

# Generating Sets of the Complete Semigroups of Binary Relations Defined by Semilattices of the Class $\Sigma_2(X, 4)$

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## Abstract

In this paper, we have studied generating sets of the complete semigroups defined by  $X$ -semilattices of the class  $\Sigma_2(X, 4)$ .

## Keywords

Semigroup, Semilattice, Binary Relations, Idempotent Elements

## 1. Introduction

Let  $X$  be an arbitrary nonempty set and  $D$  be a nonempty set of subsets of the set  $X$ . If  $D$  is closed under the union, then  $D$  is called a *complete  $X$ -semilattice of unions*. The union of all elements of the set  $D$  is denoted by the symbol  $\check{D}$ .

Let  $B_X$  be the set of all binary relations on  $X$ . It is well known that  $B_X$  is a semigroup.

Let  $f$  be an arbitrary mapping from  $X$  into  $D$ . Then we denote a binary relation  $\alpha_f = \bigcup_{x \in X} (\{x\} \times f(x))$  for each  $f$ . The set of all such binary relations is denoted by  $B_X(D)$ . It is easy to prove that  $B_X(D)$  is a semigroup with respect to the product operation of binary relations. This semigroup  $B_X(D)$  is called a complete semigroup of binary relations defined by an  $X$ -semilattice of unions  $D$ . This structure was comprehensively investigated in Diasamidze [1] and [2]. We assume that  $t, y \in X$ ,  $Y \subseteq X$ ,  $\alpha \in B_X$ ,  $T \in D$  and  $\emptyset \neq D' \subseteq D$ . Then we denote following sets

$$y\alpha = \{x \in X \mid y\alpha x\}, \quad Y\alpha = \bigcup_{y \in Y} y\alpha,$$

$$\begin{aligned}
 V(D, \alpha) &= \{Y\alpha \mid Y \in D\}, \quad X^* = \{Y \mid \emptyset \neq Y \subseteq X\} \\
 Y_T^\alpha &= \{y \in X \mid y\alpha = T\}, \quad V(X^*, \alpha) = \{Y\alpha \mid \emptyset \neq Y \subseteq X\} \\
 D_i &= \{Z' \in D \mid t \in Z'\}, \quad B_0 = \{\alpha \in B_X(D) \mid V(X^*, \alpha) = D\}
 \end{aligned}$$

Let  $D = \{\check{D}, Z_1, Z_2, \dots, Z_{m-1}\}$  be finite  $X$ -semilattice of unions and  $C(D) = \{P_0, P_1, P_2, \dots, P_{m-1}\}$  be the family of pairwise nonintersecting subsets of  $X$ . If  $\varphi = \begin{pmatrix} \check{D} & Z_1 & \dots & Z_{m-1} \\ P_0 & P_1 & \dots & P_{m-1} \end{pmatrix}$  is a mapping from  $D$  on  $C(D)$ , then the equalities  $\check{D} = P_0 \cup P_1 \cup P_2 \cup \dots \cup P_{m-1}$  and  $Z_i = P_0 \cup \bigcup_{T \in D \setminus D_Z} \varphi(T)$  are valid. These equalities are called formal.

Let  $D$  be a complete  $X$ -semilattice of unions  $\alpha \in B_X$ . Then a representation of a binary relation  $\alpha$  of the form  $\alpha = \bigcup_{T \in V(X^*, \alpha)} (Y_T^\alpha \times T)$  is called quasinormal.

Let  $P_0, P_1, P_2, \dots, P_{m-1}$  be parameters in the formal equalities,  $\beta \in B_X(D)$ ,  $\bar{\beta}_2$  be mapping from  $X \setminus \check{D}$  to  $D$ . Then  $\bar{\beta} = \bigcup_{i=0}^{m-1} \left( P_i \times \bigcup_{t \in P_i} t\beta \right) \cup \bigcup_{t' \in X \setminus \check{D}} (\{t'\} \times \bar{\beta}_2(t'))$  is called subquasinormal representation of  $\beta$ . It can be easily seen that the following statements are true.

- a)  $\bar{\beta} \in B_X(D)$ .
- b)  $\bigcup_{i=0}^{m-1} \left( P_i \times \bigcup_{t \in P_i} t\beta \right) \subseteq \beta$  and  $\beta = \bar{\beta}$  for some  $\bar{\beta}_2$ .
- c) Subquasinormal representation of  $\beta$  is quasinormal.
- d)  $\bar{\beta}_1 = \begin{pmatrix} P_0 & P_1 & \dots & P_{m-1} \\ P_0\bar{\beta} & P_1\bar{\beta} & \dots & P_{m-1}\bar{\beta} \end{pmatrix}$  is mapping from  $C(D)$  on  $D \cup \{\emptyset\}$ .

$\bar{\beta}_1$  and  $\bar{\beta}_2$  are respectively called normal and complement mappings for  $\beta$ .

Let  $\alpha \in B_X(D)$ . If  $\alpha \neq \delta \circ \beta$  for all  $\delta, \beta \in B_X(D) \setminus \{\alpha\}$  then  $\alpha$  is called external element. Every element of the set  $B_0 = \{\alpha \in B_X(D) \mid V(X^*, \alpha) = D\}$  is an external element of  $B_X(D)$ .

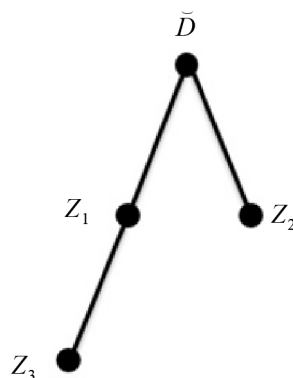
**Theorem 1. [1]** Let  $X$  be a finite set and  $\alpha, \beta \in B_X(D)$ . If  $\bar{\beta}$  is subquasinormal representation of  $\beta$  then  $\alpha \circ \beta = \alpha \circ \bar{\beta}$ .

**Corollary 1. [1]** Let  $\tilde{B}' \subseteq \tilde{B} \subseteq B_X(D)$ . If  $\alpha \neq \delta \circ \bar{\beta}$  for  $\alpha \in \tilde{B}'$ ,  $\delta \in \tilde{B} \setminus \{\alpha\}$ ,  $\bar{\beta} \in \tilde{B} \setminus \{\alpha\}$  and subquasinormal representation of  $\beta \in \tilde{B} \setminus \{\alpha\}$  then  $\alpha \neq \delta \circ \beta$ .

It is known that the set of all external elements is subset of any generating set of  $B_X(D)$  in [3].

## 2. Results

In this work by symbol  $\Sigma_{2,2}(X, 4)$  we denote all semilattices  $D = \{Z_3, Z_2, Z_1, \check{D}\}$  of the class  $\Sigma_2(X, 4)$  which the intersection of minimal elements  $Z_3 \cap Z_2 = \emptyset$ . This semilattices graphic is given in **Figure 1**. By using formal equalities, we have  $Z_3 \cap Z_2 = P_0 = \emptyset$ . So, the formal equalities of the semilattice  $D$  has a form



**Figure 1.** Graphic of semilattice  $D = \{Z_3, Z_2, Z_1, D\}$  which the intersection of minimal elements  $Z_3 \cap Z_2 = \emptyset$ .

$$\begin{aligned}
 \bar{D} &= P_1 \cup P_2 \cup P_3 \\
 Z_1 &= P_2 \cup P_3 \\
 Z_2 &= P_1 \cup P_3 \\
 Z_3 &= P_2
 \end{aligned} \tag{1}$$

Let  $\delta, \bar{\beta} \in B_X(D)$ . If quasinormal representation of binary relation  $\delta$  has a form  $\delta = (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \bar{D})$  then

$$\delta \circ \bar{\beta} = (Y_3^\delta \times Z_3 \bar{\beta}) \cup (Y_2^\delta \times Z_2 \bar{\beta}) \cup (Y_1^\delta \times Z_1 \bar{\beta}) \cup (Y_0^\delta \times \bar{D} \bar{\beta})$$

We denote the set

$$\begin{aligned}
 B_{32} &= \{ \alpha \in B_X(D) \mid V(X^*, \alpha) = \{Z_3, Z_2, \bar{D}\} \} \\
 B_{21} &= \{ \alpha \in B_X(D) \mid V(X^*, \alpha) = \{Z_2, Z_1, \bar{D}\} \} \\
 B_{31} &= \{ \alpha \in B_X(D) \mid V(X^*, \alpha) = \{Z_3, Z_1\} \} \\
 \tilde{B}_{32} &= \{ \alpha \in B_{32} \mid \alpha = (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2), Y_3^\alpha \cup Y_2^\alpha = X, Y_3^\alpha \cap Y_2^\alpha = \emptyset \} \\
 \tilde{B}_{21} &= \{ \alpha \in B_{21} \mid \alpha = (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1), Y_2^\alpha \cup Y_1^\alpha = X, Y_2^\alpha \cap Y_1^\alpha = \emptyset \}
 \end{aligned}$$

It is easy to see that

$$B_0 \cap B_{32} = B_0 \cap B_{21} = B_0 \cap B_{31} = B_{21} \cap B_{32} = B_{31} \cap B_{32} = B_{21} \cap B_{31} = \emptyset.$$

**Lemma 2.** Let  $D = \{Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_{2,2}(X, 4)$ . Then following statements are true for the sets  $B_0, B_{32}, \tilde{B}_{32}$ .

- a) If  $\alpha = (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \bar{D})$  for some  $Y_3^\alpha, Y_1^\alpha, Y_0^\alpha \notin \emptyset$ , then  $\alpha$  is product of some elements of the set  $B_0$ .
- b) If  $\beta_0 = (Z_3 \times Z_3) \cup ((X \setminus Z_3) \times Z_2)$ , then  $(B_0 \circ \beta_0) \cup \tilde{B}_{32} = B_{32}$ .
- c) If  $\sigma_1 = (Z_2 \times Z_2) \cup ((X \setminus Z_2) \times Z_1)$ , then  $(B_0 \circ \sigma_1) \cup \tilde{B}_{21} = B_{21}$ .
- d) If  $\sigma_1 = (Z_2 \times Z_2) \cup ((X \setminus Z_2) \times Z_1)$ , then  $B_{32} \circ \sigma_1 = B_{21}$ .
- e) If  $\sigma_0 = (Z_3 \times Z_3) \cup ((X \setminus Z_3) \times Z_1)$ , then  $B_{32} \circ \sigma_0 = B_{31}$ .
- f) Every element of the set  $B_{32}$  is product of elements of the set  $B_0 \cup \tilde{B}_{32}$ .
- g) Every element of the set  $B_{21}$  is product of elements of the set

$$B_0 \cup \tilde{B}_{32} \cup \{\sigma_1\}.$$

*Proof.* It will be enough to show only  $a$ ,  $b$  and  $g$ . The rest can be similarly seen.

a. Let  $\alpha = (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \tilde{D})$  for some  $Y_3^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\emptyset\}$ ,  $\delta, \bar{\beta} \in B_0$ . Then quasnormal representation of  $\delta$  has a form

$$\delta = (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \tilde{D})$$

where  $Y_3^\delta, Y_1^\delta, Y_0^\delta \notin \{\emptyset\}$ . We suppose that

$$\bar{\beta} = (P_2 \times Z_3) \cup (P_1 \times Z_2) \cup (P_3 \times Z_1) \cup \bigcup_{t' \in X \setminus \tilde{D}} (\{t'\} \times \bar{\beta}_2(t'))$$

where  $\bar{\beta}_1 = \begin{pmatrix} \emptyset & P_1 & P_2 & P_3 \\ \emptyset & Z_2 & Z_3 & Z_1 \end{pmatrix}$  is normal mapping for  $\bar{\beta}$  and  $\bar{\beta}_2$  is complement mapping of the set  $X \setminus \tilde{D}$  on the set  $\tilde{D}$ . So,  $\bar{\beta} \in B_0$  since  $V(X^*, \bar{\beta}) = D$ . From the equalities (2.1) and definition of  $\bar{\beta}$

$$\begin{aligned} Z_3 \bar{\beta} &= P_2 \bar{\beta} = Z_3 \\ Z_2 \bar{\beta} &= (P_1 \cup P_3) \bar{\beta} = P_1 \bar{\beta} \cup P_3 \bar{\beta} = Z_2 \cup Z_1 = \tilde{D} \\ Z_1 \bar{\beta} &= (P_2 \cup P_3) \bar{\beta} = P_2 \bar{\beta} \cup P_3 \bar{\beta} = Z_3 \cup Z_1 = Z_1 \\ \tilde{D} \bar{\beta} &= (P_1 \cup P_2 \cup P_3) \bar{\beta} = P_1 \bar{\beta} \cup P_2 \bar{\beta} \cup P_3 \bar{\beta} = Z_2 \cup \tilde{D} \cup Z_1 = \tilde{D} \\ \delta \circ \bar{\beta} &= (Y_3^\delta \times Z_3 \bar{\beta}) \cup (Y_2^\delta \times Z_2 \bar{\beta}) \cup (Y_1^\delta \times Z_1 \bar{\beta}) \cup (Y_0^\delta \times \tilde{D} \bar{\beta}) \\ &= (Y_3^\delta \times \tilde{D}) \cup (Y_2^\delta \times \tilde{D}) \cup (Y_1^\delta \times \tilde{D}) \cup (Y_0^\delta \times \tilde{D}) \\ &= (Y_3^\delta \times \tilde{D}) \cup (Y_1^\delta \times \tilde{D}) \cup ((Y_2^\delta \cup Y_0^\delta) \times \tilde{D}) = \alpha. \end{aligned}$$

b. Let  $\alpha \in B_0 \circ \beta_0 \cup \tilde{B}_{32}$ . Then  $\alpha \in B_0 \circ \beta_0$  or  $\alpha \in \tilde{B}_{32}$ . If  $\alpha \in B_0 \circ \beta_0$  then  $\alpha = \delta \circ \beta_0$  for some  $\delta \in B_0$ . In this case we have

$$\delta = (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \tilde{D})$$

where  $Y_3^\delta, Y_1^\delta, Y_0^\delta \notin \{\emptyset\}$ . Also

$$\begin{aligned} \alpha = \delta \circ \beta_0 &= (Y_3^\delta \times Z_3 \beta_0) \cup (Y_2^\delta \times Z_2 \beta_0) \cup (Y_1^\delta \times Z_1 \beta_0) \cup (Y_0^\delta \times \tilde{D} \beta_0) \\ &= (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \tilde{D}) \\ &= (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2) \cup ((Y_1^\delta \cup Y_0^\delta) \times \tilde{D}) \in B_{32} \setminus \tilde{B}_{32} \end{aligned}$$

is satisfied. So, we have  $(B_0 \circ \beta_0) \cup \tilde{B}_{32} \subseteq B_{32}$ . On the other hand, if  $\alpha \in \tilde{B}_{32} \subseteq B_{32}$  then  $(B_0 \circ \beta_0) \cup \tilde{B}_{32} \subseteq B_{32}$  is satisfied. Conversely, if  $\alpha \in B_{32}$  then quasnormal representation of  $\alpha$  has a form

$$\alpha = (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_0^\alpha \times \tilde{D})$$

where  $Y_3^\alpha, Y_2^\alpha, Y_0^\alpha \notin \{\emptyset\}$  or  $Y_3^\alpha, Y_2^\alpha \notin \{\emptyset\}$  and  $Y_0^\alpha = \emptyset$ . We suppose that  $Y_3^\alpha, Y_2^\alpha \notin \{\emptyset\}$ . In this case, we have

$$\begin{aligned} \delta \circ \beta_0 &= (Y_3^\delta \times Z_3 \beta_0) \cup (Y_2^\delta \times Z_2 \beta_0) \cup (Y_0^\delta \times Z_1 \beta_0) \\ &= (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2) \cup (Y_0^\delta \times \tilde{D}) = \alpha \end{aligned}$$

for  $\delta = (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_0^\alpha \times Z_1) \in B_0$ . So, we have  $B_{32} \subseteq (B_0 \circ \beta_0) \cup \tilde{B}_{32}$ . Now suppose that  $Y_3^\alpha, Y_2^\alpha \notin \{\emptyset\}$  and  $Y_0^\alpha = \emptyset$ . In this case, we have

$\alpha \in \tilde{B}_{32} \subseteq (B_0 \circ \beta_0) \cup \tilde{B}_{32}$ . So,  $(B_0 \circ \beta_0) \cup \tilde{B}_{32} = B_{32}$ .

g. From the statement c, we have that  $(B_0 \circ \beta_0) \cup \tilde{B}_{32} = B_{32}$  where  $\beta_0 \in \tilde{B}_{32}$  by definition of  $\beta_0$ . Thus, every element of the set  $B_{32}$  is product of elements of the set  $B_0 \cup \tilde{B}_{32}$ .

**Lemma 3.** Let  $D = \{Z_3, Z_2, Z_1, \tilde{D}\} \in \Sigma_{2,2}(X, 4)$ . If  $|X \setminus \tilde{D}| \geq 1$  then the following statements are true.

- a) If  $\alpha = X \times \tilde{D}$  then  $\alpha$  is product of elements of the set  $B_0$ .
- b) If  $\alpha = X \times Z_1$  then  $\alpha$  is product of elements of the set  $B_0$ .
- c) If  $\alpha = (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1)$  for some  $Y_3^\alpha, Y_1^\alpha \neq \emptyset$ , then  $\alpha$  is product of elements of the  $B_0$ .
- d) If  $\alpha = (Y_3^\alpha \times Z_3) \cup (Y_0^\alpha \times \tilde{D})$  for some  $Y_3^\alpha, Y_0^\alpha \neq \emptyset$ , then  $\alpha$  is product of elements of the  $B_0$ .
- e) If  $\alpha = (Y_2^\alpha \times Z_2) \cup (Y_0^\alpha \times \tilde{D})$  for some  $Y_2^\alpha, Y_0^\alpha \neq \emptyset$ , then  $\alpha$  is product of elements of the  $B_0$ .
- f) If  $\alpha = (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \tilde{D})$  for some  $Y_1^\alpha, Y_0^\alpha \neq \emptyset$ , then  $\alpha$  is product of elements of the  $B_0$ .

*Proof.* c. Let quasinormal representation of  $\alpha$  has a form  $\alpha = (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1)$  where  $Y_3^\alpha, Y_1^\alpha \neq \{\emptyset\}$ . By definition of the semilattice  $D$ ,  $|X| \geq 3$ . We suppose that  $|Y_3^\alpha| \geq 1$  and  $|Y_1^\alpha| \geq 2$ . In this case, we suppose that

$$\bar{\beta} = (P_2 \times Z_3) \cup ((P_1 \cup P_3) \times Z_1) \cup \bigcup_{t' \in X \setminus \tilde{D}} (\{t'\} \times \bar{\beta}_2(t'))$$

where  $\bar{\beta}_1 = \begin{pmatrix} \emptyset & P_1 & P_2 & P_3 \\ \emptyset & Z_1 & Z_3 & Z_1 \end{pmatrix}$  is normal mapping for  $\bar{\beta}$  and  $\bar{\beta}_2$  is comple-

ment mapping of the set  $X \times \tilde{D}$  on the set  $\tilde{D} \setminus \{Z_3, Z_1\} = \{Z_2\}$  (by suppose  $|X \setminus \tilde{D}| \geq 1$ ). So,  $\bar{\beta} \in B_0$  since  $V(X^*, \bar{\beta}) = D$ . Also,  $Y_3^\delta = Y_3^\alpha$  and  $Y_2^\delta \cup Y_1^\delta \cup Y_0^\delta = Y_1^\delta$  since  $|Y_3^\delta| \geq 1, |Y_2^\delta| \geq 1, |Y_1^\delta| \geq 1, |Y_0^\delta| \geq 0$ . From the equalities (2.1) and definition of  $\bar{\beta}$  we obtain that

$$\begin{aligned} Z_3 \bar{\beta} &= P_2 \bar{\beta} = Z_3 \\ Z_2 \bar{\beta} &= (P_1 \cup P_3) \bar{\beta} = P_1 \bar{\beta} \cup P_3 \bar{\beta} = Z_1 \cup Z_1 = Z_1 \\ Z_1 \bar{\beta} &= (P_2 \cup P_3) \bar{\beta} = P_2 \bar{\beta} \cup P_3 \bar{\beta} = Z_3 \cup Z_1 = Z_1 \\ \tilde{D} \bar{\beta} &= (P_1 \cup P_2 \cup P_3) \bar{\beta} = P_1 \bar{\beta} \cup P_2 \bar{\beta} \cup P_3 \bar{\beta} = Z_1 \cup Z_3 \cup Z_1 = Z_1 \\ \delta \circ \bar{\beta} &= (Y_3^\delta \times Z_3 \bar{\beta}) \cup (Y_2^\delta \times Z_2 \bar{\beta}) \cup (Y_1^\delta \times Z_1 \bar{\beta}) \cup (Y_0^\delta \times \tilde{D} \bar{\beta}) \\ &= (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_1) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times Z_1) \\ &= (Y_3^\delta \times Z_3) \cup ((Y_2^\delta \cup Y_1^\delta \cup Y_0^\delta) \times Z_1) = \alpha \end{aligned}$$

Now, we suppose that  $|Y_3^\alpha| \geq 2$  and  $|Y_1^\alpha| \geq 1$ . In this case, we suppose that

$$\bar{\beta} = ((P_2 \cup P_3) \times Z_3) \cup (P_1 \times Z_1) \cup \bigcup_{t' \in X \setminus \tilde{D}} (\{t'\} \times \bar{\beta}_2(t'))$$

where  $\bar{\beta}_1 = \begin{pmatrix} \emptyset & P_1 & P_2 & P_3 \\ \emptyset & Z_1 & Z_3 & Z_3 \end{pmatrix}$  is normal mapping for  $\bar{\beta}$  and  $\bar{\beta}_2$  is comple-

ment mapping of the set  $X \times \tilde{D}$  on the set  $\tilde{D} \setminus \{Z_3, Z_1\} = \{Z_2\}$  (by suppose

$|X \setminus \bar{D}| \geq 1$ ). So,  $\bar{\beta} \in B_0$  since  $V(X^*, \bar{\beta}) = D$ . Also,  $Y_3^\delta \cup Y_1^\delta = Y_3^\alpha$  and  $Y_2^\delta \cup Y_0^\delta = Y_1^\alpha$  since  $|Y_3^\delta| \geq 1, |Y_2^\delta| \geq 1, |Y_1^\delta| \geq 1, |Y_0^\delta| \geq 0$ . From the equalities (2.1) and definition of  $\bar{\beta}$  we obtain that

$$\begin{aligned} Z_3 \bar{\beta} &= P_2 \bar{\beta} = Z_3 \\ Z_2 \bar{\beta} &= (P_1 \cup P_3) \bar{\beta} = P_1 \bar{\beta} \cup P_3 \bar{\beta} = Z_1 \cup Z_3 = Z_1 \\ Z_1 \bar{\beta} &= (P_2 \cup P_3) \bar{\beta} = P_2 \bar{\beta} \cup P_3 \bar{\beta} = Z_3 \cup Z_3 = Z_3 \\ \bar{D} \bar{\beta} &= (P_1 \cup P_2 \cup P_3) \bar{\beta} = P_1 \bar{\beta} \cup P_2 \bar{\beta} \cup P_3 \bar{\beta} = Z_1 \cup Z_3 \cup Z_3 = Z_1 \\ \delta \circ \bar{\beta} &= (Y_3^\delta \times Z_3 \bar{\beta}) \cup (Y_2^\delta \times Z_2 \bar{\beta}) \cup (Y_1^\delta \times Z_1 \bar{\beta}) \cup (Y_0^\delta \times \bar{D} \bar{\beta}) \\ &= (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_1) \cup (Y_1^\delta \times Z_3) \cup (Y_0^\delta \times Z_1) \\ &= ((Y_3^\delta \cup Y_1^\delta) \times Z_3) \cup ((Y_2^\delta \cup Y_0^\delta) \times Z_1) = \alpha \end{aligned}$$

**Lemma 4.** Let  $D = \{Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_{2,2}(X, 4)$ ,  $\sigma_0 = (Z_3 \times Z_3) \cup ((X \setminus Z_3) \times Z_1)$  and  $\sigma_1 = (Z_2 \times Z_2) \cup ((X \setminus Z_2) \times Z_1)$ . If  $X = \bar{D}$  then the following statements are true

- a) If  $\alpha = (Y_3^\alpha \times Z_3) \cup (Y_0^\alpha \times \bar{D})$  for some  $Y_3^\alpha, Y_0^\alpha \notin \{\emptyset\}$ , then  $\alpha$  is product of elements of the  $B_0 \cup B_{32}$ .
- b) If  $\alpha = (Y_2^\alpha \times Z_2) \cup (Y_0^\alpha \times \bar{D})$  for some  $Y_2^\alpha, Y_0^\alpha \notin \{\emptyset\}$ , then  $\alpha$  is product of elements of the  $B_{32} \cup \{\sigma_1\}$ .
- c) If  $\alpha = (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \bar{D})$  for some  $Y_1^\alpha, Y_0^\alpha \notin \{\emptyset\}$ , then  $\alpha$  is product of elements of the  $B_{32} \cup \{\sigma_0, \sigma_1\}$ .

*Proof.* First, remark that  $Z_3 \sigma_0 = Z_3$ ,  $Z_2 \sigma_0 = \bar{D} \sigma_0 = Z_1$ ,  $Z_3 \sigma_1 = Z_1$ ,  $Z_2 \sigma_1 = Z_2$ ,  $\bar{D} \sigma_1 = \bar{D}$ .

a. Let  $\alpha = (Y_3^\alpha \times Z_3) \cup (Y_0^\alpha \times \bar{D})$  for some  $Y_3^\alpha, Y_0^\alpha \notin \emptyset$ . In this case, we suppose that

$$\delta = (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2) \cup (Y_0^\delta \times \bar{D})$$

and

$$\beta_1 = (Z_3 \times Z_3) \cup ((Z_1 \setminus Z_3) \times Z_1) \cup ((X \setminus Z_1) \times \bar{D})$$

where  $Y_3^\delta, Y_2^\delta \notin \{\emptyset\}$ . It is easy to see that  $\delta \in B_{32}$  and  $\beta_1$  is generating by elements of the  $B_0$  by statement b of Lemma 2. Also,  $Y_3^\delta = Y_3^\alpha$  and  $Y_2^\delta \cup Y_0^\delta = Y_0^\alpha$  since  $Z_3 \bar{\beta} = Z_3$ ,  $Z_2 \bar{\beta} = \bar{D} \bar{\beta} = \bar{D}$  and  $|Y_3^\delta| \geq 1, |Y_2^\delta| \geq 1, |Y_0^\delta| \geq 0$ . So,  $\alpha$  is product of elements of the  $B_0 \cup B_{32}$ .  $\square$

**Lemma 5.** Let

$$D = \{Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_{2,2}(X, 4)$$

and

$$\sigma_1 = (Z_2 \times Z_2) \cup ((X \setminus Z_2) \times Z_1).$$

If  $|X \setminus \bar{D}| \geq 1$  then  $S_1 = B_0 \cup \bar{B}_{32} \cup \{\sigma_1\}$  is an irreducible generating set for the semigroup  $B_X(D)$ .

*Proof.* First, we must prove that every element of  $B_X(D)$  is product of ele-

ments of  $S_1$ . Let  $\alpha \in B_X(D)$  and

$$\alpha = (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \bar{D})$$

where  $Y_3^\alpha \cup Y_2^\alpha \cup Y_1^\alpha \cup Y_0^\alpha = X$  and  $Y_3^\alpha \cap Y_2^\alpha = \emptyset$ , ( $0 \leq i \neq j \leq 3$ ). We suppose that  $|V(X^*, \alpha)| = 1$ . Then we have  $V(X^*, \alpha) \in \{\{Z_3\}, \{Z_2\}, \{Z_1\}, \{\bar{D}\}\}$ . If

$V(X^*, \alpha) \in \{\{Z_3\}, \{Z_2\}, \{Z_1\}\}$  then  $\alpha = X \times Z_3$  or  $\alpha = X \times Z_2$  or  $\alpha = X \times Z_1$ .

Quasinormal representations of  $\delta, \beta_1, \beta_2$  and  $\beta_3$  has form

$$\delta = (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \bar{D})$$

$$\beta_1 = (\bar{D} \times Z_3) \cup ((X \setminus \bar{D}) \times Z_2)$$

$$\beta_2 = (\bar{D} \times Z_2) \cup ((X \setminus \bar{D}) \times Z_1)$$

$$\beta_3 = (\bar{D} \times Z_1) \cup ((X \setminus \bar{D}) \times Z_2)$$

where  $Y_3^\delta, Y_2^\delta, Y_1^\delta \notin \{\emptyset\}$ . So,  $\delta \in B_0$ ,  $\beta_1 \in \tilde{B}_{32}$  and  $\beta_2, \beta_3 \in B_{21}$  since  $|X \setminus \bar{D}| \geq 1$ . From the definition of  $\delta, \beta_1, \beta_2$  and  $\beta_3$  we obtain that

$$\begin{aligned} \delta \circ \beta_1 &= (Y_3^\delta \times Z_3 \beta_1) \cup (Y_2^\delta \times Z_2 \beta_1) \cup (Y_1^\delta \times Z_1 \beta_1) \cup (Y_0^\delta \times \bar{D} \beta_1) \\ &= (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_3) \cup (Y_1^\delta \times Z_3) \cup (Y_0^\delta \times Z_3) \\ &= (Y_3^\delta \cup Y_2^\delta \cup Y_1^\delta \cup Y_0^\delta) \times Z_3 = X \times Z_3 \end{aligned}$$

$$\begin{aligned} \delta \circ \beta_2 &= (Y_3^\delta \times Z_3 \beta_2) \cup (Y_2^\delta \times Z_2 \beta_2) \cup (Y_1^\delta \times Z_1 \beta_2) \cup (Y_0^\delta \times \bar{D} \beta_2) \\ &= (Y_3^\delta \times Z_2) \cup (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_2) \cup (Y_0^\delta \times Z_2) \\ &= (Y_3^\delta \cup Y_2^\delta \cup Y_1^\delta \cup Y_0^\delta) \times Z_2 = X \times Z_2 \end{aligned}$$

$$\begin{aligned} \delta \circ \beta_3 &= (Y_3^\delta \times Z_3 \beta_3) \cup (Y_2^\delta \times Z_2 \beta_3) \cup (Y_1^\delta \times Z_1 \beta_3) \cup (Y_0^\delta \times \bar{D} \beta_3) \\ &= (Y_3^\delta \times Z_1) \cup (Y_2^\delta \times Z_1) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times Z_1) \\ &= (Y_3^\delta \cup Y_2^\delta \cup Y_1^\delta \cup Y_0^\delta) \times Z_1 = X \times Z_1 \end{aligned}$$

That means,  $X \times Z_1, X \times Z_2$  and  $X \times Z_3$  are generated by  $B_0 \cup \tilde{B}_{32}$ ,  $B_0 \cup B_{21}$  and  $B_0 \cup B_{21}$ , respectively. By using statement g and h of Lemma 3, we have  $X \times Z_1, X \times Z_2$  and  $X \times Z_3$  are generated by  $B_0 \cup \tilde{B}_{32} \cup \{\sigma_1\}$ . On the other hand, if  $V(X^*, \alpha) = \{\bar{D}\}$  then  $\alpha = X \times \bar{D}$ . By using statement a of Lemma 3, we have  $\alpha$  is product of some elements of  $B_0$ .

So,  $S_1$  is generating set for the semigroup  $B_X(D)$ . Now, we must prove that  $S_1 = B_0 \cup \tilde{B}_{32} \cup \{\sigma_1\}$  is irreducible. Let  $\alpha \in S_1$ .

If  $\alpha \in B_0$  then  $\alpha \neq \sigma \circ \tau$  for all  $\sigma, \tau \in B_X(D) \setminus \{\alpha\}$  from Lemma 2. So,  $\alpha \neq \sigma \circ \tau$  for all  $\sigma, \tau \in S_1 \setminus \{\alpha\}$ . That means,  $\alpha \notin B_0$ .

If  $\alpha \in \tilde{B}_{32}$  then the quasinormal representation of  $\alpha$  has form  $\alpha = (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2)$  for some  $Y_3^\alpha, Y_2^\alpha \notin \emptyset$ . Let  $\alpha = \delta \circ \beta$  for some  $\delta, \beta \in S_1 \setminus \{\alpha\}$ .

We suppose that  $\delta \in B_0 \setminus \{\alpha\}$  and  $\beta \in S_1 \setminus \{\alpha\}$ . By definition of  $B_0$ , quasinormal representation of  $\delta$  has form

$$\delta = (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \bar{D})$$

where  $Y_3^\delta, Y_2^\delta, Y_1^\delta \notin \{\emptyset\}$ . By using  $Z_3 \subset Z_1 \subset \tilde{D}$  and  $Z_2 \subset \tilde{D}$  we have  $Z_3\beta$  and  $Z_2\beta$  are minimal elements of the semilattice  $\{Z_3\beta, Z_2\beta, Z_1\beta, \tilde{D}\beta\}$ . Also, we have

$$\begin{aligned} (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) &= \alpha = \delta \circ \beta \\ &= (Y_3^\delta \times Z_3\beta) \cup (Y_2^\delta \times Z_2\beta) \cup (Y_1^\delta \times Z_1\beta) \cup (Y_0^\delta \times \tilde{D}\beta) \end{aligned}$$

Since  $Z_3$  and  $Z_2$  are minimal elements of the semilattice  $\{Z_3, Z_2, \tilde{D}\}$ , this equality is possible only if  $Z_3 = Z_3\beta$ ,  $Z_2 = Z_2\beta$  or  $Z_3 = Z_2\beta$ ,  $Z_2 = Z_3\beta$ . By using formal equalities and  $P_3\beta, P_2\beta, P_1\beta \in D$ , we obtain

$$\begin{aligned} Z_3 &= Z_3\beta = P_2\beta \quad \text{and} \quad Z_2 = Z_2\beta = P_1\beta = P_3\beta \\ Z_2 &= Z_3\beta = P_2\beta \quad \text{and} \quad Z_3 = Z_2\beta = P_1\beta = P_3\beta \end{aligned}$$

respectively. Let  $Z_3 = P_2\beta$  and  $Z_2 = P_1\beta = P_3\beta$ . If  $\bar{\beta}$  is sub-quasinormal representation of  $\beta$  then  $\delta \circ \beta = \delta \circ \bar{\beta}$  and

$$\bar{\beta} = ((P_1 \cup P_3) \times Z_2) \cup (P_2 \times Z_3) \cup \bigcup_{t' \in X \setminus \tilde{D}} (\{t'\} \times \bar{\beta}_2(t'))$$

where  $\bar{\beta}_1 = \begin{pmatrix} \emptyset & P_1 & P_2 & P_3 \\ \emptyset & Z_2 & Z_3 & Z_2 \end{pmatrix}$  is normal mapping for  $\bar{\beta}$  and  $\bar{\beta}_2$  is complement mapping of the set  $X \times \tilde{D}$  on the set  $\tilde{D} = \{Z_3, Z_2, Z_1\}$ . From formal equalities, we obtain

$$\bar{\beta} = (Z_2 \times Z_2) \cup (Z_3 \times Z_3) \cup \bigcup_{t' \in X \setminus \tilde{D}} (\{t'\} \times \bar{\beta}_2(t')) \in S_1 \setminus \{\alpha\}$$

and by using  $Z_1 \cap Z_2 \neq \emptyset, Z_3 \cup Z_2 = D$  and  $|Y_1^\delta \cup Y_0^\delta| \geq 1$ , we have

$$\begin{aligned} \delta \circ \bar{\beta} &= (Y_3^\delta \times Z_3\bar{\beta}) \cup (Y_2^\delta \times Z_2\bar{\beta}) \cup (Y_1^\delta \times Z_1\bar{\beta}) \cup (Y_0^\delta \times \tilde{D}\bar{\beta}) \\ &= (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times \tilde{D}) \cup (Y_0^\delta \times \tilde{D}) \\ &= (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2) \cup ((Y_1^\delta \cup Y_0^\delta) \times \tilde{D}) \neq \alpha \end{aligned}$$

This contradicts with  $\alpha = \delta \circ \beta$ . So,  $\delta \notin B_0 \setminus \{\alpha\}$ .

Now, we suppose that  $\delta \in \tilde{B}_{32} \setminus \{\alpha\}$  and  $\beta \in S_1 \setminus \{\alpha\}$ . Similar operations are applied as above, we obtain  $\delta \notin \tilde{B}_{32} \setminus \{\alpha\}$ .

Now, we suppose that  $\delta = \sigma_1$  and  $\beta \in S_1 \setminus \{\alpha\}$ . Similar operations are applied as above, we obtain  $\delta \neq \sigma_1$ .

That means  $\alpha \neq \delta \circ \beta$  for any  $\alpha \in \tilde{B}_{32}$  and  $\delta, \beta \in S_1 \setminus \{\alpha\}$ .

If  $\alpha = \sigma_1$ , then by the definition of  $\sigma_1$ , quasinormal representation of  $\alpha$  has a form  $\alpha = (Z_2 \times Z_2) \cup ((X \setminus Z_2) \times Z_1)$ . Let  $\alpha = \delta \circ \beta$  for some  $\delta, \beta \in S_1 \setminus \{\sigma_1\}$ .

We suppose that  $\delta \in B_0 \setminus \{\sigma_1\}$  and  $\beta \in S_1 \setminus \{\sigma_1\}$ . By definition of  $B_0$ , quasinormal representation of  $\delta$  has form

$$\delta = (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \tilde{D})$$

where  $Y_3^\delta, Y_2^\delta, Y_1^\delta \notin \{\emptyset\}$ . By using  $Z_3 \subset Z_1 \subset \tilde{D}$  and  $Z_2 \subset \tilde{D}$  we have  $Z_3\beta$  and  $Z_2\beta$  are minimal elements of the semilattice  $\{Z_3\beta, Z_2\beta, Z_1\beta, \tilde{D}\beta\}$ . Also,



we have

$$\begin{aligned} (Z_2 \times Z_2) \cup ((X \setminus Z_2) \times Z_1) &= \alpha = \delta \circ \beta \\ &= (Y_3^\delta \times Z_3\beta) \cup (Y_2^\delta \times Z_2\beta) \cup (Y_1^\delta \times Z_1\beta) \cup (Y_0^\delta \times \bar{D}\beta) \end{aligned}$$

From  $Z_2$  and  $Z_1$  are minimal elements of the semilattice  $\{Z_2, Z_1, \bar{D}\}$ , this equality is possible only if  $Z_2 = Z_3\beta$ ,  $Z_1 = Z_2\beta$  or  $Z_2 = Z_2\beta$ ,  $Z_1 = Z_3\beta$ . By using formal equalities, we obtain

$$\begin{aligned} Z_2 = Z_3\beta = P_2\beta \quad \text{and} \quad Z_1 = Z_2\beta = P_1\beta \cup P_3\beta \\ Z_1 = Z_3\beta = P_2\beta \quad \text{and} \quad Z_2 = Z_2\beta = P_1\beta = P_3\beta \end{aligned}$$

respectively. Let  $Z_2 = P_2\beta$  and  $Z_1 = P_1\beta \cup P_3\beta$  where  $P_1\beta, P_3\beta \in \{Z_3, Z_1\}$ . Then subquasinormal representation of  $\beta$  has one of the form

$$\begin{aligned} \bar{\beta}^1 &= (P_1 \times Z_3) \cup (P_2 \times Z_2) \cup (P_3 \times Z_1) \cup \bigcup_{t' \in X \setminus \bar{D}} (\{t'\} \times \bar{\beta}_2(t')) \\ \bar{\beta}^2 &= (P_3 \times Z_3) \cup (P_2 \times Z_2) \cup (P_1 \times Z_1) \cup \bigcup_{t' \in X \setminus \bar{D}} (\{t'\} \times \bar{\beta}_2(t')) \\ \bar{\beta}^3 &= (P_2 \times Z_2) \cup ((P_1 \cup P_3) \times Z_1) \cup \bigcup_{t' \in X \setminus \bar{D}} (\{t'\} \times \bar{\beta}_2(t')) \end{aligned}$$

where

$$\bar{\beta}_1 = \begin{pmatrix} \emptyset & P_1 & P_2 & P_3 \\ \emptyset & Z_3 & Z_2 & Z_1 \end{pmatrix}, \quad \bar{\beta}_2 = \begin{pmatrix} \emptyset & P_1 & P_2 & P_3 \\ \emptyset & Z_1 & Z_2 & Z_3 \end{pmatrix}, \quad \bar{\beta}_3 = \begin{pmatrix} \emptyset & P_1 & P_2 & P_3 \\ \emptyset & Z_1 & Z_2 & Z_1 \end{pmatrix}$$

are normal mapping for  $\bar{\beta}$ ,  $\bar{\beta}_2$  is complement mapping of the set  $X \times \bar{D}$  on the set  $\bar{D} = \{Z_3, Z_2, Z_1\}$  and  $\delta \circ \beta = \delta \circ \bar{\beta}_1$ . From formal equalities, we obtain

$$\begin{aligned} \bar{\beta}^1 &= ((Z_2 \setminus Z_1) \times Z_3) \cup ((Z_1 \setminus Z_2) \times Z_2) \cup ((Z_2 \setminus Z_1) \times Z_1) \cup \bigcup_{t' \in X \setminus \bar{D}} (\{t'\} \times \bar{\beta}_2(t')) \\ \bar{\beta}^2 &= ((Z_2 \cap Z_1) \times Z_3) \cup ((Z_1 \setminus Z_2) \times Z_2) \cup ((Z_2 \setminus Z_1) \times Z_1) \cup \bigcup_{t' \in X \setminus \bar{D}} (\{t'\} \times \bar{\beta}_2(t')) \\ \bar{\beta}^3 &= ((Z_1 \setminus Z_2) \times Z_2) \cup (Z_2 \times Z_1) \cup \bigcup_{t' \in X \setminus \bar{D}} (\{t'\} \times \bar{\beta}_2(t')) \end{aligned}$$

and by using  $|Y_1^\delta \cup Y_0^\delta| \geq 1$ , we have

$$\begin{aligned} \delta \circ \bar{\beta}^1 &= \delta \circ \bar{\beta}^2 = \delta \circ \bar{\beta}^3 \\ &= (Y_3^\delta \times Z_3\bar{\beta}^1) \cup (Y_2^\delta \times Z_2\bar{\beta}^1) \cup (Y_1^\delta \times Z_1\bar{\beta}^1) \cup (Y_0^\delta \times \bar{D}\bar{\beta}^1) \\ &= (Y_3^\delta \times Z_2) \cup (Y_2^\delta \times Z_1) \cup (Y_1^\delta \times \bar{D}) \cup (Y_0^\delta \times \bar{D}) \\ &= (Y_3^\delta \times Z_2) \cup (Y_2^\delta \times Z_1) \cup ((Y_1^\delta \cup Y_0^\delta) \times \bar{D}) \neq \alpha \end{aligned}$$

This contradicts with  $\alpha = \delta \circ \beta$ . So,  $\delta \notin B_0 \setminus \{\sigma_1\}$ .

Now, we suppose that  $\delta \in \tilde{B}_{32} \setminus \{\sigma_1\}$  and  $\beta \in S_1 \setminus \{\sigma_1\}$ . Similar operations are applied as above, we obtain  $\delta \notin \tilde{B}_{32} \setminus \{\sigma_1\}$ .

That means  $\alpha \neq \delta \circ \beta$  for any  $\alpha \in \tilde{B}_{32}$  and  $\delta, \beta \in S_1 \setminus \{\alpha\}$ . □

**Lemma 6.** Let  $D = \{Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_{2,2}(X, 4)$ ,  $\sigma_0 = (Z_3 \times Z_3) \cup ((X \setminus Z_3) \times Z_1)$  and  $\sigma_1 = (Z_2 \times Z_2) \cup ((X \setminus Z_2) \times Z_1)$ . If  $X = \bar{D}$  then  $S_2 = B_0 \cup \tilde{B}_{32} \cup \{\sigma_0, \sigma_1\}$  is irreducible generating set for the semigroup  $B_X(D)$ .

**Theorem 7.** Let  $D = \{Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_{2,2}(X, 4)$ ,

$\sigma_0 = (Z_3 \times Z_3) \cup ((X \setminus Z_3) \times Z_1)$  and  $\sigma_1 = (Z_2 \times Z_2) \cup ((X \setminus Z_2) \times Z_1)$ . If  $X$  is a finite set and  $|X| = n$  then the following statements are true

a) If  $|X \setminus \bar{D}| \geq 1$  then  $|B_0 \cup \tilde{B}_{32} \cup \{\sigma_1\}| = 4^n - 3^{n+1} + 2^{n+2} - 2$

b) If  $X = \bar{D}$  then  $|B_0 \cup \tilde{B}_{32} \cup \{\sigma_0, \sigma_1\}| = 4^n - 3^{n+1} + 2^{n+2} - 1$

*Proof.* Let

$$S_n = \{\varphi_i \mid \varphi_i : M = \{1, 2, \dots, n\} \rightarrow M = \{1, 2, \dots, n\}, \text{ one to one mapping}\}$$

be a group,  $\varphi_{i_1}, \varphi_{i_2}, \dots, \varphi_{i_m} \in S_n$  ( $m \leq n$ ) and  $Y_{\varphi_1}, Y_{\varphi_2}, \dots, Y_{\varphi_m}$  be partitioning of

$X$ . It is well known that  $k_n^m = |\{Y_{\varphi_1}, Y_{\varphi_2}, \dots, Y_{\varphi_m}\}| = \sum_{i=1}^m \frac{(-1)^{m+i}}{(i-1)!(m-i)!}$ . If  $m = 2, 3, 4$

then we have

$$k_n^2 = 2^{n-1} - 1$$

$$k_n^3 = \frac{1}{2} \cdot 3^{n-1} - 2^{n-1} + \frac{1}{2}$$

$$k_n^4 = \frac{1}{6} \cdot 4^{n-1} - \frac{1}{2} \cdot 3^{n-1} + \frac{1}{2} \cdot 2^{n-1} - \frac{1}{6}$$

If  $Y_{\varphi_1}, Y_{\varphi_2}$  are any two elements of partitioning of  $X$  and

$\bar{\beta} = (Y_{\varphi_1} \times T_1) \cup (Y_{\varphi_2} \times T_2)$  where  $T_1, T_2 \in D$  and  $T_1 \neq T_2$ , then the number of different binary relations  $\bar{\beta}$  of semigroup  $B_X(D)$  is equal to

$$2 \cdot k_n^2 = 2^n - 2 \tag{2}$$

If  $Y_{\varphi_1}, Y_{\varphi_2}, Y_{\varphi_3}$  are any three elements of partitioning of  $X$  and

$\bar{\beta} = (Y_{\varphi_1} \times T_1) \cup (Y_{\varphi_2} \times T_2) \cup (Y_{\varphi_3} \times T_3)$  where  $T_1, T_2, T_3$  are pairwise different elements of  $D$ , then the number of different binary relations  $\bar{\beta}$  of semigroup  $B_X(D)$  is equal to

$$6 \cdot k_n^3 = 3^n - 3 \cdot 2^n + 3 \tag{3}$$

If  $Y_{\varphi_1}, Y_{\varphi_2}, Y_{\varphi_3}, Y_{\varphi_4}$  are any four elements of partitioning of  $X$  and

$\bar{\beta} = (Y_{\varphi_1} \times T_1) \cup (Y_{\varphi_2} \times T_2) \cup (Y_{\varphi_3} \times T_3) \cup (Y_{\varphi_4} \times T_4)$  where  $T_1, T_2, T_3, T_4$  are pairwise different elements of  $D$ , then the number of different binary relations  $\bar{\beta}$  of semigroup  $B_X(D)$  is equal to

$$24 \cdot k_n^4 = 4^n - 4 \cdot 3^n + 3 \cdot 2^n - 4 \tag{4}$$

Let  $\alpha \in B_0$ . Quasinormal representation of  $\alpha$  has form

$$\alpha = (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \bar{D})$$

where  $Y_3^\alpha, Y_2^\alpha, Y_1^\alpha \notin \{\emptyset\}$ . Also,  $Y_3^\alpha, Y_2^\alpha, Y_1^\alpha$  or  $Y_3^\alpha, Y_2^\alpha, Y_1^\alpha, Y_0^\alpha$  are partitioning of  $X$  for  $|X| \geq 4$ . By using Equations (2.3) and (2.4) we obtain

$$|B_0| = 4^n - 3^{n+1} + 3 \cdot 2^n - 1$$

Let  $\alpha \in \tilde{B}_{32}$ . Quasinormal representation of  $\alpha$  has form

$\alpha = (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2)$  where  $Y_3^\alpha, Y_2^\alpha \notin \{\emptyset\}$ . Also,  $Y_3^\alpha, Y_2^\alpha$  are partitioning of  $X$ . By using (2.2) we obtain

$$|\tilde{B}_{32}| = 2^n - 2$$

So, we have

$$|B_0 \cup \tilde{B}_{32} \cup \{\sigma_1\}| = 4^n - 3^{n+1} + 2^{n+2} - 2$$

$$|B_0 \cup \tilde{B}_{32} \cup \{\sigma_0, \sigma_1\}| = 4^n - 3^{n+1} + 2^{n+2} - 1$$

since  $B_0 \cap \tilde{B}_{32} = B_0 \cap \{\sigma_0, \sigma_1\} = \tilde{B}_{32} \cap \{\sigma_0, \sigma_1\} = \emptyset$ . □

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### References

- [1] Diasamidze, Y. and Makharadze, S. (2013) Complete Semigroups of Binary Relations. Kriter Yay Nevi, Istanbul.
- [2] Diasamidze, Y., Ayd, N.N. and Erdoğan, A. (2016) Generating Set of the Complete Semigroups of Binary Relations. *Applied Mathematics*, **7**, 98-107. <https://doi.org/10.4236/am.2016.71009>
- [3] Bakuridze, A., Diasamidze, Y. and Givradze, O. Generated Sets of the Complete Semigroups of Binary Relations Defined by Semilattices of the Class  $\Sigma_1(X, 2)$ . (In Press)