (trap) as a proper subset.*
- *A* bad siphon *is a siphon, which does not include a trap.*

- *An* empty siphon (trap) *is a siphon (trap), not containing a token.*
- *The* Deadlock Trap Property (DTP) *asks for every siphon to include an initially marked trap, i.e., marked at* $\boldsymbol{m}_0$*.*

The DTP can be reformulated as: minimal siphons are not bad and the maximal traps included are initially marked.

Definition 6 (Semiflows). *Let* $\mathcal{N} = \langle P, T, \boldsymbol{Pre}, \boldsymbol{Post} \rangle$ *be a net.*

- *The* token flow matrix *(or* incidence matrix *if the net is pure, i.e., self-loop free) is a matrix* $\boldsymbol{C} = \boldsymbol{Post} - \boldsymbol{Pre}$*.*
- *A* place vector *is a vector* $\boldsymbol{y} \in \mathbb{Z}^{|P|}$*; a* transition vector *is a vector* $\boldsymbol{x} \in \mathbb{Z}^{|T|}$.
- *A* P-semiflow *is a place vector* $\boldsymbol{y}$ *with* $\boldsymbol{y} \cdot \boldsymbol{C} = \boldsymbol{0}$, $\boldsymbol{y} \geq \boldsymbol{0}$, $\boldsymbol{y} \neq \boldsymbol{0}$*;*
  a T-semiflow *is a transition vector* $\boldsymbol{x}$ *with* $\boldsymbol{C} \cdot \boldsymbol{x} = \boldsymbol{0}$, $\boldsymbol{x} \geq \boldsymbol{0}$, $\boldsymbol{x} \neq \boldsymbol{0}$.
- *The* support of a semiflow $\boldsymbol{x}$*, written as* $supp(\boldsymbol{x})$*, is the set of nodes corresponding to the non-zero entries of* $\boldsymbol{x}$*.*
- *A net is* conservative *if every place belongs to the support of a P-semiflow.*
- *A net is* consistent *if every transition belongs to the support of a T-semiflow.*
- *In a* minimal semiflow $\boldsymbol{x}$*,* $supp\,(\boldsymbol{x})$ *does not contain the support of any other semiflow* $\boldsymbol{z}$*, i.e.,* $\nexists$ *semiflow* $\boldsymbol{z} : supp\,(\boldsymbol{z}) \subset supp\,(\boldsymbol{x})$*, and the greatest common divisor of* $\boldsymbol{x}$ *is 1.*
- *A* mono-T-semiflow net *(MTS net) is a consistent and conservative net that has exactly one minimal T-semiflow.*

For convenience, we give vectors (markings, semiflows) in a short-hand notation by enumerating only the non-zero entries. Finally, we recall some well-known related propositions (see for example [Mur89], [Sta90]), which might be useful for the reasoning we pursue in this paper.

Proposition 1 (Basics).

1. *An empty siphon remains empty forever. A marked trap remains marked for ever.*
2. *If* R *and* R' *are siphons (traps), then* $R \cup R'$ *is also a siphon (trap).*
3. *A minimal siphon (trap) is a P-strongly-connected component, i.e., its places are strongly connected.*
4. *A deadlocked Petri net system has an empty siphon.*
5. *Each siphon of a live net system is initially marked.*
6. *If there is a bad siphon, the DTP does not hold.*
7. *A source place* p *establishes a bad siphon* $D = \{p\}$ *on its own, and a sink place* q *a trap* $Q = \{q\}$*.*
8. *If each transition has a pre-place, then* $P^\bullet = T$*, and if each transition has a post-place, then* ${}^\bullet P = T$*. Thus, in a net without boundary transitions, the whole set of places is a siphon as well as a trap (however, not necessarily minimal ones).*
9. *For a P-semiflow* $\boldsymbol{x}$ *it holds* ${}^\bullet supp(\boldsymbol{x}) = supp(\boldsymbol{x})^\bullet$*. Thus, the support of a P-semiflow is siphon and trap as well (however, generally not vice versa).*

Proposition 2 (DTP and behavioural properties).

1. *An ordinary Petri net without siphons is live.*
2. *If $\mathcal{N}$ is ordinary and the DTP holds for $\boldsymbol{m}_0$, then $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$ is deadlock-free.*
3. *If $\mathcal{N}$ is ES and the DTP holds for $\boldsymbol{m}_0$, then $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$ is live.*
4. *Let $\mathcal{N}$ be an EFC net. $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$ is live iff the DTP holds.*

We conclude this section with a proposition from [CCS91], which might be less known.

Proposition 3 (MTS net and behavioural properties). *Liveness and deadlock-freeness coincide in mono-T-semiflow net systems.*

3 Monotonic Liveness

If a property holds for a Petri net $\mathcal{N}$ with the marking $\boldsymbol{m}_0$, and it also holds in $\mathcal{N}$ for any $\boldsymbol{m} \geq \boldsymbol{m}_0$, then it is said to be *monotonic* in the system $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$. In this paper we are especially interested in monotonic liveness.

Definition 7 (Monotonic liveness).
Let $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$ be a Petri net system. It is called monotonically live, if being live for $\boldsymbol{m}_0$, it remains live for any $\boldsymbol{m} \geq \boldsymbol{m}_0$.

We are looking for conditions, at best structural conditions, preserving liveness under arbitrary marking increase. To illustrate the problem, let's consider a classical example [Sta90], [SR02].

Example 1. The net $\mathcal{N}$ in Figure 1 is ES, conservative, consistent, and covered by one T-semiflow. It is live for the given initial marking $\boldsymbol{m}_1 = (2p_1, p_4)$. Adding a token to place p_5 yields the initial marking $\boldsymbol{m}_2 = (2p_1, p_4, p_5)$ and the net system remains live for $\boldsymbol{m}_2 \geq \boldsymbol{m}_1$. However, adding a token to p_4 yields the initial marking $\boldsymbol{m}_3 = (2p_1, 2p_4)$ and the net behaviour now contains finite firing sequences, i.e., it can run into a deadlock (dead state). Thus, the net system is not live for $\boldsymbol{m}_3 \geq \boldsymbol{m}_1$. It is not monotonically live.

How to distinguish both cases? The net has two (minimal) bad siphons $D_1 = \{p_1, p_2\}$ and $D_2 = \{p_1, p_3\}$. There is no chance to prevent these siphons from getting empty for arbitrary markings. D_1 can potentially be emptied by firing $t_2 \in D_1{}^\bullet \setminus {}^\bullet D_1$, and D_2 by firing $t_1 \in D_2{}^\bullet \setminus {}^\bullet D_2$. The latter case destroyed the liveness for $\boldsymbol{m}_3$ as it will equally occur for all initial markings allowing transition sequences containing one of the troublemakers, in this example t_1 and t_2, sufficiently often. □

One lesson learnt from the previous example is, a net does not have to make use of the additional tokens. Thus, all behaviour (set of transition sequences), which is possible for $\boldsymbol{m}$ is still possible for $\boldsymbol{m}'$, with $\boldsymbol{m} \leq \boldsymbol{m}'$. However, new tokens may allow for additional system behaviour, which is actually well-known in Petri net theory, see Proposition 4.

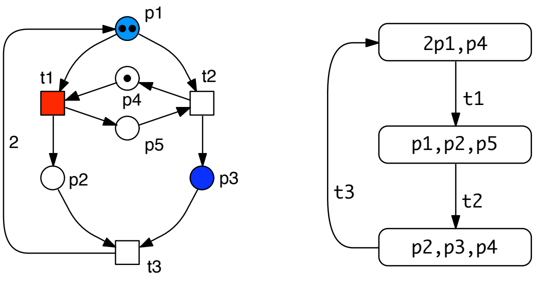

Fig. 1. A mono-T-semiflow and ES Petri net $\mathcal{N}$ and its reachability graph for the marking $\boldsymbol{m}_1 = (2p_1, p_4)$, generating the language $\mathcal{L}_{\mathcal{N}}(\boldsymbol{m}_1) = (t_1t_2t_3)^*\{\varepsilon, t_1, t_1t_2\}$. The siphon $\{p_1, p_3\}$ does not contain a trap, i.e., it is a bad siphon. If the initial marking is increased, it can potentially become empty by firing of t_1.

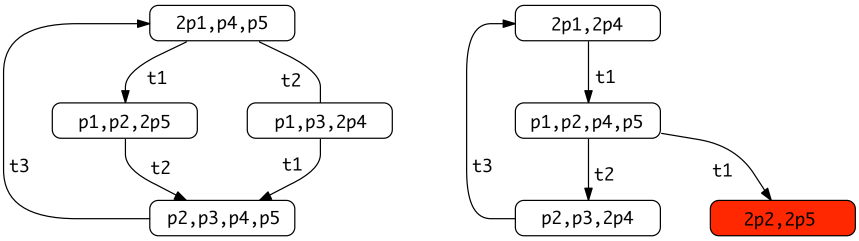

Fig. 2. Two other reachability graphs for the net $\mathcal{N}$ in Figure 1 for the initial markings $\boldsymbol{m}_2 = (2p_1, p_4, p_5)$ and $\boldsymbol{m}_3 = (2p_1, 2p_4)$; both are greater than $\boldsymbol{m}_1$. Obviously, $\mathcal{L}_{\mathcal{N}}(\boldsymbol{m}_1) \subset \mathcal{L}_{\mathcal{N}}(\boldsymbol{m}_2)$, $\mathcal{L}_{\mathcal{N}}(\boldsymbol{m}_1) \subset \mathcal{L}_{\mathcal{N}}(\boldsymbol{m}_3)$, but $\langle \mathcal{N}, \boldsymbol{m}_3 \rangle$ is not live while $\langle \mathcal{N}, \boldsymbol{m}_1 \rangle$ is live.

Proposition 4. *For any net* $\mathcal{N}$ *and two markings* $\boldsymbol{m}$ *and* $\boldsymbol{m}'$*, with* $\boldsymbol{m} \leq \boldsymbol{m}'$*, it holds* $\mathcal{L}_{\mathcal{N}}(\boldsymbol{m}) \subseteq \mathcal{L}_{\mathcal{N}}(\boldsymbol{m}')$ *[BRA83]; nevertheless,* $\langle \mathcal{N}, \boldsymbol{m} \rangle$ *may be live while* $\langle \mathcal{N}, \boldsymbol{m}' \rangle$ *not.*

Example 1 is a mono-T-semiflow net, i.e., a net, where liveness and deadlock-freeness coincide (see Proposition 3). We look briefly at Example 2 to understand that this does not generally hold if there are several T-semiflows breathing life into the net.

Example 2. The net $\mathcal{N}$ in Figure 3 is a slight extension of Example 1. It is ES, conservative, consistent and covered by two T-semiflows: $\boldsymbol{x}_1 = (t_1, t_2, t_3), \boldsymbol{x}_2 = (t_4, t_5)$. It is live for the initial marking $\boldsymbol{m}_1 = (2p_1, p_4)$.

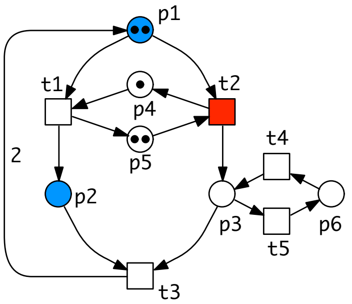

Fig. 3. An ES Petri net which is not mono-T-semiflow. It is live for the initial marking $\boldsymbol{m}_1 = (2p_1, p_4)$. The siphon $\{p_1, p_2\}$ is bad. So it can potentially become empty by firing t_2 sufficiently often. This happens for the initial marking $\boldsymbol{m}_2 = (2p_1, p_4, 2p_5)$, making the net non-live, however keeping it deadlock-free (observe that $\{p_3, p_6\}$ behaves as a trap if the firing of t_3 is blocked forever).

The net has the following minimal siphons $D_1 = \{p_1, p_2\}$, $D_2 = \{p_1, p_3, p_6\}$, and $D_3 = \{p_4, p_5\}$; the first two are bad siphons. With the initial marking $\boldsymbol{m}_2 = (2p_1, p_4, 2p_5)$, D_1 can become empty by firing twice $t_2 \in D_1{}^\bullet \setminus {}^\bullet D_1$, which destroys the liveness, without causing a dead state. The transitions t_4, t_5 are live, the others not. Thus, the net system is not live, but deadlock-free. ☐

The loss of liveness is not necessarily monotonic itself; i.e., a net may be live for $\boldsymbol{m}_1$, non-live for a marking $\boldsymbol{m}_2$ with $\boldsymbol{m}_2 \geq \boldsymbol{m}_1$, and live again for a marking $\boldsymbol{m}_3$ with $\boldsymbol{m}_3 \geq \boldsymbol{m}_2$ (which works for all examples in this paper). Liveness may also be lost by marking multiples (*homothetic markings*). Examples 1 and 2 are homothetically live, Example 3 in Section 5 not.

4 Monotonic Liveness of Ordinary Nets

Let us turn to liveness criteria suitable for our objective looking at *ordinary* nets first. Liveness criteria not relying on the marking obviously ensure monotonic liveness. Unfortunately, there are only a few.

First of all, there are some structural reduction rules, see, e.g., [Sil85], [Ber86], [Mur89], [SR99]. To give a sample, the following reduction rule is easy to accept: a source transition is live, and all its post-places are unbounded. The transition and its post-places can be deleted (for analysis purposes); the reduction can be iterated as many times it is applicable. Sometimes, this kind of reasoning allows to decide liveness (for examples, see Section 6).

Besides structural reduction we have the DTP, which in most cases does depend on the marking, but it is obviously monotonic w.r.t. the marking: if each siphon contains a marked trap at $\boldsymbol{m}$, then – of course – it contains a marked trap at $\boldsymbol{m}' \geq \boldsymbol{m}$. Thus, the DTP-related conclusions on behavioural properties in Proposition 2 are monotonic as well:

Proposition 5 (Monotonic DTP).

1. *An ordinary net without siphons is monotonically live.*
2. *An ordinary net system which holds the DTP is monotonically deadlock-free.*
3. *A live ES net system which holds the DTP is monotonically live.*
4. *An EFC net system is monotonically live iff the DTP holds.*

Proposition 5.1 can be considered as a special case of the DTP. Then, there must be source transitions (see Proposition 1.8), and the net is not strongly connected and not bounded.

Lemma 1. *Let be $\mathcal{N}$ an ordinary Petri net. If $\mathcal{N}$ is monotonically live, then there are no bad siphons.*

Proof. We will prove its reverse – if there exist a bad siphon, then the net system is not monotonically live – by contradiction. Let P_S be a bad siphon. Then there exist troublemaking transitions $\Theta_i \in P_S{}^\bullet \setminus {}^\bullet P_S$. There must be such transitions, because otherwise $P_S{}^\bullet = {}^\bullet P_S$, and then the siphon P_S would be a trap as well.

Since the net system is monotonically live, the marking of the places $P \setminus P_S$ can be increased in such a way that it will never restrict the firing of the transitions $P_S{}^\bullet$, i.e., the transitions depending on the siphon. Therefore, we can consider the subnet restricted to P_S in isolation.

We will show that the subsystem restricted to P_S can be emptied eventually by increasing the marking, hence cannot be monotonically live. Obviously we can assume that P_S is a minimal siphon. We consider two cases.

(1) The siphon has no forks (t_j is a *fork* if $|t_j{}^\bullet| > 1$). Based on the P-strongly-conectedness (see Proposition 1.3), there exists at least one path from each place $p \in P_S$ to one of the troublemakers Θ_i. Moving a token from p to ${}^\bullet\Theta_i$ does not increase the marking of any other place of P_S not belonging to the considered path. Obviously, this path can contain joins (t_j is a *join* if $|{}^\bullet t| > 1$), but we can add any tokens that are missing in the input places of the join. Firing the join, the marking of the places in the siphon is not increased. Using this process we can move the tokens from any $p \in P_S$ to some ${}^\bullet\Theta_i$, and by firing Θ_i when it is enabled, P_S can be emptied. Thus, the net system can not be monotonically live.

(2) On the contrary, let us assume that there exists at least one fork t_j and let $p_1, p_2 \in t_j{}^\bullet$ be its output places. For the same reason as discussed in case (1), there exists a directed path from both places to one or several troublemakers. If all paths from p_1 to any troublemaker Θ_i contain t_j, then they form a trap. This is impossible because siphons are assumed to be bad. By symmetry, in the case in which the paths from p_2 to troublemakers contain t_j, there exists a trap as well.

Finally, let us assume that there exists a path from p_1 to a troublemaker Θ_i and one path from p_2 to a troublemakers Θ_k, none of them containing t_j.

On both paths the same kind of reasoning can be applied (in an iterative way if several forks appear). Therefore, the siphon can be emptied even if firing t_j increases the tokens in P_S. □

Lemma 1 helps to preclude monotonic liveness for Examples 1 and 2 as well as for all other non-monotonically live examples we are aware of.

Theorem 1. *Let be $\mathcal{N}$ an ordinary Petri net. If $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$ is monotonically live, then the DTP holds.*

Proof. The structural check of the DTP can have three possible outcomes.

1. If there are no siphons, then the DTP holds trivially and the net is monotonically live (see Proposition 5.1).
2. If there are bad siphons, then the DTP does not hold for any initial marking and the net is not monotonically live (see Lemma 1).
3. If each siphon includes a trap, then the *maximal* trap P_T in every minimal siphon P_S has to be initially marked to fulfill the DTP. Because we assume liveness of the net system, there has to be at least one token in each minimal siphon (see Proposition 1.5). Let us assume that a token is not in P_T, but in a place $p \in P_S \setminus P_T$. If there exists at least one path without forks from p to a troublemaking transition $\Theta_i \in P_S{}^\bullet \setminus {}^\bullet P_S$ not containing any transition belonging to the trap, ${}^\bullet P_T$, then p can be emptied using the same reasoning as used in the proof of Lemma 1, case (1). Therefore the net can not be live. If the path from p to a troublemaking transition $\Theta_i \in P_S{}^\bullet \setminus {}^\bullet P_S$ contains a fork, then the output places of the fork will be marked when p is emptied, and the paths from the output places of the forks to the output should be considered separately.
   Finally, if *all* paths from p to the troublemaking transitions contain at least one transition ${}^\bullet P_T$, then the trap P_T is not maximal since P_T together with all places belonging to the above mentioned paths (including all non-minimal ones) from p to transitions ${}^\bullet P_T$ are also a trap. □

According to Theorem 1, the DTP establishes a necessary condition for monotonic liveness, which complements Proposition 5.3.

Corollary 1. *A live ES net system is monotonically live iff the DTP holds.*

Moreover, for those systems for which deadlock-freeness is equivalent to liveness, the DTP is a sufficient criteria for liveness monotonicity. This leads, for example, to the following theorem:

Theorem 2. *Let be $\mathcal{N}$ an ordinary mono-T-semiflow Petri net which for $\boldsymbol{m}_0$ fulfills the DTP. Then the system $\langle \mathcal{N}, \boldsymbol{m} \rangle$ is live for any $\boldsymbol{m} \geq \boldsymbol{m}_0$.*

Proof. It follows from Proposition 5.2 (DTP and deadlock-freeness monotonicity) and Proposition 3 (equivalence of liveness and deadlock freeness in mono-T-semiflow net systems). □

Therefore, the DTP is a sufficient criterion for monotonic liveness of ordinary mono-T-semiflow net systems as well. In summary, while the DTP is in general neither necessary nor sufficient for liveness, it turns out to be the case to keep alive ordinary ES nets or ordinary mono-T-semiflow nets under any marking increase.

5 Monotonic Liveness of Non-ordinary Nets

It is well-known that non-ordinary nets can be simulated under interleaving semantics by ordinary ones [Sil85] (see Figure 4 for an example). Let us look on the net structures we get by this simulation to learn how far the results for ordinary nets of Section 4 can be uplifted to non-ordinary nets.

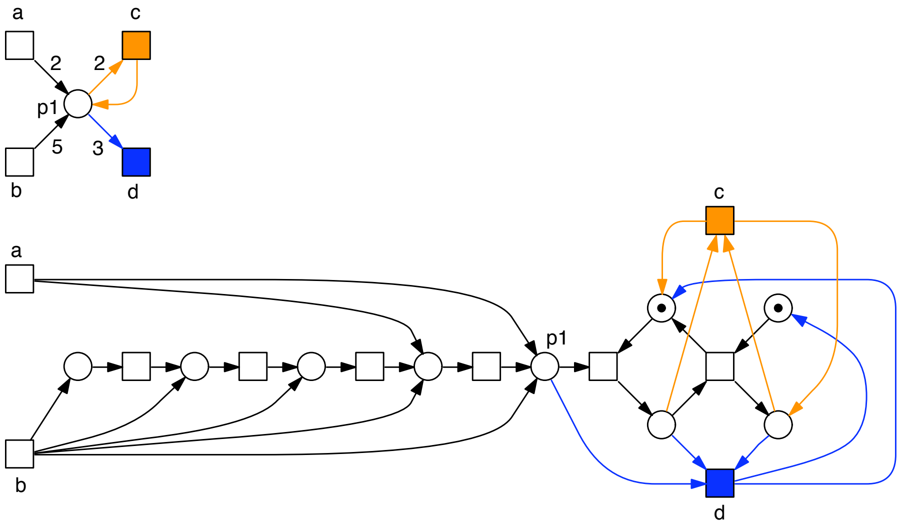

Fig. 4. A general principle to *simulate* a non-ordinary net system by an ordinary net system (here, the firing language of the second net projected on $\{a, b, c, d\}$ is always equal to that of the first) [Sil85].

Example 3. We take a non-ordinary net from [SR02] and consider its simulation by an ordinary net, which we construct according to the general principle demonstrated in Figure 4.

The two net systems in Figure 5 are conservative, consistent, and live for the given initial marking. The ordinary net on the right hand side is not ES, and it has two minimal bad siphons $\{q_1, p_1, p_{1b}, p_{1c}\}$, $\{p_2, p_1, p_{1b}, p_{1c}\}$. Thus, according to Lemma 1, it is not monotonically live. Because our simulation preserves the projection of the firing language, in particular, preserves monotonicity of liveness. Thus, we conclude that the model on the left hand side is not monotonically live. Indeed, both nets are not live for any initial marking with an even number of tokens in p_1, but live for infinitely many other markings greater than or equal to $(1, 1)$. ☐

As a consequence of firing simulation by the ordinary net systems of the non-ordinary ones (preserving always the markings of the places involved in the head of the tail and complement, here q_1 and q_{1co}), liveness monotonicity can be studied on the ordinary simulation.

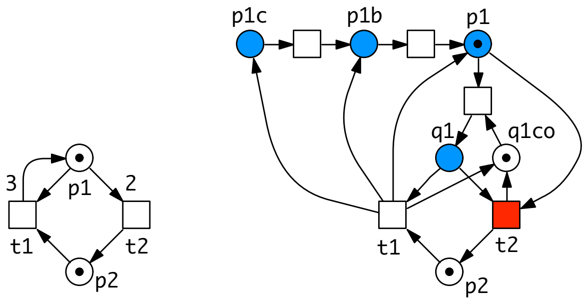

Fig. 5. A non-ordinary Petri net system and its simulation by an ordinary one. Both systems are non-live for any initial marking with an even number of tokens in p_1, and live for any other odd marking. Note that the markings of q_1 and q_{1co} should not be increased in order to keep the language simulation in the right hand model. The net system on the right has a bad siphon $\{q_1, p_1, p_{1b}, p_{1c}\}$ that can potentially become empty by firing t_2 sufficiently often.

6 Applications

We consider a variety of test cases of our benchmark repository to demonstrate the helpfulness of the DTP for biomolecular networks. The following list sketches some basic characteristics. The essential analysis results are summarized in Table 1. All models hold the DTP, they are consistent and (supposed to be) live. For non-ordinary nets, the DTP refers to its simulation by an ordinary one.

1. **Apoptosis** (size: 37 places, 45 transitions, 89 arcs) is a signal transduction network, which governs complex mechanisms to control and execute genetically programmed cell death in mammalian cells. Disturbances in the apoptotic processes may lead to various diseases. This essential part of normal physiology for most metazoan species is not really well understood; thus

there exist many model versions. The validation by Petri net invariants of the model considered here is discussed in [HKW04], [HK04].

2. **RKIP** (size: 11 places, 11 transitions, 14 arcs) models the core of the influence of the Raf-1 Kinase Inhibitor Protein (RKIP) on the Extracellular signal Regulated Kinase (ERK) signalling pathway. It is one of the standard examples used in the systems biology community. It has been introduced in [CSK+03]; the corresponding qualitative, stochastic, continuous Petri nets are scrutinized in [GH06], [HDG10].
3. **Biosensor** (size: 6 places, 10 transitions, 21 arcs) is a gene expression network extended by metabolic activity. The model is a general template of a biosensor, which can be instantiated to be adapted to specic pollutants. It is considered as qualitative, stochastic, and continuous Petri net in [GHR+08] to demonstrate a model-driven design of a self-powering electrochemical biosensor.
4. **Hypoxia** (size: 14 places, 19 transitions, 56 arcs) is one of the well-studied molecular pathways activated under hypoxia condition. It models the Hypoxia Induced Factor (HIF) pathway responsible for regulating oxygen-sensitive gene expression. The version considered here is discussed in [YWS+07]; the corresponding qualitative and continuous Petri nets are used in [HS10] to determine the core network.
5. **Lac operon** (size: 11 places, 17 transitions, 41 arcs) is a classical example of prokaryotic gene regulation. We re-use the simplified model discussed in [Wil06]. Its corresponding stochastic Petri net is considered in [HLGM09].
6. **G/PPP** (size: 26 places, 32 transitions, 76 arcs) is a simplified model of the combined glycolysis (G) and pentose phosphate pathway (PPP) in erythrocytes (red blood cells). It belongs to the classical examples of biochemistry textbooks, see e.g. [BTS02], and thus of systems biology as well. The model was first discussed using Petri net technologies in [Red94]. Its validation by Petri net invariants is shown in [HK04], and a more exhaustive qualitative analysis in [KH08].
7. **MAPK** (size: 22 places, 30 transitions, 90 arcs) models the signalling pathway of the mitogen-activated protein kinase cascade, published in [LBS00]. It is a three-stage double phosphorylation cascade; each phosphorylation/dephosphorylation step applies the mass action kinetics pattern. The corresponding qualitative, stochastic, and continuous Petri net are scrutinized in [GHL07], [HGD08].
8. **CC – Circadian clock** (size: 14 places, 16 transitions, 58 arcs) refers to the central time signals of a roughly 24-hour cycle in living entities. Circadian rhythms are used by a wide range of organisms to anticipate daily changes in the environment. The model published in [BL00] demonstrates that circadian network can oscillate reliably in the presence of stochastic biomolecular noise and when cellular conditions are altered. It is also available as PRISM model on the PRISM website (http://www.prismmodelchecker.org). Its corresponding stochastic Petri net belongs to the benchmark suite used in [SH09]. We consider here a version with inhibitor arcs modelled by co-places.

9. **Halo** (size: 37 places, 38 transitions, 138 arcs) is a cellular signaling and regulation network, describing the phototaxis in the halobacterium salinarum [NMOG03]. It models the sophisticated survival strategy, which the halobacterium developed for harsh conditions (high temperature, high salt). A light sensing system and flagellar motor switching allows the cells to swim to those places of their habitat where the best light conditions are available. The model is the result of prolonged investigations by experimentally working scientists [Mar10].
10. **Pheromone** (size: 42 places, 48 transitions, 119 arcs) is a signal transduction network of the well understood mating pheromone response pathway in *Saccharomyces cerevisiae*. The qualitative Petri net in [SHK06] extends a former ODE model [KK04]. The Petri net was validated by Petri net invariants and a partitioning of the transition set.
11. **Potato** (size: 17 places, 25 transitions, 78 arcs) describes the main carbon metabolism, the sucrose-to-starch breakdown in Solanum tuberosum (potato) tubers. The qualitative Petri net model was developed in cooperation with experimentally working scientists, experienced in ODE modelling. Its validation by Petri net invariants is discussed in [HK04], and a more detailed pathway exploration in [KJH05].

Table 1. Some biomolecular case studies; all of them hold the DTP, are consistent and live.

#	case study	multiplicities	net class	bounded	liveness shown by
1	apoptosis	ORD	ES	no	Proposition 2.1
2	RKIP	ORD	ES	yes	Proposition 2.3
3	biosensor	ORD	ES	no	Proposition 2.3
4	hypoxia	ORD	not ES	no	structural reduction
5	lac operon	HOM	not ES	no	structural reduction
6	G/PPP	HOM	not ES	no	structural reduction
7	MAPK	ORD	not ES	yes	dynamic analysis (RG)
8	CC	HOM	not ES	yes	dynamic analysis (RG)
9	halo	not HOM	not ES	yes	dynamic analysis (RG)
10	pheromone	HOM	not ES	no	by reasoning
11	potato	not HOM	not ES	no	by reasoning

Contrary, the model of signal transduction events involved in the angiogenesis processes, which is discussed in [NMC$^+$09] as a stochastic and continuous Petri net model (size: 39 places, 64 transitions, 185 arcs) is to a large extent covered by a (non-minimal) bad siphon. Thus, even if the net is live for a certain marking $\boldsymbol{m}$, there is always a larger marking $\boldsymbol{m}'$, which will allow to remove all tokens from the bad siphon. Consequently, an arbitrary marking increase will not preserve liveness.

7 Tools

The Petri nets for the case studies have been constructed using Snoopy [RMH10], a tool to design and animate or simulate hierarchical graphs, among them qualitative, stochastic and continuous Petri nets as used in the case studies in Section 6. Snoopy provides export to various analysis tools as well as import and export of the Systems Biology Markup Language (SBML).

The qualitative analyses have been made with the Petri net analysis tool Charlie [Fra09], complemented by the structural reduction rules supported by the Integrated Net Analyser INA [SR99].

8 Conclusions

We have discussed the problem of monotonic liveness, with one of the motivations originating from bio-model engineering. We have presented a new result showing the necessity of the DTP for monotonic liveness.

Moreover, we immediately know – thanks to the well-known propositions of the DTP – that ordinary ES nets are monotonically iff the DTP holds. Furthermore, we know – because the DTP monotonically ensures deadlock freeness – that for any net class, in which liveness and deadlock freeness coincide, monotonic liveness is characterized by the DTP. We have shown one instance for this case: the mono-T-semiflow nets (MTS).

We have demonstrated the usefulness of our results by applying them to a variety of biomolecular networks.

One of the remaining open issues is: what are sufficient conditions for monotonic liveness for more general net structures? While none of our test cases is an MTS net, this line might be worth being explored more carefully, e.g. by looking at FRT nets (Freely Related T-Semiflows) [CS92] and extensions.

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