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NUMERICAL RECKONING FIXED POINTS IN BUSEMANN SPACES

KIFAYAT ULLAH[1,*], MUHAMMAD ARSHAD[2], AMNA KALSOOM[2], HIKMAT NAWAZ KHAN[1]

[1]Department of Mathematics,National University of Science's and Technology, Township Campus, Bannu, Pakistan
[2]Department of Mathematics, International Islamic University, H-10, Islamabad, Pakistan

Abstract. In this paper, we prove some strong and Δ-convergence theorems for mappings, which satisfy the condition (E) in the setting of Busemann spaces via the M iteration process. Numerical examples are given to show the efficiency of the M iteration process. As an application, we investigate the well-known Delay differential equation.

Keywords. Mapping satisfying condition (E); Busemann space; Iteration process; Δ-convergence; Strong convergence.

1. INTRODUCTION

An efficient tool to find fixed points for nonexpansive mappings is the iterative method such as Mann iterative process, Ishikawa iterative process and Norr iterative process. In 1979, Reich [1] obtained, in a uniformly convex Banach space with a Frechet differentiable norm, a celebrated weak convergence result of nonexpansive mappings. In 2008, Dhompongsa and Panyanak [2] obtained Δ-convergence theorems for the Picard, Mann and Ishikawa iterations of nonexpansive mapping in the setting of $CAT(0)$ space. Also an analogue of the Kirk's fixed point theorem in class of uniformly convex metric spaces, as an extension of the class of $CAT(0)$ spaces, was given in [3, Corollary 3.10]. Some existence and approximation results for SKC mappings in Busemann spaces have been recently studied by Khan, Abbas and Nazir [4]. The approximation of mappings satisfying condition (E) in Busemann space was studied by Bagherboum [5].

The well-known Banach contraction theorem uses the Picard iterative process for fixed points of contractions. There are many other import extensions of the Picard iterative process, such as, Mann iterative process [6], Ishikawa iterative process [7], S iterative process [8], Noor iterative

*Corresponding author.
E-mail addresses: kifayatmath@yahoo.com (K. Ullah), marshadzia@iiu.edu.pk (M. Arshad), amna.kalsoom@iiu.edu.pk (A. Kalsoom), hikmatnawazkhan@gmail.com (H. N. Khan).
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process [9], Abbas iterative process [10], SP iterative process [11], S* iterative process [12], CR iterative process [13], Normal-S iterative process [14], Picard-Mann iterative process [15], Picard-S iterative process [16], Thakur New iterative process [17], K* iterative process [18], M* iterative process [19] and so on. Recently, Ullah and Arshad [20] introduced a new iterative process called M iterative process. They prove that the M iterative process is faster than the other known iterative processes, such as, the Picard-S and S iterative processes. In [20], they developed an example of mappings satisfying condition C, which is not nonexpansive and used it to show the efficiency of M iterative process. They also proved weak and strong convergence theorems for the mappings satisfying condition C in the setting of uniformly convex Banach spaces.

The purpose of this paper is to prove the effectiveness of the M iterative process with the help of numerical examples. Moreover, utilizing the M iteration process, some Δ-convergence and strong convergence theorem for the mapping satisfying condition E are proved in uniformly convex Busemann spaces.

2. Preliminaries

Let (X,d) be a metric space. A geodesic form x to y in X is a map c from closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$ and $d(c(t),c(t^{'})) = |t-t^{'}|$ for all $t,t^{'} \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image of c is called a geodesic (or metric) segment joining x and y. The space (X,d) is said to be geodesic space if every two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each x, $y \in X$, which we will denote by $[x,y]$, called the segment joining x to y.

Let $\gamma : [a,b] \to X$ be a path in a metric space X. We say that γ is an affine parameterized geodesic if either γ is a constant path or there exists a geodesic path $\gamma' : [c,d] \to X$ such that $\gamma = \gamma' \circ \Psi$, where $\Psi : [a,b] \to [c,d]$ is the unique affine homeomorphism between the intervals $[a,b]$ and $[c,d]$.

Lat X be a uniquely geodesic space. If $\gamma([a,b])$ is a geodesic segment joining x and y and $\lambda \in [0,1]$, then $z := \gamma((1-\lambda)a+\lambda b)$ is the unique point in $\gamma([a,b])$ satisfying $d(z,x) = \lambda d(x,y)$ and $d(x,z) = (1-\lambda)d(x,y)$. In the sequel, the notation $[x,y]$ is used for the geodesic segment $\gamma([a,b])$ and z is denoted by $(1-\lambda)x \oplus \lambda y$, provided that there is no possible ambiguity. A subset $C \subseteq X$ is said to be geodesically convex if C includes every geodesic segment joining any two of its point.

Let X be a geodesic metric space and let $f : X \to \mathbb{R}$ be a mapping. We say that f is convex (respectively strictly convex) if for every geodesic path $\gamma : [a,b] \to X$, the map $f \circ \gamma : [a,b] \to \mathbb{R}$ is convex (respectively strictly convex). It is know if $f : X \to \mathbb{R}$ is a function and $g : f(X) \to \mathbb{R}$ is an increasing convex (respectively strictly convex) function, then $g \circ f : X \to \mathbb{R}$ is convex (respectively strictly convex).

Definition 2.1. [21] The geodesic metric space (X,d) is called a Busemann space if for any two affine parameterized geodesics $\gamma : [a,b] \to X$ and $\gamma' : [a',b'] \to X$, the map $D_{r,r'} : [a,b] \times [a',b'] \to \mathbb{R}$ define by $D_{r,r'}(t,t') = d(\gamma(t),\gamma'(t'))$ is convex; i.e., the metric of Busemann space is convex.

The statements, which are equivalent to this definition, were presented in [22, Proposition 8.1.2]. The simplest examples of Busemann spaces are $CAT(0)$ spaces, strictly convex normed spaces (in particular, strictly convex Banach spaces and uniformly convex Banach spaces such as the L^p, l^p and $W^m{}_p$ spaces, for $1 < p < \infty$, Minkowski spaces (i.e., finite dimensional affine spaces equipped with a Finsler metric, which is invariant under translations) and the simply connected Reimannian manifolds of nonpositive sectional curvature.

In these spaces, the following conditions are fulfilled [5]:

(1) $d(z,(1-\lambda)x\oplus\lambda y) \leq (1-\lambda)d(z,x)+\lambda d(z,y)$,

(2) $d((1-\lambda)x\oplus\lambda y),(1-\lambda')x\oplus\lambda' y) = |\lambda-\lambda'|$,

(3) $(1-\lambda)x\oplus\lambda y = \lambda y\oplus(1-\lambda)x$,

(4) $d((1-\lambda)x\oplus\lambda z,(1-\lambda)y\oplus\lambda w) \leq (1-\lambda)d(x,y)+\lambda d(z,w)$,

where $x,y,z,w \in X$ and $\lambda,\lambda' \in [0,1]$. Therefore, Busemann spaces are hyperbolic spaces, which were introduced by Kohlenbach [23].

Definition 2.2. The Busemann space [5] (X,d) is said to be uniformly convex if for $r > 0$ and $\varepsilon \in (0,2]$, there exists a map δ such that, for every three points $a,x,y \in X$,

$$\left.\begin{array}{r} d(x,a) \leq r \\ d(y,a) \leq r \\ d(x,y) \geq \varepsilon r \end{array}\right\} \Longrightarrow d(\tfrac{1}{2}x\oplus\tfrac{1}{2}y,a) \leq (1-\delta)r,$$

such that, for all $\varepsilon \in (0,2]$, $\inf\{\delta : r > 0\} > 0$.

A mapping $\eta : (0,\infty)\times(0,2] \to (0,1]$ providing such a $\eta(r,\varepsilon) := \delta$ for given $r > 0$ and $\varepsilon \in (0,2]$ is called a modulus of uniform convexity.

From now on, a modulus of the uniform convexity with a decreasing modulus with respect to r (for a fixed ε) is called a monotone modulus of the uniform convexity. The following lemma shows a property of uniformly convex Busemann spaces which will be useful to obtain our main results.

Definition 2.3. A point p is called a fixed point of a mapping T if $T(p) = p$, and $F(T)$ represents the set of all fixed points of mapping T. Let C be a nonempty subset of a Busemann space X.

Lemma 2.4. *[24, Lemma 2.2] Let X be a uniformly convex Busemann space with a monotone modulus of convexity η and let $x \in X$. Let $\{\lambda_n\}$ be a sequence in $[b,c]$ for some $b,c \in (0,1)$ and $\{x_n\}$, $\{y_n\}$ be sequences in X such that, for some $r \geq 0$,*

$$\limsup_{n\to\infty} d(x_n,x) \leq r, \limsup_{n\to\infty} d(y_n,x) \leq r \text{ and } \limsup_{n\to\infty} d(\lambda_n x_n \oplus (1-\lambda_n)y_n,x) = r.$$

Then

$$\lim_{n\to\infty} d(x_n,y_n) = 0.$$

Let C be a nonempty closed convex subset of a $CAT(0)$ space X and let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we set$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x_n,x)$. The asymptotic radius of $\{x_n\}$ relative to C is given by

$$r(C,\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in C\},$$

and the asymptotic center of $\{x_n\}$ relative to C is the set

$$A(C,\{x_n\}) = \{x \in C : r(x,\{x_n\}) = r(C,\{x_n\})\}.$$

It is known that, in a $CAT(0)$ space, $A(C,\{x_n\})$ consists of exactly one point.

This set may be empty, a singleton, or may certain infinite points. In fact, if $\{ x_n\}$ converges strongly to $x \in C$, then $A_c(\{x_n\}) = \{x\}$ and $\{ x_n\}$ converges strongly to x and $x \notin C$, where $d(x,C) := \inf_{c\in C} d(x,c)$ and $A_c(\{x_n\}) = \{c \in C : d(x,c) = d(x,C)\} := P_C(x)$, which $P_C(x)$ is the projection of x on C. The existence of unique projection of x on a nonempty closed convex subset in complete Busmann space was proved, for more in [25, Proposition 8.4.8].

Now for an arbitrary bounded sequence $\{x_n\}$, $\varphi : X \to \mathbb{R}^+$ is define by $\varphi(x) = r(x,\{x_n\})$. It can be easily observed that φ is a continuous convex function. Indeed, the continuity follows from the triangle inequality convexity is an immediate consequence of convexity distance function (see, [25, Example 8.4.7(i)]).

Definition 2.5. Let $\{x_n\}$ be a bounded sequence in X and let $x \in X$. The sequence $\{x_n\}$ is said to be Δ -converge to x if the point x is the unique asypmtotic center of $\{u_n\}$, for any subsequences $\{u_n\}$ of $\{x_n\}$. Such a bounded sequence is called regular. In this case, we write $\Delta -lim_n x_n = x$ and call x the $\Delta - \lim it$ of $\{x_n\}$.

Any bounded sequence $\{x_n\}$ in X is said to be regular with respect to subset C of X if the asymptotic radii of subsequences of $\{x_n\}$ with respect to C are the same. We have the following lemma. Based on this lemma, any bounded sequence has a Δ-convergent subsequence.

Lemma 2.6. *Let X be a uniformly convex Busemann space, $\{x_n\}$ be a bounded sequence in X and C be a subset of X. Then $\{x_n\}$ has a subsequence which is regular with respect to C.*

It is clear that any strong convergence sequence is Δ-convergence. Also, if C is a closed convex subset of a Busemann apace, then Δ-convergence of any bounded sequence to x implies that $x_n \to x$ (i.e., the asymptotic center of $\{x_n\}$ with respect to C is x).

Lemma 2.7. *[11, Proposition 2.1] If C is a closed convex subset of a uniformly convex Busemann space X and $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.8. *[26, Proposition 3.7] Let C be a closed convex subset of a uniformly convex Busemann space X and let $T : C \to X$ be a mapping satisfying condition (E). Then the conditions $\{x_n\}$ Δ-converges to x and $d(Tx_n,x_n) \to 0$ implies $x \in C$ and $Tx = x$.*

Definition 2.9. If T is a mapping satisfying condition (E) and has a fixed point, than T is a quasi-nonexpansive mapping.

3. The M Iterative Process

The M Iteration process introduced by Ullah and Arshad [20] is defined as:

$$\begin{cases} x_0 \in C, \\ z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = T z_n, \\ x_{n+1} = T y_n, \end{cases} \tag{3.1}$$

where $n \geq 0$ and $\{\alpha_n\}$ is a real sequence in $[0,1]$.

TABLE 1. Sequences generated by M, Picard-S and S iteration processes for mapping f of Example 3.1.

	M	Picard-S	S
x_0	10	10	10
x_1	5.44565574132238	5.66076955440972	6.83624962725991
x_2	5.00693080112109	5.01960812810809	5.37044124393507
x_3	5.00008898476401	5.00044149419855	5.04752702755717
x_4	5.00000113904951	5.00000985609151	5.00540334935426
x_5	5.00000001457984	5.00000021998893	5.00060429338923
x_6	5.00000000018662	5.00000000491015	5.00006745514064
x_7	5.00000000000239	5.00000000010960	5.00000752819304
x_8	5.00000000000003	5.00000000000245	5.00000084014883
x_9	5.	5.00000000000006	5.00000009376064
x_{10}	5.	5.	5.00000001046369

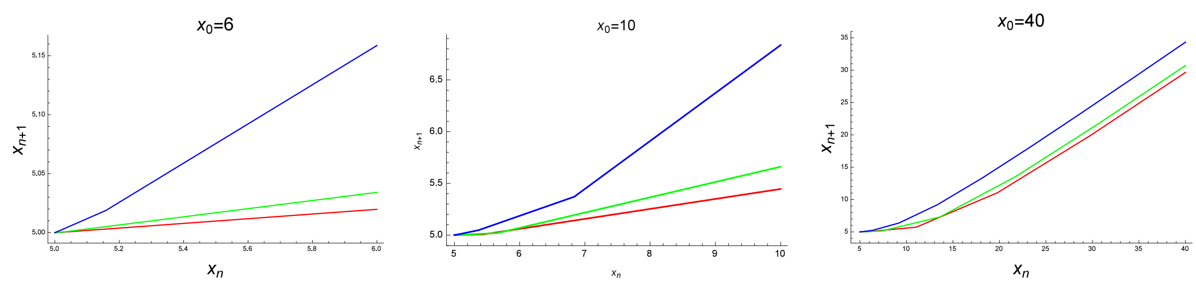

FIGURE 1. Convergence of M, Picard-S and S iteration processes to the fixed point 5 of the mapping define in Example 3.1 for different initial values.

Example 3.1. Let us define a function $T : [0,50) \to [0,50)$ by $f(x) = \sqrt{x^2 - 8x + 40}$. Clearly, T is a contraction mapping. Let $\alpha_n = 0.85$ for all n. The iterative values for initial value $x_0 = 10$ are given in Table 1. Figure 1 shows the convergence graphs for different initial values. Table and graphs shows that M iterative process is more efficient as compare to other iterative processes.

In 2008, Suzuki [27] introduced the concept of generalized nonexpansive mappings which is a condition on mappings called condition (C).

Definition 3.2. A mapping $T : C \to C$ is said to satisfy condition (C) if, for all $x, y \in C$,

$$\frac{1}{2} d(x,Tx) \le d(x,y) \text{ implies } d(Tx,Ty) \le d(x,y).$$

Definition 3.3. A mapping $T : C \to C$ is said to satisfy condition (E_μ), if, for all $x, y \in X$,

$$(x,Ty) \le \mu d(x,Tx) + d(x,y).$$

We say that T satisfies condition (E), whenever T satisfies condition (E_μ) for some $\mu \ge 1$.

Example 3.4. Define a mapping $T: C \to C$ by

$$T(x,y) = \begin{cases} (\frac{1+x}{5}, y) & 0 \leq x < \frac{1}{5}, \\ (0,y) & \frac{1}{5} \leq x \leq 1, \end{cases}$$

where $C = [0,1]^2 \subset X =(\mathrm{R}^2, d)$. Suppose that $x = (x_1, y_1)$, $y = (x_2, y_2) \in C$. We need to show that T satisfy condition (E) but does not satisfy condition (C). Let $x_1 = \frac{1}{5}$ and $x_2 = \frac{2}{5}$. Then $x = (\frac{1}{5}, y_1)$ and $y = (\frac{2}{5}, y_2)$

$d(x,Tx) = d((x_1,y_1),T(x_1,y_1)) = d((x_1,y_1),(\frac{1+x_1}{5},y_1) = d((\frac{1}{5},y_1),(\frac{1+\frac{1}{5}}{5},y_1))$
$= d((\frac{1}{5},y_1),(\frac{6}{25},y_1)) = ((\frac{1}{5}-\frac{6}{25})^2+(y_1-y_1)^2)^{\frac{1}{2}} = \frac{1}{25}$,
$\frac{1}{2}d(x,Tx) = \frac{1}{50}$,
$d(x,y) = d((x_1,y_1),(x_2,y_2)) = d((\frac{1}{5},y),(\frac{2}{5},y)) = ((\frac{1}{5}-\frac{2}{5})^2+(y-y)^2)^{\frac{1}{2}}$
$= \frac{1}{5}$.
Hence $\frac{1}{2}d(x,Tx) \leq d(x,y)$ and
$d(Tx,Ty) = d(T(x_1,y_1),T(x_2,y_2)) = d((\frac{1+x_1}{5},y),(0,y)) = d((\frac{1+\frac{1}{5}}{5},y),(0,y))$
$= d((\frac{6}{25},y),(0,y))((\frac{6}{25}-0)^2+(y-y)^2))^{\frac{1}{2}} = \frac{6}{25} > \frac{1}{5} = d(x,y)$.
Thus

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \nRightarrow d(Tx,Ty) \leq d(x,y).$$

Hence T does not satisfy condition (C). To verify that T satisfy condition (E), we consider the following cases:

Case I: ($x_1 \leq \frac{1}{5}$ and $x_2 \leq \frac{1}{5}$) or ($x_1 > \frac{1}{5}$ and $x_2 > \frac{1}{5}$).

(a). If $x_1 = \frac{1}{5}$ and $x_2 = \frac{1}{6}$, then $x = (\frac{1}{5}, y_1)$ and $y = (\frac{1}{6}, y_2)$,

$$d(x,y) = d((x_1,y_1),(x_2,y_2)) = d((\frac{1}{5},y),(\frac{1}{6},y)) = ((\frac{1}{5}-\frac{1}{6})^2+(y-y)^2))^{\frac{1}{2}} = \frac{1}{30},$$

$d(Tx,Ty) = d(T(x_1,y_1),T(x_2,y_2)) = d((\frac{1+x_1}{5},y),(\frac{1+x_2}{5},y))d((\frac{1+\frac{1}{5}}{5},y),(\frac{1+\frac{1}{6}}{5},y))$
$= d((\frac{6}{25},y),(\frac{7}{30},y)) = ((\frac{6}{25}-\frac{7}{30})^2+(y-y)^2))^{\frac{1}{2}} = \frac{1}{50}$.
Then $d(Tx,Ty) \leq d(x,y)$, which implies that

$$d(x,Ty) \leq d(x,Tx) + d(Tx,Ty) \leq d(x,Tx) + d(x,y).$$

(b). If $x_1 = \frac{2}{5}$ and $x_2 = \frac{1}{5}$, then $x = (\frac{2}{5}, y_1)$ and $y = (\frac{1}{5}, y_2)$,

$$d(x,y) = d((x_1,y_1),(x_2,y_2)) = d((\frac{2}{5},y),(\frac{1}{5},y)) = ((\frac{2}{5}-\frac{1}{5})^2+(y-y)^2))^{\frac{1}{2}} = \frac{1}{5},$$

and

$$d(Tx,Ty) = d(T(x_1,y_1),T(x_2,y_2)) = d((0,y),(0,y)) = ((0-0)^2+(y-y)^2))^{\frac{1}{2}} = 0.$$

Then $d(Tx,Ty) \leq d(x,y)$, which implies that

$$d(x,Ty) \leq d(x,Tx) + d(Tx,Ty) \leq d(x,Tx) + d(x,y).$$

Case II: ($x_1 \leq \frac{1}{5}$ and $x_2 > \frac{1}{5}$) or ($x_1 > \frac{1}{5}$ and $x_2 \leq \frac{1}{5}$)
(a). If $x_1 = \frac{1}{5}$ and $x_2 = \frac{2}{5}$, then $x = (\frac{1}{5}, y_1)$ and $y = (\frac{2}{5}, y_2)$,

$$d(y.Ty) = d((x_2,y_2),T(x_2,y_2)) = d((\frac{2}{5},y),(0,y)) = ((\frac{2}{5}-0)^2+(y-y)^2))^{\frac{1}{2}} = \frac{2}{5},$$

$$d(x,Tx) = d((x_1,y_1),T(x_1,y_1)) = d((x_1,y),(\frac{1+x_1}{5},y)) = d((\frac{1}{5},y),(\frac{1+\frac{1}{5}}{5},y))$$
$$= ((\frac{1}{5}-\frac{6}{25})^2+(y-y)^2))^{\frac{1}{2}} = \frac{1}{25},$$

and $6d(x,Tx) = \frac{6}{25}$. Then $d(y,Ty) \leq 6d(x,Tx)$, which implies that $d(x,Ty) \leq d(x,y)+d(y,Ty) \leq d(x,y)+6d(x,Tx)$.

(b). If $x_1 = \frac{2}{5}$ and $x_2 = \frac{1}{5}$, then $x = (\frac{2}{5},y_1)$ and $y = (\frac{1}{5},y_2)$,

$$d(y,Ty) = d((x_2,y_2),T(x_2,y_2)) = d((x_2,y),(\frac{1+x_2}{5},y)) = d((\frac{2}{5},y),(\frac{1+\frac{2}{5}}{5},y)) = \frac{3}{25},$$

$$d(x,Tx) = d((x_1,y_1),T(x_1,y_1)) = d((\frac{2}{5},y),(0,y)) = \frac{2}{5}.$$

Then $6d(x,Tx) = \frac{12}{5}$. Hence $d(y,Ty) \leq 6d(x,Tx)$, which implies that $d(x,Ty) \leq d(x,y)+6d(x,Tx)$. Hence T is a mapping satisfying condition (E).

4. Convergence results for a mapping satisfying condition (E)

In this section, we prove some strong and $\Delta-$convergence of the sequence generated by M iteration process for mapping satisfying condition (E) in the setting of Busemann spaces. The M iteration process in the language of Busemann space is defined as:

$$\begin{cases} x_0 \in C, \\ z_n = (1-\alpha_n)x_n \oplus \alpha_n Tx_n, \\ y_n = Tz_n, \\ x_{n+1} = Ty_n, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$.

Theorem 4.1. *Let C be a nonempty closed convex subset of a Busemann spaces X, and let $T : C \to C$ be a mapping satisfying condition (E) with $F(T) \neq \emptyset$. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (4.1), then $\lim_{n\to\infty} d(x_n,p)$ exists for any $p \in F(T)$.*

Proof. Let $p \in F(T)$ and $z \in C$. Since T mapping satisfying condition (E), we have

$$\frac{1}{2}d(p,Tp) = 0 \leq d(p,z) \text{ implies that } d(Tp,Tz) \leq d(p,z).$$

Using Definition 2.9, we have

$$\begin{aligned} d(z_n,p) &= d((1-\alpha_n)x_n \oplus \alpha_n Tx_n,p) \\ &\leq (1-\alpha_n)d(x_n,p)+\alpha_n d(Tx_n,p) \\ &\leq (1-\alpha_n)d(x_n,p)+\alpha_n[\mu d(Tp,p)+d(x_n,p)] \\ &\leq (1-\alpha_n)d(x_n,p)+\alpha_n d(x_n,p) \\ &= d(x_n,p). \end{aligned} \tag{4.2}$$

Using (4.2), we get

$$\begin{aligned} d(y_n, p) &= d(Tz_n, p) \\ &\leq [\mu d(Tp, p) + d(z_n, p)] \\ &\leq d(z_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{4.3}$$

Similarly by using (4.3), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(Ty_n, p) \\ &\leq [\mu d(Tp, p) + d(y_n, p)] \\ &\leq d(y_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{4.4}$$

This implies that $\{d(x_n, p)\}$ is bounded and non-increasing for all $p \in F(T)$. Hence $\lim_{n\to\infty}(x_n, p)$ exists, as required. □

Theorem 4.2. *Let C be a nonempty closed convex subset of a CAT(0) space X, and let $T : C \to C$ be a mapping satisfying condition (E). For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (4.1) for all $n \geq 1$, where $\{\alpha_n\}$ is a sequence of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$. Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.*

Proof. Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. Then, by Theorem 4.1, $\lim_{n\to\infty} d(x_n, p)$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty}(x_n, p) = r. \tag{4.5}$$

From (4.2) and (4.5), we have

$$\limsup_{n\to\infty} d(z_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = r. \tag{4.6}$$

By Definition 2.9, we have

$$\limsup_{n\to\infty} d(Tx_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = r. \tag{4.7}$$

On the other hand, by using the M iteration process, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(Ty_n, p) \\ &\leq [\mu d(Tp, p) + d(y_n, p)] \\ &\leq d(y_n, p) \\ &\leq d(z_n, p). \end{aligned}$$

Therefore

$$r \leq \liminf_{n\to\infty} d(z_n, p). \tag{4.8}$$

By using (4.6) and (4.8), we get

$$\begin{aligned} r &= \lim_{n\to\infty} d(z_n, p) \\ &= \lim_{n\to\infty} d(((1-\alpha_n)x_n + \alpha_n T x_n), p) \\ &= \lim_{n\to\infty} [\alpha_n d(T x_n, p) + (1-\alpha_n) d(x_n, p)]. \end{aligned} \tag{4.9}$$

From (4.5), (4.7), (4.9) and Lemma 2.6, we have that $\lim_{n\to\infty} d(T x_n - x_n) = 0$.

Conversely, suppose that $\{x_n\}$ is bounded and $lim_{n\to\infty} d(T x_n, x_n) = 0$ Than,by Lemma$\{x_n\}$ has a subsequences which is regular with respect to C. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ such that $A_c(u_n) = x$,Therefore,

$$\limsup_{n\to\infty} d(u_n, T p) \leq \limsup_{n\to\infty} [\mu d(u_n, T u_n) + d(u_n, x)] = \limsup_{n\to\infty} d(u_n, x).$$

Thus the uniqueness of asymptotic center implies that x is a fixed point of T and this completes the proof. □

Now we are in the position to prove the Δ-convergence theorem.

Theorem 4.3. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X, and let $T : C \to C$ be a mapping satisfying condition (E) with $F(T) \neq \emptyset$. Let $\{t_n\}$ and $\{s_n\}$ be sequences in $[0,1]$ so that $\alpha_n \in [a,b]$ with $0 < a \leq b < 1$. From arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ generating by (4.1). Then $\{x_n\}$ Δ-converges to a fixed point of T .*

Proof. Since $F(T) \neq \emptyset$, by Theorem 4.1, we have that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(T x_n, x_n) = 0$. We now let $w_w\{x_n\} := \bigcup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $w_w\{x_n\} \subset F(T)$. Let $u \in w_w\{x_n\}$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.7 and Lemma 2.8, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim_n \{v_n\} = v \in C$. Since $\lim_{n\to\infty} d(v_n, T v_n) = 0$. Then $v \in F(T)$ by Lemma 2.8. We claim that $u = v$. If not, since T is Suzuki generalized nonexpansive mapping and $v \in F(T)$, $\lim_n d(x_n, v)$ exists by Theorem 4.1. Then by uniqueness of asymptotic centers, we have

$$\begin{aligned} \limsup_n d(v_n, v) &< \limsup_n d(v_n, u) \\ &\leq \limsup_n d(u_n, u) \\ &< \limsup_n d(u_n, v) \\ &= \limsup_n d(x_n, v)) \\ &= \limsup_n d(v_n, v), \end{aligned}$$

which is a contradiction. Hence $u = v \in F(T)$. To show that $\{x_n\}$ Δ-converges to a fixed point of T, it is suffices to show that $w_w\{x_n\}$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 2.7 and Lemma 2.8, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim_n \{v_n = v \in C$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have seen that $c \in F(T)$. We can complete the proof by showing that $x = v$. If not, since $\{d(x_n, v)\}$ is convergent, by the uniqueness of asymptotic centers, we have

$$\limsup_n d(v_n, v) < \limsup_n d(v_n, x) \le \limsup_n d(x_n, x) < \limsup_n d(x_n, v) = \limsup_n d(v_n, v),$$

which is a contradiction. Hence the conclusion follows. □

Next we prove the strong convergence theorem.

Theorem 4.4. *Let C be a nonempty compact convex subset of a $CAT(0)$ space X, and let $T : C \to C$ be a mapping satisfying condition (E). For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (4.1) for all $n \ge 1$, where $\{\alpha_n\}$ is a sequence of real numbers in $[a, b]$ for some a, b with $0 < a \le b < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T .*

Proof. By Theorem 4.2 and Theorems 4.3 we have $\{x_n\}$ is bounded and Δ-converges to $x \in F(T)$. Suppose on the contrary that $\{x_n\}$ does not converge strongly to x. By the compact assumption, passing to subsequences if necessary, we may assume that there exists $x' \in C$ with $x' \ne x$ such that $\{x_n\}$ converge strongly to x'. Therefore, $\lim_{n\to\infty} d(x_n, x') = 0 \le \lim_{n\to\infty} d(x_n, x')$. Since x is unique asymptotic center of $\{x_n\}$, it follows that $x' \ne x$, which is contradiction. This completes the proof. □

Now we prove another strong convergence theorem using condition (I).

Theorem 4.5. *Let C be a nonempty closed convex subset of a $CAT(0)$ space X, and let $T : C \to C$ be a mapping satisfying condition (E). For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (4.1) for all $n \ge 1$, where $\{\alpha_n\}$ is a sequence of real numbers in $[a, b]$ for some a, b with $0 < a \le b < 1$ such that $F(T) \ne \emptyset$. If T satisfy condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Theorem 4.1, we have that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. So, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Assume that $\lim_{n\to\infty} d(x_n, p) = r$ for some $r \ge 0$. If $r = 0$, then the result follows. Supposing $r > 0$, from the hypothesis and condition (I), we have $f(d(x_n, F(T))) \le d(Tx_n, x_n)$. Since $F(T) \ne \emptyset$, so by Theorem 4.2, we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. So,

$$\lim_{n\to\infty} f(d(x_n, F(T))) = 0, \tag{4.10}$$

which implies $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus, we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{y_k\} \subset F(T)$ such that

$$d(x_{n_k}, y_k) < \frac{1}{2^k} \text{ for all } k \in \mathbb{N}.$$

It follows that

$$d(x_{n_{k+1}}, y_k) \le d(x_{n_k}, y_k) < \frac{1}{2^k}.$$

Hence

$$\begin{aligned} d(y_{k+1},y_k) &\leq d(y_{k+1},x_{k+1})+d(x_{k+1},y_k) \\ &\leq \frac{1}{2^{k+1}}+\frac{1}{2^k} \\ &< \frac{1}{2^{k-1}} \to 0, \text{ as } k\to\infty. \end{aligned}$$

This shows that $\{y_k\}$ is a Cauchy sequence in $F(T)$ and so it converges to a point p. Since $F(T)$ is closed, therefore $p\in F(T)$ and then $\{x_{n_k}\}$ converges strongly to p. Since $\lim_{n\to\infty} d(x_n,p)$ exists, we have that $x_n\to p\in F(T)$. □

5. Solutions of functional equations

Throughout this section, we have the space $C([a,b])$, of all continuous real-valued functions on a closed interval $[a,b]$, equipped with Chebyshev norm $\|x-y\|_\infty = \max_{t\in[a,b]}|x(t)-y(t)|$. Clearly $(C([a,b]),\|\cdot\|_\infty)$ is a Banach space, see [28]. Now, we consider a delay differential equation such that

$$x'(t)=f(t,x(t),x(t-\tau)),\ t\in[t_0,b], \tag{5.1}$$

with initial condition

$$x(t)=\rho(t),\ t\in[t_0-\tau,t_0]. \tag{5.2}$$

The following are some conditions:
(i). $t_0,b\in\mathbb{R}$, $\tau>0$;
(ii). $f\in C\left([t_0,b]\times\mathbb{R}^2,\mathbb{R}\right)$;
(iii). $\rho\in C([t_0-\tau,b],\mathbb{R})$;
(iv). there exists $L_f>0$ such that

$$|f(t,u_1,u_2)-f(t,v_1,v_2)|\leq L_f\left(\sum_{i=1}^{2}|u_i-v_i|\right),\ \forall u_i,v_i\in\mathbb{R}, i=1,2, t\in[t_0,b]; \tag{5.3}$$

(v). $2L_f(b-t_0)<1$.
By a solution of problem (5.1)-(5.2), we mean the function $p\in C([t_0-\tau,b],\mathbb{R})\cap C^1([t_0,b],\mathbb{R})$.
The problem (5.1)-(5.2) can be reconstituted in the following form of integral equation:

$$x(t)=\begin{cases}\rho(t), & t\in[t_0-\tau,t_0],\\ \rho(t_0)+\int_{t_0}^{t} f(s,x(s),x(s-\tau))ds, & t\in[t_0,b].\end{cases} \tag{5.4}$$

Now we are in a position to give the following result.

Theorem 5.1. *Let conditions (i)-(v) be satisfied. Then problem (5.1)-(5.2) has a unique solution, say, p in $C([t_0-\tau,b],\mathbb{R})\cap C^1([t_0,b],\mathbb{R})$ and (3.1) with real sequence $\{\alpha_n\}_{n=0}^{\infty}$ in $[0,1]$ satisfying $\sum_{n=0}^{\infty}\alpha_n=\infty$, converges to p.*

Proof. Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by M I.P (3.1) for a map $f : C([t_0 - \tau, b], \mathbb{R}) \to C([t_0 - \tau, b], \mathbb{R})$, defined by:

$$f(x(t)) = \begin{cases} \rho(t), & t \in [t_0 - \tau, t_0], \\ \rho(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau))ds, & t \in [t_0, b]. \end{cases}$$

Denote the fixed point of f by p. We will show that $x_n \to p$ as $n \to \infty$. For $t \in [t_0 - \tau, t_0]$, it is easy to show that $x_n \to p$ when $n \to \infty$.

Next we prove, for $t \in [t_0, b]$,

$$\begin{aligned}
\|z_n - p\|_\infty &\le (1-\alpha_n)\|x_n - p\|_\infty + \alpha_n \|fx_n - fp\|_\infty \\
&= (1-\alpha_n)\|x_n - p\|_\infty + \alpha_n \max_{t\in[t_0-\tau,b]} |fx_n(t) - fp(t)| \\
&= (1-\alpha_n)\|x_n - p\|_\infty \\
&\quad + \alpha_n \max_{t\in[t_0-\tau,b]} \left| \begin{array}{c} \rho(t_0) + \int_{t_0}^{t} f(s, x_n(s), x_n(s-\tau))ds \\ -\rho(t_0) - \int_{t_0}^{t} f(s, p(s), p(s-\tau))ds \end{array} \right| \\
&= (1-\alpha_n)\|x_n - p\|_\infty \\
&\quad + \alpha_n \max_{t\in[t_0-\tau,b]} \left| \begin{array}{c} \int_{t_0}^{t} f(s, x_n(s), x_n(s-\tau)) \\ -f(s, p(s), p(s-\tau))ds \end{array} \right| \\
&\le (1-\alpha_n)\|x_n - p\|_\infty \\
&\quad + \alpha_n \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f \left(\begin{array}{c} |x_n(s) - p(s)| \\ + |x_n(s-\tau) - p(s-\tau)| \end{array} \right) ds \\
&\le (1-\alpha_n)\|x_n - p\|_\infty \\
&\quad + \alpha_n \int_{t_0}^{t} L_f \left(\begin{array}{c} \max_{s\in[t_0-\tau,b]} |x_n(s) - p(s)| \\ + \max_{s\in[t_0-\tau,b]} |x_n(s-\tau) - p(s-\tau)| \end{array} \right) ds \\
&= (1-\alpha_n)\|x_n - p\|_\infty + \alpha_n \int_{t_0}^{t} L_f \left(\|x_n - p\|_\infty + \|x_n - p\|_\infty\right) ds \\
&\le (1 - \alpha_n(1 - 2L_f(b - t_0)))\|x_n - p\|_\infty.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|y_n - p\|_\infty &= \|fz_n - fp\|_\infty \\
&= \max_{t\in[t_0-\tau,b]} \left| \int_{t_0}^{t} f(s, z_n(s), z_n(s-\tau)) - f(s, p(s), p(s-\tau))ds \right| \\
&\le \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f \left(|z_n(s) - p(s)| + |z_n(s-\tau) - p(s-\tau)|\right) ds \\
&\le 2L_f(b - t_0)\|z_n - p\|_\infty \\
&\le 2L_f(b - t_0)(1 - \alpha_n(1 - 2L_f(b - t_0)))\|x_n - p\|_\infty.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}\|x_{n+1}-p\|_\infty &= \|fy_n - fp\|_\infty \\ &= \max_{t\in[t_0-\tau,b]}\left|\int_{t_0}^{t} f(s,y_n(s),y_n(s-\tau)) - f(s,p(s),p(s-\tau))ds\right| \\ &\le \max_{t\in[t_0-\tau,b]}\int_{t_0}^{t} L_f\left(|y_n(s)-p(s)|+|y_n(s-\tau)-p(s-\tau)|\right)ds \\ &\le 2L_f(b-t_0)\|y_n-p\|_\infty \\ &\le \left(2L_f(b-t_0)\right)^2(1-\alpha_n(1-2L_f(b-t_0)))\|x_n-p\|_\infty .\end{aligned}$$

By using assumption (v), we get that

$$\|x_{n+1}-p\|_\infty \le (1-\alpha_n(1-2L_f(b-t_0)))\|x_n-p\|_\infty . \tag{5.5}$$

Inductively, we have

$$\|x_{n+1}-p\|_\infty \le \prod_{k=0}^{n}(1-\alpha_n(1-2L_f(b-t_0)))\|x_0-p\|_\infty . \tag{5.6}$$

By using assumption (v), we have $1-\alpha_n(1-2L_f(b-t_0))<1$. Also, we know that $1-x\le e^{-x}$ for every $x\in[0,1]$. It follows that

$$\|x_{n+1}-p\|_\infty \le \frac{\|x_0-p\|_\infty}{e^{(1-2L_f(b-t_0))\sum_{k=0}^{n}\alpha_k}}. \tag{5.7}$$

Taking the limit of both sides of the above inequality yields $\lim_{n\to\infty}\|x_n-p\|_\infty=0$, i.e., $x_n\to p$ for $n\to\infty$, i.e., M iterations converges to the solution of problem (5.1)-(5.2). □

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