Edge covering of acyclic graphs

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Abstract. A procedure for counting and generating edge covers of acyclic graphs is presented. The procedure splits acyclic graphs of rooted trees composed of simple path graphs for which exhibiting and counting covers can be achieved by linear procedures. The procedure is therefore the foundation of a linear algorithm for edge counting on acyclic graphs. It is also believed that the procedure will serve as the basis for a more complex algorithm for counting edge covers on arbitrary simple graphs.

Keywords: edge covering, graph theory, trees

1 Introduction

This paper is concerned with the *edge covering* problem on acyclic graphs [1, 2]. The problem is #P-complete [3–7] and the paper will also consider a combination of both cyclic and acyclic graphs. Such graphs are a combination of simple paths and trees involving a special configuration. The edge covering problem may be divided on counting covers, exhibiting covers (which build a precise covering for graphs), and the minimum and maximum cover problem. The paper is concerned only with counting covers for acyclic graphs and the procedure will show how to precisely recover covers from certain sequences of numbers.

2 Preliminares

The following conventions and results are in common use and can be looked at [8]. A graph is a pair $G = (V, E)$, where V is a set of *vertices* and E is a set of *edges* that associates any pair of vertices not necessarily distint. The number of vertices and edges is denoted by $v(G)$ and $e(G)$, respectively. The parametre $v(G)$ is called the order whereas $e(G)$ will be the size of the graph. Each vertex v of a graph has associated a number $d_G(v)$, called the degree of v, and it counts the number of edges incident on v. A vertex of degree zero is called an isolated vertex. It will be denoted by $\delta(G)$, $\Delta(G)$ the minimum and maximum degree of the vertices of G, respectively.

A *subgraph* of a graph G is a graph G'such that $V' \subseteq V$ and $E' \subseteq E$. Subgraphs can be computed by deleting edges and vertices. If $e \in E$, e can simply be removed from graph G, yielding a subgraph denoted by $G\backslash e$; this is obviously the graph $(V, E-e)$. Analogously, if $v \in V$, $G\backslash v$ is the graph $(V-v, E')$ where $E' \subseteq E$. The process of vertex deletion implies removing all edges incident on the vertex to be removed.

A *spanning subgraph* is a subgraph computed by deleting only edges while keeping all its vertices, that is, if $S \subset E$ is a subset of E, then a spanning subgraph of $G = (V, E)$ is $G'{=}G\backslash S$.

A *path* in a graph is a linear sequence of adjacent vertices, whereas a *cycle* is a sequence of vertices that can be arranged in a cyclic sequence. A path and a cycle are simple graphs, where simple graph means a graph that has no loops and two adjacent vertices are connected by one and only one edge.

An *acyclic* graph is a graph that contains no cycles. The connected acyclic graphs are called *trees*, and a connected graph is a graph that for any two pair of vertices there exists a path connecting them. It is not difficult to infer that in a tree there is a unique path connecting any two pair of vertices. A *rooted tree* $T(v)$ is a tree T with a given vertex v, called the root of T. The vertices in a tree with degree equal to one are called *leafs*. A path graph and a cycle graph that contain n vertices will be denoted by P_n, C_n, respectively. The path graph containing n vertices can be precisely defined as the graph $P = (V_P, E_P)$, where $V_P = \{v_i | i \in \mathbb{N}^n\}$ and $E_P = \{e_i | e_i = v_i v_{i+1}, i \in \mathbb{N}^{n-1}\}$. The notation $v_i v_{i+s}$ represents the edge connecting vertex v_i to vertex v_{i+1}. The cycle graph with n vertices can analogously be defined except that $v_1 = v_n$.

An *edge cover*, or simply an e-cover of a graph G is the edge set of a spanning subgraph of G. In other words, an edge cover of $G = (V_G, E_G)$ is a subset $E_G^{'} \subseteq E_G$ that covers all vertices of G. Therefore, the *edge cover* counting problem can be simply stated as follows: Given a simple graph G, count the total number of different edge covers of G. Let us denote the set of *e-covers* for a graph G as $\mathcal{E}_G$, therefore the number of *e-covers* is $|\mathcal{E}_G|$.

3 Joining and splitting trees

3.1 Joining graphs

The *join* concept defined here is different from the join defined in the literature [8], in the sense that *join* in this context means simply the "union" of two graphs trough a vertex or set of vertices.

Definition 1. *The* join*, denoted by* $\vee$*, of two graphs trough a pair of vertices is defined as follows. Let* $G = (V_G, E_G)$*,* $H = (V_H, E_H)$ *be two graphs such that* $V_G \cap V_H = \emptyset$*,* $E_G \cap E_H = \emptyset$*,* $v \in V_G$ *and* $u \in V_H$*. The* join *operation is obtained by choosing* $w \notin V_G$*,* $w \notin V_H$ *therefore by relabeling* u*,* v *in both graphs* G *and* H *to* w *in such a way that we get the sets of vertices* $V_G^{'}$*,* $V_H^{'}$ *and edges* $E_G^{'}$*,* $E_H^{'}$*, thus*

$$G \vee H = (V_G^{'} \cup V_H^{'}, E_G^{'} \cup E_H^{'}) \tag{3.1}$$

Clearly, the set $V_{G \vee H} = V_G^{'} \cup V_H^{'}$ *of vertices contains the vertex* w *such that* $d_{G\vee H}(w){=}d_G(v){+}d_H(u)$*, this means that the incident edges to* w *are those that were incident to both* u *and* v*.*

The *join* operation of two or more graphs by a vertex or set of vertices can be attained by successive application of definition (1) as follows: Let $S = \{u_i | i \in \mathbb{N}^k\} \subseteq V_G$ be a subset of vertices for a given graph G and $H = \{H_i | i \in \mathbb{N}^k\}$ a family of graphs such that H_i will be *joined* through u_i, $v_i \in V_{H_i}$ for every $i \in \mathbb{N}^k$, then the *join* of G with H can be achieved as follows

$$G \vee H = (\cdots((G \vee H_1) \vee \cdots \vee H_{k-1})H_k = G \bigvee_{i \in \mathbb{N}^k} H_i \tag{3.2}$$

The above procedure will be useful to define subgraphs of rooted trees by *joining* path graphs through a single vertex.

3.2 Basic subtrees (*b-subtree*)

Let $T(u)$ be a rooted tree, with root vertex u, since in general rooted tree have vertices $v \neq u$ with $d_T(v) \geq 3$ we will define *b-subtrees* with root vertex v as the *join* of path graphs.

Notice that the following definition applies for connected as well as disconnected paths only. Let us called the interior of a connected path P which starts and ends at vertices u, v to be the set $V_P^\circ = \left\{ w \in V_P \,\middle|\, w \neq u, v \right\}$, and the family $\mathcal{F}_v$ of all paths starting at vertex $v \in V_T$ of arbitrary length $e(P)$ as the set $\mathcal{F}_v = \{P | d_P(x) = 2 \text{ for all } x \in V_P^\circ\}$.

Let us define the *b-subtree* M_v at vertex v of the rooted tree $T(u)$ as the *join*

$$M_v = \bigvee_{P \in \mathcal{F}_v} P \tag{3.3}$$

The set $V_M \subset V_T$ does not contain only vertices of degree two since they are somehow connected via a path P of T. Thus, in general, the ending vertices of P have arbitrary degree. The vertices $v \neq w \in V_{M_v}$ with $d_T(w) \geq 3$ will be called free vertices of M_v in the sense that M_v can only be *joined* through those vertices to another *b-subtrees*. The vertices of degree two are part of V_P°, $P \in \mathcal{F}_v$ whereas of degree one are just *leaves*. Free vertices can be isolated and non-isolated vertices, and the *join* has to be well defined in the sense that non-isolated vertices are not to be *joined*.

3.3 Tree graphs as joins of *b-subtrees*

In order to count *e-cover* for a given tree T, it is necessary to split the tree into *b-subtrees*. It will be shown that is much simpler to calculate *e-covers* for *b-subtrees*, then joining them together will result in *e-covers* for trees. To split the tree T into *b-subtrees* we will take all vertices $v \in V_T$ such that $d_T(v) \geq 3$ and define subtrees as the *join* M_v in the same way as in (3.3).

The split of T. A vertex w is a neighbour vertex of v if there is a path $P \in \mathcal{F}_v$ such that $w \in V_P$, $d_T(w) \geq 3$ and $d_T(v) \geq 3$. This obviously means that w and v are connected via P. The set $N_v = \{w \in V_T | w \text{ is neighbour of } v\}$ is the set of all neighbour vertices of v.

It is well known that for any tree $T(u)$, $v \in V_{T(u)}$ there exist a unique path P connecting u and v. This characteristic of trees allow us to identify vertices by placing them on levels. Initially, the root vertex will be, by definition, at level 0. Any other vertex v with $d_T(v) \geq 3$ will be at level s if the set V_P° of the unique path P connecting v and u, has exactly $s-1$ vertices of degree greater than or equal to three. For example, if $v \in N_u$ then v is at level $s = 1$, because by definition any $w \in V_P^\circ$ has degree equal to 2, therefore containing no vertices of degree greater than or equal to three, for the path P connecting v and u. Therefore, it can be defined the set $V_T^s = \{v \in V_T | v \text{ is at level } s, d_T(v) \geq 3\}$, and the following relation: vertex v *is below* w ($v \prec w$) if $v \in V_T^s$ and $w \in V_T^t$ with $s < t$. Obviously, $u \prec w$ for any vertex $w \in V_{T(u)}$ with $d_T(w) \geq 3$.

Definition 2. *A w-branch of $T(u)$ with root vertex w is an acyclic sugbraph $S(w)$ of $T(u)$ such that V_S is the set of all $v \in V_T$ and there exists a unique P connecting v to w and $u \notin V_P$. The path P is unique and $E_P \subseteq E_S \subseteq E_T$.*

The following lemma summarize the splitting process of a tree graph.

Lemma 1. *Every rooted tree $T(u)$ at u is the* join *operation of b-subtree graphs.*

Proof. Since $T(u)$ is finite then there exist an integer s such that none of $w \in V_T^s$ has vertices of degree greater than or equal to three *above*. Thus, $N_w = \emptyset$ and the end vertex w' of P for every $P \in \mathcal{F}_w$ holds $d_T(w') = 1$. If the *join* of $\mathcal{F}_w$ is taken we get $M_w = \bigvee_{P \in \mathcal{F}_w} P$ yielding only one free vertex which is w itself. Now, for all $w \in V_T^s$ there is at least one neighbour vertex $v \in V_T^{s-1}$ of w which is the root vertex of a *b-subtree* M_v and w is a free vertex of M_v as well. Clearly, $M_v = bigvee_{P \in \mathcal{F}_v} P$ therefore we can *join* M_v to M_w through w as $M_v \vee M_w$. In this way, we can take all $w \in N_v$ and perform the *join* operation in such a way that the only free vertex left of $M_v \vee M_w$ is v itself. Since v is arbitrary, this procedure guarantees that all neighbours of vertices at level $s-1$ have been *joined* to a *b-subtree*. The procedure is performed for every level and every branch of T until the root vertex is reached by decreasing s by one unit on each step. The *join* operation to get the original tree graph $T(u)$ can be summarize in one single expression, thus

$$T(u) = \left[M_u \bigvee_{w' \in N_u} M_{w'} \right] \bigvee \cdots \bigvee_{v \in V_T^{s-1}} \left[M_v \bigvee_{w \in N_v} M_w \right] \tag{3.4}$$

The number of connections on each step depend on the cardinality of N_v, V_T^s for all s.□

4 *e-covers* on *b-subtrees*

4.1 Mapping *e-covers* onto sequences of 0's and 1's

Let us consider the family set $\mathcal{P}$ of paths graphs of arbitrary degree, connected or disconnected. Now, if $P \in \mathcal{P}$ and $P = v_0e_1v_1 \cdots v_{i-1}\hat{e}_iv_i \cdots v_{n-2}e_{n-1}v_{n-1}$, thus by removing the labels for vertices in P we end up with a sequence $e_1 \cdots \hat{e}_i \cdots e_{n-1}$ of edges representing also the path P as well. The hat symbol $\hat{}$ at e_i means that P is a disconnected path at position i, in other words, the edge e_i does not appear at P. The family will also contain paths with two or more edges removed, however a path with two consecutive edges removed cannot belong to $\mathcal{P}$. For instance, the path $e_1\hat{e}_2\hat{e}_3e_4e_5 \notin \mathcal{P}$ whereas $e_1\hat{e}_2e_3\hat{e}_4e_5$ does belong to $\mathcal{P}$. In summary, a path P belongs to the family $\mathcal{P}$ if V_P° has no isolated vertices but the starting and ending vertices of P can or cannot be isolated vertices.

Let us also define the set $\mathcal{W}$ of sequences ω of 0's and 1's of arbitrary length. For example, $01110, 011110 \in \mathcal{W}$, etc. The length for a sequence $\omega \in \mathcal{W}$ will be denoted by $\mathcal{L}(\omega)$, and it is defined as the numbers of 1's appearing in ω plus the number of 0's in ω.

In order to count and build *e-cover* for paths P by using sequences of 0'1 and 1's a map $\Psi : \mathcal{P} \longrightarrow \mathcal{W}$ is defined. If we have any partition $I \subseteq \mathbb{N}^n$ where the edge e_i, $i \in I$ does not appear in P, which can be indicated by the symbol $\hat{e}_i$ then Ψ can be defined by replacing those edges appearing in P with 1 and those that do not appear in P with 0, therefore $\Psi : \mathcal{P} \longrightarrow \mathcal{W}$ is defined as $e_1 \cdots \hat{e}_i \cdots e_n \longrightarrow \overset{(1)}{1} \cdots \overset{(i)}{0} \cdots \overset{(n)}{1}$ for all $i \in I$.

It is obvious that any sequence $\omega \in \mathcal{W}$ can be represented by $1^{q_1}0^{q_2}\cdots$, where $q_1, q_2, ... \in \mathbb{N}$ are arbitrary numbers, indicating the number of times that 0 or 1 must appear in the sequence. Of course, if $q_j = 0$ means that the bit with such exponent has to be removed from the sequence. The set $\mathcal{W}$, however, is going to be restricted to four types of finite length sequences only, they are of the form $1\cdots1$, $0\cdots1$, $1\cdots0$ and $0\cdots0$, which will be called of type α, β, γ and ξ, respectively. If we set $K = \{\alpha, \beta, \gamma, \xi\}$ we will be referring to sequences of type $\kappa \in K$ as ω^κ. Type ω^γ, for instance, can be described as those sequences that on first position the bit 1 must appear at least once whereas 0 must also appear at least once on position $\mathcal{L}(\omega^\gamma)$. Similar descriptions apply for the other three types of sequences. The sequences ω^α,...,ω^ξ will aditionally have the property that two consecutive zeros are not allowed to happen in the sequences. Therefore, the four type of sequences ω^κ can be represented by $1^p01^{q_1}0\cdots01^{q_{s-1}}01^t$, where $p, t \geq 0$, $q_i \geq 1$, $i \geq 1$ and the exponent in 0 is always 1.

Definition 3. *The sequence $\omega^\kappa \in \mathcal{W}$ of type $\kappa \in K$ can be represented by the sequence $1^p01^{q_1}0\cdots01^{q_{s-1}}01^t$ where $p, t \geq 0$, $q_i \geq 1$, $1 \leq i \leq s-1$, s is the number of 0's appearing in ω^κ and*

$$\mathcal{L}(\omega^\kappa) = \begin{cases} (p+t)+s & \text{if } 1 \leq s < 2 \\ (p+t)+s+\sum_{i=1}^{s-1} q_i & \text{if } s \geq 2 \end{cases} \tag{4.1}$$

The sets of sequences of type ω^κ are denoted by $\mathcal{W}^\kappa$ for all $\kappa \in K$; to make emphasis on the length of the sequences, $\mathcal{W}_n^\kappa$ where $n = \mathcal{L}(\omega^\kappa)$ will be the set of sequences of type κ and length n. From now on, $\mathcal{W}$ will be restricted to the above four types, that is, let us assumed that $\mathcal{W} = \bigcup_{\kappa \in K} \mathcal{W}^\kappa$ where $\mathcal{W}^\kappa = \bigcup_{n \geq 1} \mathcal{W}_n^\kappa$. The union is a disjoint union since $\bigcap_{n \geq 1} \mathcal{W}_n^\kappa = \emptyset$ for all κ such that $\mathcal{W}_n^\kappa \neq \emptyset$.

It is clear that $\mathcal{E}_P \subseteq \mathcal{P}$; since Ψ is well defined then if it can be proved that $\mathcal{E}_P = \Psi^{-1}(\mathcal{W}_{e(P)}^\alpha)$ then it can be inferred that the calculation of *e-covers* for P is equivalent to the computation of set of sequences of type ω^α.

Lemma 2. *Let $P \in \mathcal{P}$ be a path of length $e(P)$, $\mathcal{W}_{e(P)}^\kappa$ the set of sequences of type ω^κ of length $e(P)$ with $\kappa \in K$ then $\bigcup_{\kappa \in K} \Psi^{-1}(\mathcal{W}_{e(P)}^\kappa)$ is the set of e-covers of P.*

Proof. It follows from definition (3) and Eq. (4.1).

In lemma (2), it is allowed to have path graphs having the start, the end or both vertices to be isolated vertices. From lemma (2), it is obvious that $|\mathcal{E}_P| = |\mathcal{W}_{e(P)}^\alpha|$ therefore finding and counting *e-covers* for path graphs is equivalent to find and count sequences of type $\omega \in \mathcal{W}_{e(P)}^\kappa$. For an example of how Ψ operates on paths, see the example shown in Fig.(1).

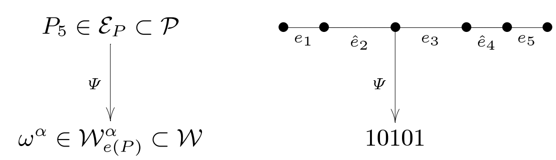

Fig. 1. Example of a path P with edges e_2 and e_4 removed and its corresponding sequence via the map Ψ.

4.2 Counting sequences

It will be seen, that it suffices to calculate $\mathcal{W}_n^\alpha$ for all $n \in \mathbb{N}$, since the other types of sequences can be calculated from it. Computing $|\mathcal{W}_n^\alpha|$ is based on the partition of an integer n into r parts. The problem of integer partitions is being around since G. H. Hardy and Ramanujan gave an asymptotic approximation to the number of partitions of an integer. This paper is not concerned with the theoretical implications of integer partitions, but to the pure calculation task of partitions and its number instead. Therefore, the algorithm described in [9] will be used to exhibit all partitions of a given number n, that yields a sequence of type ω^ξ.

Definition 4. *A partition of a number l into $s-1$ parts is a $s-1$-tuple $\mathbf{q} = (q_1, ..., q_{s-1})$ such that $\sum_i q_i = l$. The q_i can be repeated within the sum $\sum_i q_i$.*

If $\{q_1, ..., q_m\}$ *is the set of all distint* q_j *in* $\mathbf{q}$ *for some* m*, and* $\Lambda = \{\lambda_{q_1}, ..., \lambda_{q_m}\}$ *is the set of all integers such that* q_i *appears* λ_{q_i} *times within* $\sum_i q_i$ *then*

$$l = \sum_i \lambda_{q_i} q_i \ , \quad s = 1 + \sum_{\lambda \in \Lambda} \lambda \tag{4.2}$$

$\sum\limits_i \lambda_{q_i} q_i = \sum\limits_i q_i$ *will be denoted by* $|\mathbf{q}|$ *and* $\dim(\mathbf{q}) = s - 1$ *and* $\mathcal{Q}_{ls} = \{\mathbf{q} \mid |\mathbf{q}| = l, \dim \mathbf{q} = s - 1\}$.

The generation of partitions of an integer is not an easy task. The research on the subject has produced functions from which asymptotic approximations can be given to count partitions, i.e., asymptotic approximations to $p(l, s-1)$ which counts the total number of different partitions with respect to $|\mathbf{q}|$.

A different matter is to exhibit explicitly the partitions of an integer which is also difficult, even though the authors in [9] propose an algorithm to do so. The function $p(l, s-1)$, on the other hand returns the number of different partitions of l into $s-1$ parts, let us say $l = q_1 + \cdots + q_{s-1}$. It is therefore necessary to take into account a combination of the q_i for a given partition. Let us assume that we have calculated all different partitions of l into $s-1$ parts and that they are given by the sequences of numbers $q_1, ..., q_{s-1}$ while its number is given by $p(l, s-1)$. The q_i are also assumed to repeat in the partition λ_{q_i} times and that the q_i have been ordered such that $q_1 \leq \cdots \leq q_m$ thus $l = \sum \lambda_{q_i} q_i$, $1 \leq i \leq m$ for some $m \in \mathbb{N}$, and $\sum_{i=1}^{m} \lambda_{q_i} = s - 1$. It is well know, that the total number of non-repeated partitions generated for a given sequence $q_1, ..., q_{s-1}$, denoted by $\sharp\mathbf{q}$, is given by $\sharp\mathbf{q} = (\sum_{i=1}^{m} \lambda_{q_i})! \Big/ \prod_{i=1}^{m} (\lambda_{q_i})!$ which takes into account $q_1, ..., q_{s-1}$ itself. The set of all different m-tuples generated from $q_1, ..., q_{s-1}$ will be denoted by $\widehat{\mathbf{q}} = \{\mathbf{q}_\sigma | |\mathbf{q}_\sigma| = l, \dim(\mathbf{q}_\sigma) = s - 1\}$ where $\mathbf{q}_\sigma = (q_{\sigma(1)}, ..., q_{\sigma(s-1)})$ and $\sigma \in S_s \subset S_n$. The set S_n is the usual set of permutations of $\mathbb{N}^n$ whereas S_s is the set of permutations of the set $\mathbb{N}^{s-1}$ such that $|S_s| = \sharp\mathbf{q}$. Clearly, $\sigma \in S_s$ such that $\sigma(i) = i$ for all i implies that $\mathbf{q} \in \widehat{\mathbf{q}}$ and $\mathcal{Q}_{ls} = \bigcup_{\mathbf{q} \in \widehat{\mathbf{q}}} \mathbf{q}$.

To make an agreement with sequences we will define in advanced two sets of integers, namely L_{ns}, $\widehat{s}_n$ for a given $n \in \mathbb{N}$ as $L_{ns} = \{s-1, ..., n-s-2\}$, $\widehat{s}_n = \{2, ..., s_n\}$ for some integer $s \in \widehat{s}_n$, $s_n \in \mathbb{N}$. The number s_n is particularly special since it counts the number of zeros appearing in a sequence of type ω^κ and strongly depends on the number n. For such reason its calculation is done in lemma (3). Thus, by restricting l to the range L_{ns} and $s \in \widehat{s}_n$ we define the following sets in terms of $\mathcal{Q}_{ls}$ therefore

$$\widehat{\mathcal{Q}}_{s_n} = \bigcup_{s \in \widehat{s}_n} \bigcup_{l \in L_{ns}} \mathcal{Q}_{ls} = \bigcup_{s \in \widehat{s}_n} \widehat{\mathcal{Q}}_s \tag{4.3}$$

There is a map between the set $\widehat{\mathcal{Q}}_{s_n}$ and set of sequences $\mathcal{W}^\xi$, defined by $\Phi : \widehat{\mathcal{Q}}_{s_n} \longrightarrow \mathcal{W}^\xi$ which maps any $\mathbf{q} = (\lambda_{q_1} q_1, ..., \lambda_{q_N} q_N) \in \widehat{\mathcal{Q}}_{s(n)}$ to $\omega \in \mathcal{W}^\xi$ of the form $0\, \overbrace{1^{q_1} 0 \cdots 0 1^{q_1}}^{\lambda_1} 0 \cdots 0\, \overbrace{1^{q_N} \cdots 0 1^{q_N}}^{\lambda_N} 0$. This map will be used in section (4.2) to calculate $\mathcal{W}_n^\xi$.

Cardinality of $\mathcal{W}$. The following lemma gives the theoretical estimation of the integer s_n, which at the same time will allow us to calculate the cardinality of the sets $\mathcal{W}^\kappa$, $\kappa \in K$ and therefore *e-covers* for any path graph P of arbitrary size.

Lemma 3. *Let $\omega^\alpha \in \mathcal{W}_n^\alpha$ be a sequence of type α, $\omega^\alpha = 1^p\omega^\xi 1^t$, $n = \mathcal{L}(\omega^\alpha) = p + t + \mathcal{L}(\omega^\xi)$ with $\omega^\xi \in \mathcal{W}^\xi$ then $1 \le \mathcal{L}(\omega^\xi) \le n-2$ and the maximum number s_n of zeros appearing in ω^ξ is given by $s_n = \frac{n-2}{2}$ if n is even and $s_n = \frac{n-1}{2}$ if n is odd. Therefore, $\mathcal{W}_{n-p-t}^\xi = \Phi(\widehat{\mathcal{Q}}_{s_n})$, $|\mathcal{W}_{n-p-t}^\xi| = |\widehat{\mathcal{Q}}_{s_n}|$ and $\mathcal{W}_n^\alpha = \left\{1^p\omega^\xi 1^t \,\middle|\, \omega^\xi \in \mathcal{W}_{n-p-t}^\xi\right\}$.*

Proof. If $\omega \in \mathcal{W}_n^\alpha$ and $\omega = 1^p\omega^\xi 1^t$ for some $\omega^\xi \in \mathcal{W}^\xi$ and $n = \mathcal{L}(\omega^\alpha) = p + t + \mathcal{L}(\omega^\xi)$ then the maximum value that $\mathcal{L}(\omega^\xi)$ can have is when $p = t = 1$ which is $n-2$. This means that $s + \sum_i q_i \le n-2$. Now the minimum value that $\sum_i q_i$ can take is when $q_i = 1$ for all $i = 1, ..., s-1$, thus we have that $s + (s-1) \le s + \sum_i q_i \le n-2$ from which immediately follows that $(s-1) \le \sum_i q_i \le n - s - 2$. If n is odd, the maximum number of zeros s attain its maximum when $q_i = 1$ for all i and $s + \sum_i q_i = 2s - 1$ is odd. Since, n is odd thus $n-2$ is odd too, then $2s - 1 = 2k + 1 - 2 = 2k - 1$ which implies that $s = k$. It turns out that $s = \frac{n-1}{2}$. If now $n = 2k$ is even then $n - 2 = 2(k-1)$ is even. The only way to get $s + \sum_i q_i$ even and s reaches its maximum value it is necessary that $q_i = 2$ for some i and the others $s-2$ terms are equal to 1, thus $s + \sum_i q_i = s + (s-2) + 2 = 2(k-1)$ from which immediately follows that $s = \frac{n-2}{2}$. The relationship $|\mathcal{W}_{n-p-t}^\xi| = |\widehat{\mathcal{Q}}_{s_n}|$ follows from the fact that $\mathcal{W}_{n-p-t}^\xi = \Phi(\widehat{\mathcal{Q}}_{s_n})$ and definition (4.3). It is obvious that $\mathcal{W}_n^\alpha = \left\{1^p\omega^\xi 1^t \,\middle|\, \omega^\xi \in \mathcal{W}_{n-p-t}^\xi\right\}$. □

The cardinality of $\mathcal{W}_n^\alpha$ is not straightforward since p, t must be taken into account. Its calculation is performed in the next lemma which is the main result, together with the above lemma, for computing *e-covers* for paths of arbitrary length.

Lemma 4. *The cardinality of $\mathcal{W}_n^\alpha$ can be calculated as*

$$|\mathcal{W}_n^\alpha| = 1 + \sum_{s=1}^{s_n} \Bigg(\sum_{\mathbf{q} \in \widehat{\mathcal{Q}}_s} [n - 1 - (s + |\mathbf{q}|)] \Bigg) \tag{4.4}$$

Proof. By using lemma (3), every $\omega^\alpha \in \mathcal{W}_n^\alpha$ can be written down as $\omega^\alpha = 1^p\omega^\xi 1^t$ for some $p, t \in \mathbb{N}$, and fixed $\omega^\xi \in \mathcal{W}_{n-p-t}^\xi$ such that $n = p + t + \mathcal{L}(\omega^\xi)$. This is equivalent to $t = n - p - \mathcal{L}(\omega^\xi)$ then $\omega^\alpha = 1^p\omega^\xi 1^{n-p-\mathcal{L}(\omega^\xi)}$. Since $\omega^\alpha \in \mathcal{W}_n^\alpha$, $n = \mathcal{L}(\omega^\alpha) < \infty$ and $p \ge 1$ then p must reach a maximum value, which is attained when $n - p - \mathcal{L}(\omega^\xi) = 1$, that is $p_{\max} = n - \mathcal{L}(\omega^\xi) - 1$. In other words, the number of times that ω^ξ can be shifted to the right within ω^α in order to generate a new sequence from ω^ξ is $n - \mathcal{L}(\omega^\xi) - 1$. Let us define the following set $\widehat{\omega^\xi} = \left\{1^p \omega^\xi 1^{n-p-\mathcal{L}(\omega^\xi)} \,\middle|\, 1 \le p \le n-1-\mathcal{L}(\omega^\xi)\right\}$, clearly $\mathcal{L}(a) = n$ for all $a \in \widehat{\omega^\xi}$ and $|\widehat{\omega^\xi}| = n-1-\mathcal{L}(\omega^\xi)$. It turns out that $\mathcal{W}_n^\alpha = \bigcup_{\omega^\xi \in \mathcal{W}_{n-p-t}^\xi} \widehat{\omega^\xi}$.

From lemma (3) the maximum number of zeros that ω^ξ can have is s_n therefore

$$
\begin{aligned}
|\mathcal{W}_n^\alpha| &= 1 + \sum_{s=1}^{s_n} \sum_{\omega^\xi \in \mathcal{W}^\xi_{\mathcal{L}(\omega^\xi)}} |\widehat{\omega^\xi}| \\
&= 1 + \sum_{s=1}^{s_n} \sum_{\omega \in \mathcal{W}^\xi_{\mathcal{L}(\omega^\xi)}} [n-1-\mathcal{L}(\omega^\xi)] \\
&= 1 + \sum_{s=1}^{s_n} \sum_{\mathbf{q} \in \widehat{\mathcal{Q}}_s} [n-1-\mathcal{L}(\Phi(\mathbf{q}))] \\
&= 1 + \sum_{s=1}^{s_n} \sum_{\mathbf{q} \in \widehat{\mathcal{Q}}_s} [n-1-(s+|\mathbf{q}|)] \qquad (4.5)
\end{aligned}
$$

The number 1 appearing in in Eq. (4.5) is counting the only sequence where no zeros are present. This is $\omega = 1^p 1^t$ with $\mathcal{L}(\omega) = n$. □

Lemma 5. *$|\mathcal{W}_n^\gamma| = |\mathcal{W}_{n-1}^\alpha|$, $|\mathcal{W}_n^\beta| = |\mathcal{W}_{n-1}^\alpha|$ and $|\mathcal{W}_n^\xi| = |\mathcal{W}_{n-2}^\alpha|$*

Proof. It follows from lemma (4). □

Theorem 1. *Let P a path of size $e(P)$ then $\mathcal{E}_P = \bigcup_{\kappa \in K} \Psi^{-1}(\mathcal{W}^\kappa_{e(P)})$ and $|\mathcal{E}_P| = \sum_{\kappa \in K} |\mathcal{W}^\kappa_{e(P)}|$.*

Proof. It follows from lemmas (2), (3), (4) and (5).

4.3 *e-covers* for *b-subtrees*

Let us consider a path $P = e_1 \cdots e_{n-1}$ of size $n = e(P)$. In order to count *e-covers* for *b-subtrees* we will consider *e-covers* for P and subpaths of P. That is, we will count the *e-covers* for P, $P \backslash e_1$, $P \backslash e_2$ and $P \backslash \{e_1, e_2\}$, from theorem (1) this is equivalent to consider all sequences of type α, β, γ and ξ. For every P path we associate a 4-tuple $\langle |\alpha_P|, |\beta_P|, |\gamma_P|, |\xi_P| \rangle$, where $|\alpha_P| = |\mathcal{W}_n^\alpha|$, $|\beta_P| = |\mathcal{W}_n^\beta| = |\mathcal{W}_{n-1}^\alpha|$, $|\gamma_P| = |\mathcal{W}_n^\gamma| = |\mathcal{W}_{n-1}^\alpha|$ and $|\xi_P| = |\mathcal{W}_n^\xi| = |\mathcal{W}_{n-2}^\alpha|$. The set α_P, is the set of *e-covers* for P when v_1, v_n are not isolated vertices; β_P is the set of *e-covers* for $P \backslash e_1$, that is when vertex v_1 is isolated; similarly γ_P, ξ_P are sets of *e-covers* when both v_n and v_1, v_n are isolated vertices, respectively. From the above discussion we can define a function $\Upsilon : \mathcal{M}_v \longrightarrow \mathbb{N}^4$, that for paths P becomes $\Upsilon(P) = \langle |\alpha_P|, |\beta_P|, |\gamma_P|, |\xi_P| \rangle$ whereas for *b-subtrees* is defined in the following theorem. From definition of the numbers $|\alpha_P|$,...,$|\xi_P|$ it is clear that $|\alpha_p| + |\beta_P| + |\gamma_P| + |\xi_P| = |\mathcal{E}_P|$

Theorem 2. *Let M_v be an b-subtree that is $M_v = \bigvee_{P\in\mathcal{F}_v} P$ such that all free vertices of M_v have been covered either by a path P or another b-subtree then*

$$\Upsilon(M_v) = \left\langle \prod_{P\in\mathcal{F}_v} |\mathcal{E}_P| - \prod_{P\in\mathcal{F}_v} |\xi_P|, \prod_{P\in\mathcal{F}_v} |\xi_P|, 0, 0 \right\rangle \tag{4.6}$$

The first coordinate in Eq. (4.6) gives the total number of *e-covers* if M_v where a single graph, that is the v vertex was the root vertex. The second coordinate in Eq. (4.6) gives the total number of *e-covers* in such a way that v remains an isolated vertex. This means that M_v has to be *joined* to another *b-subtree* via the vertex v from *below*.

5 *e-covers* for trees

The calculation of *e-covers* for trees will be given inductively on the number of vertices with degree greater than or equal to three. If $T = T(u)$, a given tree graph with root vertex u is non-trivial in the sense that $d_T(u) \geq 3$ and at least one $v \in V_T$ holds $d_T(v) \geq 3$ then the set $\mathcal{E}_T$ can be inductively constructed by joining *b-subtrees*. Let us consider the sets V_T^{s-1} and V_T^s for some $s \in \mathbb{N}$; for every $v \in V_T^{s-1}$ there are partitions $I, J \subseteq Q_s = \{1, .., |V_T^s|\}$ such that $v_i \in N_v$ and $v \prec v_i$ for all $i \in I \cup J$. Additionally, some of the v_i are free vertices, let us say v_i for all $i \in I$ whereas for all $i \in J$, v_i are leaves which correspond to vertices belonging to $N_v \backslash \{v_i\}_{i\in I}$. If we denote by P the path that ends at v and by P_i the path that starts at v and ends at v_i then $\Upsilon(P) = \langle|\alpha_P|, |\beta_P|, |\gamma_P|, |\xi_P|\rangle$, $\Upsilon(P_i) = \langle|\alpha_{P_i}|, |\beta_{P_i}|, |\gamma_{P_i}|, |\xi_{P_i}|\rangle$ for $i \in I$ and $\Upsilon(P_i) = \left\langle|\alpha_{P_i}|, |\beta_{P_i}|, 0, 0\right\rangle$ for $i \in J$. Clearly, we have that $\mathcal{F}_v = \{P_i\}_{i\in I\cup J}$, therefore

$$M_v = \bigvee_{i\in I\cup J} P_i, \;\; \mathcal{E}_{M_v} = \bigcup_{\substack{i\in I\cup J\\ \kappa\in K}} \mathcal{W}^{\kappa}_{e(P_i)} \tag{5.1}$$

where $K = \{\alpha, \beta, \gamma, \xi\}$. The join of M_v and the path P is done according to rule (3.4), that is $P \bigvee_v M_v = P \vee (\bigvee_{i\in I\cup J} P_i)$, however counting *e-covers* of the resulting *b-subtree* is done according to the following recursive procedure. Let us make $\mathbf{t}_\mathrm{P} = \Upsilon(P)$ and $\mathbf{t}_\mathrm{M} = \Upsilon(M_v) = \langle\alpha_M, \beta_M, \gamma_M, \xi_M\rangle$ and define the following contracting functions $c_\alpha(\mathbf{t}_\mathrm{P}, \mathbf{t}_\mathrm{M}) = (\alpha_P + \beta_P)(\alpha_M + \gamma_M) - \beta_P\gamma_M$, $c_\beta(\mathbf{t}_\mathrm{P}, \mathbf{t}_\mathrm{M}) = (\alpha_P + \beta_P)(\beta_M + \xi_M) - \beta_P\xi_M$, $c_\gamma(\mathbf{t}_\mathrm{P}, \mathbf{t}_\mathrm{M}) = (\gamma_P + \xi_P)(\alpha_M + \gamma_M) - \gamma_M\xi_P$ and finally $c_\xi(\mathbf{t}_\mathrm{P}, \mathbf{t}_\mathrm{M}) = (\gamma_P + \xi_P)(\beta_M + \xi_M) - \xi_P\xi_M$. We can therefore define a recursive function to count *e-covers* of the *join* of a path P and a *b-subtree* M_v as follows

$$c(\mathbf{t}_\mathrm{P}, \mathbf{t}_\mathrm{M}) = \left\langle c_\alpha(\mathbf{t}_\mathrm{P}, \mathbf{t}_\mathrm{M}), c_\beta(\mathbf{t}_\mathrm{P}, \mathbf{t}_\mathrm{M}), c_\gamma(\mathbf{t}_\mathrm{P}, \mathbf{t}_\mathrm{M}), c_\xi(\mathbf{t}_\mathrm{P}, \mathbf{t}_\mathrm{M})\right\rangle \tag{5.2}$$

The main idea in the process of *join* and counting *e-covers* for tree graphs lies on the *join* operation, the recursive relation (4.6), Eq. (5.1) and Eq. (5.2). The *join* of P and the *b-subtree* graph M_v means the replacement of $\Upsilon(P)$, the 4-tuple before the *join* operation associated to P, by $c(\mathbf{t}_{\mathrm{P}}, \mathbf{t}_{\mathrm{M_v}})$ after performing the *join* operation.

The above discussion leads to the following theorem, which summarize the process of building and counting *e-cover* for a given tree graph.

Theorem 3. *Given a tree graph T the set $\mathcal{E}_T$ can be calculated by means of theorem (1), (2), Eq. (5.1) and the recurrence relation (5.2) therefore can also be obtained the cardinality of $\mathcal{E}_T$.*

Example 1. To illustrate the usage of theorem (3), the tree graph shown in Fig. (2) is considered. The graph $T(u)$ has two vertices v and w of degree three

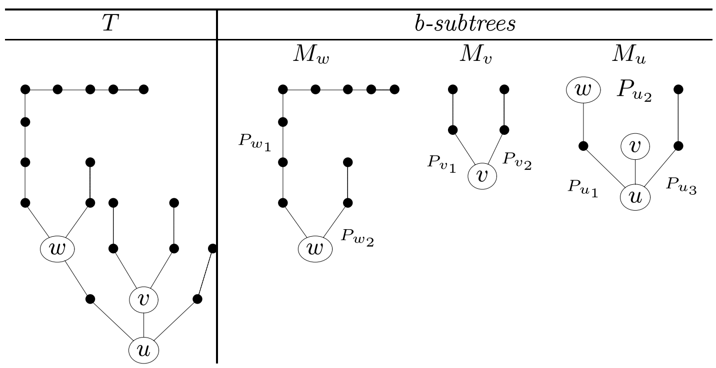

Fig. 2. A rooted tree $T(u)$ with root vertex u and its decomposition into *b-subtrees* M_v, M_w and M_u.

therefore $T(u)$ is the *join* of *b-subtrees*, namely M_w, M_v and M_u as shown in Fig. (2), that is $T = (M_u \vee_v M_v) \vee_w M_w$ where $M_w = P_{w_1} \vee P_{w_2}$, $M_v = P_{v_1} \vee P_{v_2}$ and $M_u = P_{u_1} \vee P_{u_2} \vee P_{u_3}$. For illustration purposes, calculation will be done only for P_{w_1} since it is the largest path in T that shows the technique developed in the paper. First of all, $e(P_{w_1}) = 8$ thus $\mathcal{W}_8^{\kappa}$ with $\kappa \in K$ have to be obtained. From lemma (3), $s_8 = 3$, $\widehat{s}_8 = \{2, 3\}$ then the sets L_{8s} are $L_{82} = \{1, 2, 3, 4\}$, $L_{83} = \{2, 3\}$. Firstly, by definition of sequences of type ω^{ξ} we have $\mathcal{W}_{s+j}^{\xi} = \{\omega^{\xi} = 01^{q_1} \cdots 1^{q_{s-1}}0 | \sum_{i=1}^{s-1} q_i = j\}$ with $s \in \widehat{s}_8$ and $j \in L_{8s}$. Therefore, $\mathcal{W}_{2+j}^{\xi} = \{\omega^{\xi} | q_1 = j\} = \{01^j0 | j \in L_{82}\}$, whereas $\mathcal{W}_{3+j}^{\xi} = \{\omega^{\xi} | q_1 + q_2 = j, j \in L_{83}\}$. The partitions of integers 2 and 3 into two parts is the set $\widehat{\mathcal{Q}}_3 = \{(1, 1), (1, 2), (2, 1)\}$ therefore $\mathcal{W}_{2+j}^{\xi} = \{01^j0\} = \{010, .., 011110\}$ and $\mathcal{W}_{3+j}^{\xi} = \{01010, 010110, 011010\}$.

The $\mathcal{W}_8^{\alpha}$ can be calculated as $1^p\omega^{\xi}1^t$ where p, t as in lemma (3) and $\omega^{\xi} \in \bigcup_{s,j} \mathcal{W}_{s+j}^{\xi}$. Thus $\mathcal{W}_8^{\alpha}$ becomes the list of sequences 11111111, 1**0**111111,...,111111**0**1, 1**010**1111, 11**010**111, 111**010**11, 1111**010**1, 1**0110**111, 11**0110**11, 111**0110**1 1**01110**11, 11**01110**1, 1**011110**1, 1**01010**11, 11**01010**1, 1**010110**1, 1**011010**1 from which results by counting them $|\mathcal{W}_8^\alpha| = 21$. On the other hand, because we are to *join* P_{w_1} with M_u at w we have to consider sequences of type β which is obviously the set $\mathcal{W}_8^\beta$ whose cardinality is 13. The sequences of type ξ used in the calculations are in bold type format to highlight its usage. From theorem (4.6), $\Upsilon(P_{w_1}) = \langle 21, 13, 0, 0\rangle$. Similarly, $\Upsilon(P_{w_2}) = \langle 1, 1, 0, 0\rangle$, $\Upsilon(P_{v_1}) = \langle 1, 1, 0, 0\rangle$, $\Upsilon(P_{v_2}) = \langle 1, 1, 0, 0\rangle$, $\Upsilon(P_{u_1}) = \langle 1, 1, 1, 0\rangle$, $\Upsilon(P_{u_2}) = \langle 1, 0, 0, 1\rangle$, $\Upsilon(P_{u_3}) = \langle 1, 1, 0, 0\rangle$. Once again from theorem (4.6), $\Upsilon(M_w) = \langle (21+13)(1+1) - (13)(1), (13)(1), 0, 0\rangle = \langle 55, 13, 0, 0\rangle$, $\Upsilon(M_v) = \left\langle 3, 1, 0, 0\right\rangle$. In order to *join* M_w to P_{u_1} and M_v to P_{u_2} Eq. (5.2) is used to replace $\Upsilon(P_{u_1})$, $\Upsilon(P_{u_2})$ with the new values. Thus, $\Upsilon(P_{u_1}) = c(\Upsilon(P_{u_1}), \Upsilon(M_w)) = \langle 123, 68, 0, 0\rangle$, $\Upsilon(P_{u_2}) = c(\Upsilon(P_{u_2}), \Upsilon(M_v)) = \langle 4, 3, 0, 0\rangle$. To conclude the exercise we calculate $\Upsilon(M_u)$ by using Eq. (4.6) as $\Upsilon(M_u) = \langle (123+68)(4+3)(1+1) - (68)(3)(1), (68)(3)(1), 0, 0\rangle = \langle 2470, 204, 0, 0\rangle$. Thus $|\mathcal{E}_T| = 2470$ since the 204 number is counting the combinations when u is not covered. The number of *e-covers* for T is too big to write down every single cover of T, however they can be built from the sequences above.

6 Conclusions

A procedure for counting *e-covers* for acyclic graphs has been presented. It is believed that the procedure could be implemented into a linear algorithm albeit the procedure to built covers for a given tree graph is much more complex.

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