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Numerical Solutions of Three-Dimensional Coupled Burgers' Equations by Using Some Numerical Methods

Fatheah Ahmad Alhendi[1], Aisha Abdullah Alderremy[2]

[1]Department of Mathematics, King Abdulaziz University, Jeddah, Kingdom of Saudi Arabia
[2]Department of Mathematics, King Khaild University, Abha, Kingdom of Saudi Arabia
Email: falhendi@kau.edu.sa, aaldramy@kku.edu.sa

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Abstract

In this paper, we found the numerical solution of three-dimensional coupled Burgers' Equations by using more efficient methods: Laplace Adomian decomposition method, Laplace transform homotopy perturbation method, variational iteration method, variational iteration decomposition method and variational iteration homotopy perturbation method. Example is examined to validate the efficiency and accuracy of these methods and they reduce the size of computation without the restrictive assumption to handle nonlinear terms and it gives the solutions rapidly.

Keywords

Three-Dimensional Coupled Burgers' Equations, Laplace Transform, Adomian Decomposition, Homotopy Perturbation, Variational Iteration Method

1. Introduction

The Burgers Equation was first presented by Bateman [1] and treated later by J. M. Burgers (1895-1981) then it is widely named as Burgers' Equation [2]. Burgers' Equation is nonlinear partial differential equation of second order which is used in various fields of physical phenomena such as boundary layer behaviour, shock weave formation, turbulence, the weather problem, mass transport, traffic flow and acoustic transmission [3] [4]. In addition, coupled Burgers' Equations has played an important role in many physical applications such as hydrodynamic turblence, vorticity transport, skock wave, dispersion in porous media and wave processes. In particular, the three-dimentional coupled Burgers' Equations are important in large scale structure formation in the universe [5] [6]. In order to make a great application for burgers' Equations, many researchers have been interested in solving it by various techniques. Analytic solution of one-dimensional Burgers' Equation is got by many standard methods such as Backland transformation method, differential transformation method and tanh-coth method [6], while an analytical solution of two-dimensional coupled Burgers' Equations is first presented by Fletcher using the Hopf-Cole transformation [7] and an analytical solution of three-dimensional coupled Burgers' Equations is derived by Srivastava *et al* using the variable of separable and Hopf-Cole transformation [6]. The finite difference, finite element, spectral methods, Adomian decomposition method, the variational iteration method, homotopy perturbation method HPM and Eulerian-Lagrangian method, etc. gave an numerical solution of one- and two-dimentional Burgers' Equations [3] [8]-[15].

Shukla *et al* are proposed a numerical solutions of three-dimensional coupled viscous Burgers' Equations by using a modified cubic B-spline differential quadrature method [5].

The motive of this paper is to find the numerical solution of three-dimensional coupled Burgers' Equations by using more efficiently methods: Laplace Adomian decomposition method, Laplace transform homotopy perturbation method, the variational iteration method, variational iteration decomposition method and variational iteration homotopy perturbation method. We consider three-dimensional couple Burgers' Equations as the following [6]:

$$\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right),\\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=\frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right),\\
&\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=\frac{1}{R}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right).
\end{aligned} \tag{1}$$

with the initial conditions:

$$\begin{aligned}
&u_0=u(x,y,z,0)=g_1(x,y,z),\\
&v_0=v(x,y,z,0)=g_2(x,y,z),\ (x,y,z)\in\Omega\\
&w_0=w(x,y,z,0)=g_3(x,y,z).
\end{aligned} \tag{2}$$

and the boundary conditions:

$$\begin{aligned}
&u(x,y,z,t)=f_1(x,y,z,t),\\
&v(x,y,z,t)=f_2(x,y,z,t),\ x,y,z\in\Gamma,t>0\\
&w(x,y,z,t)=f_3(x,y,z,t).
\end{aligned} \tag{3}$$

where $\Omega=\{(x,y,z)\mid a_1\le x\le b_1,a_2\le y\le b_2,a_3\le z\le b_3\}$ and Γ is its boundary, $u(x,y,z,t)$ and $v(x,y,z,t)$ are the velocity components to be determined, g_1, g_2, g_3, f_1, f_2 and f_3 are known functions and R is the Reynolds number.

This paper is organized into five sections. Each method is in one section. We showed an overview of these methods then explained methodology and finally illustrated the methods by using examples. It is clear to see that numerical methods are reasonably in good covenant with the exact solution.

2. Laplace Adomian Decomposition Method

The Laplace transform **LT** is an integral transform discovered by Pierre-Simon Laplace. LT is a very powerful technique for solving ordinary and partial differential Equations, which transforms the original differential equation into an elementary algebraic expression [16].

Definition: the Laplace transform of $f(t)$ where $t \geq 0$, denoted by $\mathcal{L}f(t) = F(s)$, is given by:

$$\mathcal{L}f(t) = F(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

Adomian decomposition method **ADM** is proposed in 1980 by Grorge Adomian. ADM has been encountered much attention in recent years in applied mathematics in general and in solving Burgers' Equations in particular. A wide class of linear and nonlinear, ordinary and partial differential Equations solved easily and more accurately via ADM. It has successfully used to handle most type of physical models of partial differential Equations without dependence on linearization or any restrictive assumptions that may change physical behavior of the models under study [3] [17] [18] [19].

ADM consists of decomposing the unknown functions of any equations into a sums of an infinite number of components defined by the decomposition series:

$$u(x_1, x_2, \cdots, x_n) = \sum_{i=0}^{\infty} u_i(x_1, x_2, \cdots, x_n) \tag{4}$$

where $u_i(x_1, x_2, \cdots, x_n), i \geq 0$ are to be determined in recursive manner. The nonlinear term $N(u)$ can be expressed by an infinite series of Adomian polynomial A_m which is given as:

$$N(u) = \sum_{i=0}^{\infty} A_i(u_0, u_1, u_2, \cdots), \tag{5}$$

where Adomian polynomial A_m using the form

$$A_m = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}. \tag{6}$$

There is a growing interest of researchers has been to study the Adomian decomposition method ADM [3] [20] [21] [22] [23]. In this work, we will use Laplace transform-Adomian decomposition method (LT-ADM) introduced by Khuri khuri [24]. Some time it is known as Laplace Adomian decomposition method (LADM). This numerical technique explains how the Laplace transform may be used to approximate the solutions of the nonlinear partial differential equations (PDEs) including Burgers' Equations with the decomposition method [11] [24] [25] [26].

2.1. Methodology of LT-ADM for Three-Dimensional Couple Burgers' Equations

We consider the system (1) and apply LT on both side of it:

$$\begin{aligned}
&\mathcal{L}\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=\mathcal{L}\left[\frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\right],\\
&\mathcal{L}\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right]=\mathcal{L}\left[\frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)\right],\\
&\mathcal{L}\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right]=\mathcal{L}\left[\frac{1}{R}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)\right].
\end{aligned} \tag{7}$$

We can write (7) as:

$$\begin{aligned}
&\mathcal{L}\left[\frac{\partial u}{\partial t}\right]=\mathcal{L}\left[-\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)+\frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\right],\\
&\mathcal{L}\left[\frac{\partial v}{\partial t}\right]=\mathcal{L}\left[-\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)+\frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)\right],\\
&\mathcal{L}\left[\frac{\partial w}{\partial t}\right]=\mathcal{L}\left[-\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)+\frac{1}{R}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)\right].
\end{aligned} \tag{8}$$

By using (2) we get:

$$\begin{aligned}
&su(x,y,z,s)-u_0=\mathcal{L}\left[-\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)+\frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\right],\\
&sv(x,y,z,s)-v_0=\mathcal{L}\left[-\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)+\frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)\right],\\
&sw(x,y,z,s)-w_0=\mathcal{L}\left[-\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)+\frac{1}{R}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)\right].
\end{aligned} \tag{9}$$

Using inverse Laplace transform on both sides of (9), we have

$$\begin{aligned}
&u(x,y,z,t)=u_0+\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left[-\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)+\frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\right]\right\},\\
&v(x,y,z,t)=v_0+\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left[-\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)+\frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)\right]\right\},\\
&w(x,y,z,t)=w_0+\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left[-\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)+\frac{1}{R}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)\right]\right\}.
\end{aligned} \tag{10}$$

from (4) we can write the solutions as:

$$\begin{aligned}
&u(x,y,z,t)=\sum_{n=0}^{\infty}u_n(x,y,z,t),\\
&v(x,y,z,t)=\sum_{n=0}^{\infty}v_n(x,y,z,t),\\
&w(x,y,z,t)=\sum_{n=0}^{\infty}w_n(x,y,z,t).
\end{aligned} \tag{11}$$

Now, we assume the nonlinear terms as:

$$\begin{aligned}
&u\frac{\partial u}{\partial x}=\sum_{n=0}^{\infty}A_{(1,n)}(u),\ v\frac{\partial u}{\partial y}=\sum_{n=0}^{\infty}A_{(2,n)}(u,v),\ w\frac{\partial u}{\partial z}=\sum_{n=0}^{\infty}A_{(3,n)}(u,w),\\
&u\frac{\partial v}{\partial x}=\sum_{n=0}^{\infty}B_{(1,n)}(u,v),\ v\frac{\partial v}{\partial y}=\sum_{n=0}^{\infty}B_{(2,n)}(v),\ w\frac{\partial v}{\partial z}=\sum_{n=0}^{\infty}B_{(3,n)}(v,w),\\
&u\frac{\partial w}{\partial x}=\sum_{n=0}^{\infty}C_{(1,n)}(u,w),\ v\frac{\partial w}{\partial y}=\sum_{n=0}^{\infty}C_{(2,n)}(u,w),\ w\frac{\partial w}{\partial z}=\sum_{n=0}^{\infty}C_{(3,n)}(w).
\end{aligned}\tag{12}$$

where $A(i,n), B(i,n)$ and $C(i,b); i:=1,2,3$ are the Adomian polynomial given as (6). From (11) and (12) into (10), we have:

$$\begin{aligned}
&\sum_{n=0}^{\infty}u_n(x,y,z,t)=u_0+\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left[-\left(\sum_{n=0}^{\infty}A_{(1,n)}+\sum_{n=0}^{\infty}A_{(2,n)}+\sum_{n=0}^{\infty}A_{(3,n)}\right)+\frac{1}{R}\Delta\sum_{n=0}^{\infty}u_n\right]\right\},\\
&\sum_{n=0}^{\infty}v_n(x,y,z,t)=v_0+\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left[-\left(\sum_{n=0}^{\infty}B_{(1,n)}+\sum_{n=0}^{\infty}B_{(2,n)}+\sum_{n=0}^{\infty}B_{(3,n)}\right)+\frac{1}{R}\Delta\sum_{n=0}^{\infty}v_n\right]\right\},\\
&\sum_{n=0}^{\infty}w_n(x,y,z,t)=w_0+\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left[-\left(\sum_{n=0}^{\infty}C_{(1,n)}+\sum_{n=0}^{\infty}C_{(2,n)}+\sum_{n=0}^{\infty}C_{(3,n)}\right)+\frac{1}{R}\Delta\sum_{n=0}^{\infty}w_n\right]\right\}.
\end{aligned}\tag{13}$$

where $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$. Then, using (2), (6), (12) into (13), we have

$$\begin{aligned}
&u=u_0+u_1+u_2+u_3+\cdots,\\
&v=v_0+v_1+v_2+v_3+\ldots,\\
&w=w_0+w_1+w_2+w_3+\cdots.
\end{aligned}\tag{14}$$

2.2. Example

Consider (1) if the exact solution is [5]:

$$\begin{aligned}
&u^*(x,y,z,t)=\frac{-2}{R}\left(\frac{1+\cos(x)\sin(y)\sin(z)\mathrm{e}^{-t}}{1+x+\sin(x)\sin(y)\sin(z)\mathrm{e}^{-t}}\right),\\
&v^*(x,y,z,t)=\frac{-2}{R}\left(\frac{\sin(x)\cos(y)\sin(z)\mathrm{e}^{-t}}{1+x+\sin(x)\sin(y)\sin(z)\mathrm{e}^{-t}}\right),\\
&w^*(x,y,z,t)=\frac{-2}{R}\left(\frac{\sin(x)\sin(y)\cos(z)\mathrm{e}^{-t}}{1+x+\sin(x)\sin(y)\sin(z)\mathrm{e}^{-t}}\right).
\end{aligned}\tag{15}$$

To solve this example by LT-ADM, we follow the methodology which discussed in subsection (2.1). The accuracy of LT-ADM for the three-dimensional coupled Burgers' Equations agrees very well with the exact solution and absolute errors are very small for the current choice of x, y, z and t. The results are shown in Tables 1-3 for $R = 100$, $x = 0.1$, $y = 0.02$ and $z = 0.03$.

3. Laplace Transform Homotopy Perturbation Method

Homotopy perturbation method **HPM** was first proposed by He. HPM is combination of traditional perturbation method and homotopy method. The important advantage of HPM is that the nonlinear term can be easily handled. It is easy to calculate the solution

Table 1. The absolute error (AEs) of $u(x,y,z,t)$ by LT-ADM for example 2.2.

t	$u^*(x,y,z,t)$	$u(x,y,z,t)$	$\lvert u^* - u\rvert$
0	−0.01819168001	−0.01819168001	0
0.002	−0.01819166033	−0.01819167942	1.909×10^{-8}
0.004	−0.01819164064	−0.01819167883	3.819×10^{-8}
0.006	−0.01819162103	−0.01819167823	5.720×10^{-8}
0.008	−0.01819160143	−0.01819167764	7.621×10^{-8}
0.010	−0.01819158189	−0.01819167705	9.516×10^{-8}

Table 2. The AEs of $v(x,y,z,t)$ by LT-ADM for example 2.2.

t	$v^*(x,y,z,t)$	$v(x,y,z,t)$	$\lvert v^* - v\rvert$
0	−0.00005443257070	−0.00005443257070	0
0.002	−0.00005432382028	−0.00005442930556	1.0548528×10^{-7}
0.004	−0.00005421528708	−0.00005442604171	2.1075463×10^{-7}
0.006	−0.00005410697072	−0.00005442277913	3.1580841×10^{-7}
0.008	−0.00005399887076	−0.00005441951783	4.2064707×10^{-7}
0.010	−0.00005389098678	−0.00005441625782	5.2527104×10^{-7}

Table 3. The AEs of $w(x,y,z,t)$ by LT-ADM for example 2.2.

t	$w^*(x,y,z,t)$	$w(x,y,z,t)$	$\lvert w^* - w\rvert$
0	−0.00003628233106	−0.00003628233106	0
0.002	−0.00003620984286	−0.00003628015467	7.031181×10^{-8}
0.004	−0.00003613749944	−0.00003627797914	1.4047970×10^{-7}
0.006	−0.00003606530058	−0.00003627580447	2.1050389×10^{-7}
0.008	−0.00003599324596	−0.00003627363065	2.8038469×10^{-7}
0.010	−0.00003592133528	−0.00003627145770	3.5012242×10^{-7}

with this method. Linear or nonlinear ODEs and PDEs are studied successfully by using LT-HPM [17], [27] [28] [29] [30] [31]. To figure out how HPM works [32], consider n-dimensional Burgers' equation

$$\frac{\partial u_i}{\partial t}+\sum_{j=1}^{n}u_j\frac{\partial u_i}{\partial x_j}=\mu\Delta u_i,\ \ i=1,2,\cdots,n \tag{16}$$

We construct the following homotopy:

$$(1-p)\left[\frac{\partial u_{(i,k)}}{\partial t}-\frac{\partial u_{(i,0)}}{\partial t}\right]+p\left[\frac{\partial u_i}{\partial t}+\sum_{j=1}^{n}u_{(j,k)}\frac{\partial u_{(i,k)}}{\partial x_j}-\mu\Delta u_{(i,k)}\right]=0. \tag{17}$$

or

$$\frac{\partial u_{(i,k)}}{\partial t}-\frac{\partial u_{(i,0)}}{\partial t}+p\left[\frac{\partial u_{(i,0)}}{\partial t}+\sum_{j=1}^{n}u_{(j,k)}\frac{\partial u_{(i,k)}}{\partial x_j}-\mu\Delta u_{(i,k)}\right]=0. \tag{18}$$

where $\Delta=\frac{\partial^2}{x_1^2}+\frac{\partial^2}{x_2^2}+\cdots+\frac{\partial^2}{x_n^2}$, $i=1,2,\cdots,n$, $k=1,2,\cdots$, $p\in[0,1]$ is an embedding parameter while $u_{(1,0)},u_{(2,0)},\cdots,u_{(n,0)}$ are initial approximations of (16). Assume the solution of (16) has the form

$$u_i=\sum_{\ell=0}^{\infty}p^{\ell}u_{(i,\ell)}\left(x_j,t\right),\quad i,j=1,2,\cdots,n. \tag{19}$$

Now, substituting u_i from Equation (19) in Equation (18) and comparing coefficients of terms with identical powers of *p*, we get:

$$\begin{aligned}
&p^0:\frac{\partial u_{(i,0)}}{\partial t}-\frac{\partial u_{(i,0)}}{\partial t}=0\\
&p^1:\frac{\partial u_{(i,0)}}{\partial t}+u_{(j,0)}\frac{\partial u_{(i,0)}}{\partial x_j}-\mu\Delta u_{(i,0)},\\
&p^2:u_{(j,1)}\frac{\partial u_{(i,1)}}{\partial x_j}-\mu\Delta u_{(i,1)}\\
&p^3:u_{(j,2)}\frac{\partial u_{(i,2)}}{\partial x_j}-\mu\Delta u_{(i,2)},\\
&\ \ \vdots
\end{aligned} \tag{20}$$

The solution is

$$u_i\left(x_j,t\right)=u_{(i,0)}+u_{(i,1)}+u_{(i,2)}+\cdots. \tag{21}$$

In [33], Aminikhah presented LT-HPM to solve nonlinear Blasius' viscous flow equation. In [34], some application examples of LT-HPM for nonlinear ODEs with Dirichlet, mixed, and Neumann boundary conditions were presented. It has used to solve PDEs then to solve Burgers' Equations [12] [14] [30] [31] [34].

Now, we are going to study LT-HPM for (1) as following.

3.1. Methodology of LT-HPM for Three-Dimensional Couple Burgers' Equations

We consider the system (1) and apply HPM on it. We construct the following homotopy:

$$\begin{aligned}&(1-p)\left(\frac{\partial U}{\partial t}-\frac{\partial u_0}{\partial t}\right)+p\left(\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z}-\frac{1}{R}\Delta U\right)=0,\\&(1-p)\left(\frac{\partial V}{\partial t}-\frac{\partial v_0}{\partial t}\right)+p\left(\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+W\frac{\partial V}{\partial z}-\frac{1}{R}\Delta V\right)=0,\\&(1-p)\left(\frac{\partial W}{\partial t}-\frac{\partial w_0}{\partial t}\right)+p\left(\frac{\partial W}{\partial t}+U\frac{\partial W}{\partial x}+V\frac{\partial W}{\partial y}+W\frac{\partial W}{\partial z}-\frac{1}{R}\Delta W\right)=0.\end{aligned}\tag{22}$$

where u_0, v_0 and w_0 are the initial approximation values of (1), and U, V and W have the following forms:

$$U=\sum_{i=0}^{\infty}p^i u_i\left(x,y,z,t\right),V=\sum_{i=0}^{\infty}p^i v_i\left(x,y,z,t\right),W=\sum_{i=0}^{\infty}p^i w_i\left(x,y,z,t\right).\tag{23}$$

we can write (22) as:

$$\begin{aligned}&\frac{\partial U}{\partial t}=\frac{\partial u_0}{\partial t}-p\frac{\partial u_0}{\partial t}-p\left(U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z}-\frac{1}{R}\Delta U\right),\\&\frac{\partial V}{\partial t}=\frac{\partial v_0}{\partial t}-p\frac{\partial v_0}{\partial t}-p\left(U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+W\frac{\partial V}{\partial z}-\frac{1}{R}\Delta V\right),\\&\frac{\partial W}{\partial t}=\frac{\partial w_0}{\partial t}-p\frac{\partial w_0}{\partial t}+p\left(U\frac{\partial W}{\partial x}+V\frac{\partial W}{\partial y}+W\frac{\partial W}{\partial z}-\frac{1}{R}\Delta W\right).\end{aligned}\tag{24}$$

Applying LT on both side of (24)

$$\begin{aligned}&\mathcal{L}\left\{\frac{\partial U}{\partial t}\right\}=\mathcal{L}\left\{\frac{\partial u_0}{\partial t}-p\frac{\partial u_0}{\partial t}-p\left(U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z}-\frac{1}{R}\Delta U\right)\right\},\\&\mathcal{L}\left\{\frac{\partial V}{\partial t}\right\}=\mathcal{L}\left\{\frac{\partial v_0}{\partial t}-p\frac{\partial v_0}{\partial t}-p\left(U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+W\frac{\partial V}{\partial z}-\frac{1}{R}\Delta V\right)\right\},\\&\mathcal{L}\left\{\frac{\partial W}{\partial t}\right\}=\mathcal{L}\left\{\frac{\partial w_0}{\partial t}-p\frac{\partial w_0}{\partial t}-p\left(U\frac{\partial W}{\partial x}+V\frac{\partial W}{\partial y}+W\frac{\partial W}{\partial z}-\frac{1}{R}\Delta W\right)\right\}.\end{aligned}\tag{25}$$

By applying inverse of LT on (25), we get

$$\begin{aligned}&U\left(x,y,z,t\right)\\&=\mathcal{L}^{-1}\left\{\frac{1}{s}\left[u_0+\mathcal{L}\left\{\frac{\partial u_0}{\partial t}-p\frac{\partial u_0}{\partial t}-p\left(U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z}-\frac{1}{R}\Delta U\right)\right\}\right]\right\},\\&V\left(x,y,z,t\right)\\&=\mathcal{L}^{-1}\left\{\frac{1}{s}\left[v_0+\mathcal{L}\left\{\frac{\partial v_0}{\partial t}-p\frac{\partial v_0}{\partial t}-p\left(U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+W\frac{\partial V}{\partial z}-\frac{1}{R}\Delta V\right)\right\}\right]\right\},\\&W\left(x,y,z,t\right)\\&=\mathcal{L}^{-1}\left\{\frac{1}{s}\left[w_0+\mathcal{L}\left\{\frac{\partial w_0}{\partial t}-p\frac{\partial w_0}{\partial t}-p\left(U\frac{\partial W}{\partial x}+V\frac{\partial W}{\partial y}+W\frac{\partial W}{\partial z}-\frac{1}{R}\Delta W\right)\right\}\right]\right\}.\end{aligned}\tag{26}$$

By substituting U, V and W from (23) in (26) and comparing coefficients of terms with identical powers of p:

$$p^0: \begin{cases} u_0(x,y,z) = \mathcal{L}^{-1}\left\{\frac{1}{s}\left[u_0 + \mathcal{L}\left\{\frac{\partial u_0}{\partial t}\right\}\right]\right\}, \\ v_0(x,y,z) = \mathcal{L}^{-1}\left\{\frac{1}{s}\left[v_0 + \mathcal{L}\left\{\frac{\partial v_0}{\partial t}\right\}\right]\right\}, \\ w_0(x,y,z) = \mathcal{L}^{-1}\left\{\frac{1}{s}\left[w_0 + \mathcal{L}\left\{\frac{\partial w_0}{\partial t}\right\}\right]\right\}, \end{cases}$$

$$p^1: \begin{cases} u_1(x,y,z,t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\left[\mathcal{L}\left\{-\frac{\partial u_0}{\partial t} - u_0\frac{\partial u_0}{\partial x} - v_0\frac{\partial u_0}{\partial y} - w_0\frac{\partial u_0}{\partial z} + \frac{1}{R}\Delta u_0\right\}\right]\right\}, \\ v_1(x,y,z,t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\left[\mathcal{L}\left\{-\frac{\partial v_0}{\partial t} - u_0\frac{\partial v_0}{\partial x} - v_0\frac{\partial v_0}{\partial y} - w_0\frac{\partial v_0}{\partial z} + \frac{1}{R}\Delta v_0)\right\}\right]\right\}, \\ w_1(x,y,z,t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\left[\mathcal{L}\left\{-\frac{\partial w_0}{\partial t} - u_0\frac{\partial w_0}{\partial x} - v_0\frac{\partial w_0}{\partial y} - w_0\frac{\partial w_0}{\partial z} + \frac{1}{R}\Delta w_0)\right\}\right]\right\}. \end{cases} \tag{27}$$

$$p^2: \begin{cases} u_2(x,y,z,t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\left[\mathcal{L}\left\{-u_0\frac{\partial u_1}{\partial x} - u_1\frac{\partial u_0}{\partial x} - v_0\frac{\partial u_1}{\partial y} - v_1\frac{\partial u_0}{\partial y} - w_0\frac{\partial u_1}{\partial z} - w_1\frac{\partial u_0}{\partial z} + \frac{1}{R}\Delta u_1\right\}\right]\right\}, \\ v_2(x,y,z,t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\left[\mathcal{L}\left\{-u_0\frac{\partial v_1}{\partial x} - u_1\frac{\partial v_0}{\partial x} - v_0\frac{\partial v_1}{\partial y} - v_1\frac{\partial v_0}{\partial y} - w_0\frac{\partial v_1}{\partial z} - w_1\frac{\partial v_0}{\partial z} + \frac{1}{R}\Delta v_1\right\}\right]\right\}, \\ w_2(x,y,z,t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\left[\mathcal{L}\left\{-u_0\frac{\partial w_1}{\partial x} - u_1\frac{\partial w_0}{\partial x} - v_0\frac{\partial w_1}{\partial y} - v_1\frac{\partial w_0}{\partial y} - w_0\frac{\partial w_1}{\partial z} - w_1\frac{\partial w_0}{\partial z} + \frac{1}{R}\Delta w_1\right\}\right]\right\}. \end{cases}$$

$$\vdots$$

Finally, the approximate solutions are:

$$\begin{aligned} u &= u_0 + u_1 + u_2 + \cdots, \\ v &= v_0 + v_1 + v_2 + \cdots, \\ w &= w_0 + w_1 + w_2 + \cdots. \end{aligned} \tag{28}$$

3.2. Example

To solve (1) by LT-HPM, we follow the methodology which discussed in subsection (3.1). The results in **Tables 4-6** show that LT-HPM is more effective and high accuracy when compared with the exact solutions for $R = 100$, $x = 0.1$, $y = 0.02$ and $z = 0.03$.

Table 4. The AEs of $u(x,y,z,t)$ by LT-HPM for example 3.2.

t	$u^*(x,y,z,t)$	$u(x,y,z,t)$	$\lvert u^* - u\rvert$
0	−0.01819168001	−0.01819168001	0
0.002	−0.01819166033	−0.01819168060	2.027×10^{-8}
0.004	−0.01819164064	−0.01819168119	4.055×10^{-8}
0.006	−0.01819162103	−0.01819168179	6.076×10^{-8}
0.008	−0.01819160143	−0.01819168238	8.095×10^{-8}
0.010	−0.01819158189	−0.01819168297	1.0108×10^{-7}

Table 5. The AEs of $v(x,y,z,t)$ by LT-HPM for example 3.2.

t	$v^*(x,y,z,t)$	$v(x,y,z,t)$	$\lvert v^* - v\rvert$
0	−0.00005443257070	−0.00005443257070	0
0.002	−0.00005432382028	−0.00005443583647	1.1201619×10^{-7}
0.004	−0.00005421528708	−0.00005443910222	2.2381514×10^{-7}
0.006	−0.00005410697072	−0.00005444236796	3.3539724×10^{-7}
0.008	−0.00005399887076	−0.00005444563368	4.4676292×10^{-7}
0.010	−0.00005389098678	−0.00005444889938	5.5791260×10^{-7}

Table 6. The AEs of $w(x,y,z,t)$ by LT-HPM for example 3.2.

t	$w^*(x,y,z,t)$	$w(x,y,z,t)$	$\lvert w^* - w\rvert$
0	−0.00003628233106	−0.00003628233106	0
0.002	−0.00003620984286	−0.00003628450787	7.466501×10^{-8}
0.004	−0.00003613749944	−0.00003628668465	1.4918521×10^{-7}
0.006	−0.00003606530058	−0.00003628886141	2.2356083×10^{-7}
0.008	−0.00003599324596	−0.00003629103816	2.9779220×10^{-7}
0.010	−0.00003592133528	−0.00003629321487	3.7187959×10^{-7}

4. The Variational Iteration Method

The variational iteration method (VIM) was proposed by Ji-Huan He in 1997 [35] [36] [37] [38]. The VIM has been applied successfully for the most important problems in physically important phenomena including Burgers' Equation [13] [39] [40] [41] [42] The VIM solve linear or nonlinear ODEs and PDEs without needing small parameter or linearization and by few iterations lead to high accurate solutions.

To explain the basic concepts of the VIM, we consider n-dim of burgers' Equation (16). We can write the correction functional for it as [32]:

$$u_{n+1} = u_n + \int_0^t \lambda_i(\xi)\left[\frac{\partial u_i}{\partial \xi} + \sum_{j=1}^{n} \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} - \mu\Delta\tilde{u}_i\right]\mathrm{d}\xi. \tag{29}$$

where $i = 1,2,\cdots,n$, $u = u(x_i,\xi)$, λ_i are the general Lagrangian multipliers which can be find via variational theory, $\tilde{u}_i$ are restricted variation which means $\delta\tilde{u}_i = 0$. The solution is given by

$$u_i(x_j,t) = \lim_{n\to\infty} u_{(i,n)}(x_j,t), \quad j = 1,2,\cdots,n. \tag{30}$$

4.1. Methodology of VIM for Three-Dimensional Couple Burgers' Equations

Consider the system (1), we can construct a the following correction functional:

$$\begin{aligned} u_{n+1}(x,y,z,t) &= u_n + \int_0^t \lambda_1 \left(\frac{\partial u}{\partial \xi} + \tilde{u}\frac{\partial \tilde{u}}{\partial x} + \tilde{v}\frac{\partial \tilde{u}}{\partial y} + \tilde{w}\frac{\partial \tilde{u}}{\partial z} - \frac{1}{R}\Delta \tilde{u} \right) \mathrm{d}\xi, \\ v_{n+1}(x,y,z,t) &= v_n + \int_0^t \lambda_2 \left(\frac{\partial v}{\partial \xi} + \tilde{u}\frac{\partial \tilde{v}}{\partial x} + \tilde{v}\frac{\partial \tilde{v}}{\partial y} + \tilde{w}\frac{\partial \tilde{v}}{\partial z} - \frac{1}{R}\Delta \tilde{v} \right) \mathrm{d}\xi, \\ w_{n+1}(x,y,z,t) &= w_n + \int_0^t \lambda_3 \left(\frac{\partial w}{\partial \xi} + \tilde{u}\frac{\partial \tilde{w}}{\partial x} + \tilde{v}\frac{\partial \tilde{w}}{\partial y} + \tilde{w}\frac{\partial \tilde{w}}{\partial z} - \frac{1}{R}\Delta \tilde{w} \right) \mathrm{d}\xi. \end{aligned} \quad (31)$$

$\lambda_1 = \lambda_2 = \lambda_3 = -1$. Then, the iteration formulae are given as:

$$\begin{aligned} u_{n+1}(x,y,z,t) &= u_n - \int_0^t \left(\frac{\partial u}{\partial \xi} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - \frac{1}{R}\Delta u \right) \mathrm{d}\xi, \\ v_{n+1}(x,y,z,t) &= v_n - \int_0^t \left(\frac{\partial v}{\partial \xi} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} - \frac{1}{R}\Delta v \right) \mathrm{d}\xi, \\ w_{n+1}(x,y,z,t) &= w_n - \int_0^t \left(\frac{\partial w}{\partial \xi} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - \frac{1}{R}\Delta w \right) \mathrm{d}\xi. \end{aligned} \quad (32)$$

4.2. Example

To solve (1) by VIM, we follow the methodology which discussed in subsection (4.1). The results in Tables 7-9 are shown that the efficiency and accuracy of the VIM. It reduces the size of computation without the restrictive assumption to handle nonlinear terms and it gives the solutions rapidly.

5. Variational Iteration Decomposition Method

The variational iteration decomposition method (VIDM) is technique combination of two the most powerful mathematical methods for solving a large class of differential Equations, namely variational iteration method and Adomian decomposition method. In 2007 VIDM has been used to solve quadratic Riccati differential Equation problems

Table 7. The AEs of $u(x,y,z,t)$ by VIM for example 4.2.

t	$u^*(x,y,z,t)$	$u(x,y,z,t)$	$\lvert u^* - u \rvert$
0	−0.01819168001	−0.01819168085	8.4×10^{-10}
0.002	−0.01819166033	−0.01819168057	2.024×10^{-8}
0.004	−0.01819164064	−0.01819167882	3.818×10^{-8}
0.006	−0.01819162103	−0.01819167984	5.881×10^{-8}
0.008	−0.01819160143	−0.01819167803	7.660×10^{-8}
0.010	−0.01819158189	−0.01819167741	9.552×10^{-8}

Table 8. The AEs of $v(x,y,z,t)$ by VIM for example 4.2.

t	$v^*(x,y,z,t)$	$v(x,y,z,t)$	$\lvert v^* - v\rvert$
0	−0.00005443257070	−0.00005443257572	5.02×10^{-12}
0.002	−0.00005432382028	0.00005442930575	1.0548547×10^{-7}
0.004	−0.00005421528708	0.00005442603014	2.1074306×10^{-7}
0.006	−0.00005410697072	0.00005442276920	3.1579848×10^{-7}
0.008	−0.00005399887076	0.00005441950206	4.2063130×10^{-7}
0.010	−0.00005389098678	0.00005441623773	5.2525095×10^{-7}

Table 9. The AEs of $w(x,y,z,t)$ by VIM for example 4.2.

t	$w^*(x,y,z,t)$	$w(x,y,z,t)$	$\lvert w^* - w\rvert$
0	−0.00003628233106	−0.00003628232124	9.82×10^{-12}
0.002	−0.00003620984286	−0.00003628012665	7.028379×10^{-8}
0.004	−0.00003613749944	−0.00003627797169	1.4047225×10^{-7}
0.006	−0.00003606530058	−0.00003627578326	2.1048268×10^{-7}
0.008	−0.00003599324596	−0.00003627365724	2.8041128×10^{-7}
0.010	−0.00003592133528	−0.00003627142962	3.5009434×10^{-7}

[43]. Noor *et al.* [44] [45] used this method for solving eighth-order boundary value problems, sixth-order boundary value problems and higher dimensional initial boundary value problems. Grover and Tomer solved twelfth order boundary value problems by using VIDM. In 2013, the fractional Riccati differential Equation is solved by VIM by using Adomian polynomials for nonlinear terms [46].

To illustrate the general concept of VIDM by using (5), (6) in (29), hence, we have the correction functional for (16) as:

$$u_{n+1} = u_n + \int_0^t \lambda_i \left[\frac{\partial u_i}{\partial \xi} + \sum_{j=1}^{n} A_j - \mu \Delta \tilde{u}_i \right] \mathrm{d}\xi. \tag{33}$$

We are solved three-dimensional coupled Burgers' Equations by using VIDM as following.

5.1. Methodology of VIDM for Three-Dimensional Couple Burgers' Equations

We consider the system (1). Next, by using (11) and (12) in (32), we obtain the iterative scheme to find the approximate solutions by VIDM as following:

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$$u_{n+1}(x,y,z,t) = u_n - \int_0^t \left(\frac{\partial u}{\partial \xi} + \sum_{n=0}^{\infty} A_{(1,n)}(u) + \sum_{n=0}^{\infty} A_{(2,n)}(u,v) + \sum_{n=0}^{\infty} A_{(3,n)}(u,w) - \frac{1}{R}\Delta u \right) \mathrm{d}\xi,$$
$$v_{n+1}(x,y,z,t) = v_n - \int_0^t \left(\frac{\partial v}{\partial \xi} + \sum_{n=0}^{\infty} B_{(1,n)}(u,v) + \sum_{n=0}^{\infty} B_{(2,n)}(v) + \sum_{n=0}^{\infty} B_{(3,n)}(v,w) - \frac{1}{R}\Delta v \right) \mathrm{d}\xi, \quad (34)$$
$$w_{n+1}(x,y,z,t) = w_n - \int_0^t \left(\frac{\partial w}{\partial \xi} + \sum_{n=0}^{\infty} C_{(1,n)}(u,w) + \sum_{n=0}^{\infty} C_{(2,n)}(v,w) + \sum_{n=0}^{\infty} C_{(3,n)}(w) - \frac{1}{R}\Delta w \right) \mathrm{d}\xi.$$

5.2. Example

To solve (1) by VIDM by using (34). The numerical results in Tables 10-12 show that VIDM is an effective and powerful method to find better results.

Table 10. The AEs of $u(x,y,z,t)$ by VIDM for example 5.2.

t	$u^*(x,y,z,t)$	$u(x,y,z,t)$	$\lvert u^*-u \rvert$
0	−0.01819168001	−0.01819168001	0
0.002	−0.01819166033	−0.01819167942	1.909×10^{-8}
0.004	−0.01819164064	−0.01819167883	3.819×10^{-8}
0.006	−0.01819162103	−0.01819167823	5.720×10^{-8}
0.008	−0.01819160143	−0.01819167764	7.621×10^{-8}
0.010	−0.01819158189	−0.01819167705	9.516×10^{-8}

Table 11. The AEs of $v(x,y,z,t)$ by VIDM for example 5.2.

t	$v^*(x,y,z,t)$	$v(x,y,z,t)$	$\lvert v^*-v \rvert$
0	−0.00005443257070	−0.00005443257070	0
0.002	−0.00005432382028	−0.00005442930556	1.0548528×10^{-7}
0.004	−0.00005421528708	−0.00005442604171	2.1075463×10^{-7}
0.006	−0.00005410697072	−0.00005442277913	3.1580841×10^{-7}
0.008	−0.00005399887076	−0.00005441951783	4.2064707×10^{-7}
0.010	−0.00005389098678	−0.00005441625782	5.2527104×10^{-7}

Table 12. The AEs of $w(x,y,z,t)$ by VIDM for example 5.2.

t	$w^*(x,y,z,t)$	$w(x,y,z,t)$	$\lvert w^*-w \rvert$
0	−0.00003628233106	−0.00003628233106	0
0.002	−0.00003620984286	−0.00003628015467	7.031181×10^{-7}
0.004	−0.00003613749944	−0.00003627797914	1.4047970×10^{-7}
0.006	−0.00003606530058	−0.00003627580447	2.1050389×10^{-7}
0.008	−0.00003599324596	−0.00003627363065	2.8038469×10^{-7}
0.010	−0.00003592133528	−0.00003627145770	3.5012242×10^{-7}

6. Variational Iteration Homotopy Perturbation Method

The variational iteration homotopy perturbation method (VIHPM) is combination of two well-known methods, namely variational iteration method and homotopy perturbation method. VIHPM has been applied in [32] [47]-[52] for solving a large class of differential Equations.

To illustrate the concept of VIHPM [32], we consider (16) and assume the solution of (16) has the form

$$\begin{aligned} v_i &= \sum_{\ell=0}^{\infty} p^{\ell} u_{(i,\ell)}\left(x_j,t\right) = v_i, \quad i,j=1,2,\cdots,n, \\ v_j &= \sum_{\ell=0}^{\infty} p^{\ell} u_{(j,\ell)}\left(x_j,t\right) = v_j, \quad i,j=1,2,\cdots,n. \end{aligned} \tag{35}$$

from (35), (16) can be written as:

$$\frac{\partial v_i}{\partial t} + \sum_{j=1}^{n} v_j \frac{\partial v_i}{\partial x_j} = \mu \Delta v_i, \; i=1,2,\cdots,n. \tag{36}$$

we can from the correction functional for(36) we can write

$$v_{i+1} = v_0 + p\int_0^t \lambda_i(\xi)\left[-\sum_{j=1}^{n} \tilde{v}_j \frac{\partial \tilde{v}_i}{\partial x_j} + \mu \Delta \tilde{v}_i\right]\mathrm{d}\xi. \tag{37}$$

where $i=1,2,\cdots,n$, $v = v\left(x_i,\xi\right)$, from (35) in (37) and by comparing the coefficients of like powers of p, we get

$$\begin{aligned} &p^0 : u_{(i,0)}\left(x_j,0\right) = f_i\left(x_j,0\right), \\ &p^1 : u_{(i,1)}\left(x_j,t\right) = \int_0^t \lambda_i(\xi)\left[-u_{(j,0)}\left(x_j,\xi\right)\frac{\partial u_{(i,0)}\left(x_j,\xi\right)}{\partial x_j} + \mu\Delta u_{(i,0)}\left(x_j,\xi\right)\right]\mathrm{d}\xi, \\ &p^2 : u_{(i,2)}\left(x_j,t\right) = \int_0^t \lambda_i(\xi)\left[-u_{(j,1)}\left(x_j,\xi\right)\frac{\partial u_{(i,1)}\left(x_j,\xi\right)}{\partial x_j} + \mu\Delta u_{(i,1)}\left(x_j,\xi\right)\right]\mathrm{d}\xi, \\ &p^3 : u_{(i,3)}\left(x_j,t\right) = \int_0^t \lambda_i(\xi)\left[-u_{(j,2)}\left(x_j,\xi\right)\frac{\partial u_{(i,2)}\left(x_j,\xi\right)}{\partial x_j} + \mu\Delta u_{(i,2)}\left(x_j,\xi\right)\right]\mathrm{d}\xi, \\ &\vdots \end{aligned} \tag{38}$$

The approximate solutions are give by

$$u_i\left(x_j,t\right) = u_{(i,0)} + u_{(i,1)} + u_{(i,2)} + \cdots. \tag{39}$$

we used this method to solve three-dimensional coupled Burgers' Equations as followed.

6.1. Methodology of VIHPM for Three-Dimensional Couple Burgers' Equations

we consider the correction functional (31) with $\lambda_1 = \lambda_2 = \lambda_3 = -1$ by assuming that

$$U = \sum_{i=0}^{n} p^i u_i\left(x,y,z,t\right), \; V = \sum_{i=0}^{n} p^i v_i\left(x,y,z,t\right), \; W = \sum_{i=0}^{n} p^i w_i\left(x,y,z,t\right). \tag{40}$$

by VIHPM, we have

$$\begin{aligned}
u_0 + pu_1 + p^2u_2 + \cdots = u_0 &- p\int_0^t -\left(u_0 + pu_1 + p^2u_2 + \cdots\right)\frac{\partial}{\partial x}\left(u_0 + pu_1 + p^2u_2 + \cdots\right)\mathrm{d}\xi \\
&- p\int_0^t -\left(v_0 + pv_1 + p^2v_2 + \cdots\right)\frac{\partial}{\partial y}\left(u_0 + pu_1 + p^2u_2 + \cdots\right)\mathrm{d}\xi \\
&- p\int_0^t -\left(w_0 + pw_1 + p^2w_2 + \cdots\right)\frac{\partial}{\partial z}\left(u_0 + pu_1 + p^2u_2 + \cdots\right)\mathrm{d}\xi \\
&- p\int_0^t \frac{1}{R}\Delta\left(u_0 + pu_1 + p^2u_2 + \cdots\right)\mathrm{d}\xi, \\
v_0 + pv_1 + p^2v_2 + \cdots = v_0 &- p\int_0^t -\left(u_0 + pu_1 + p^2u_2 + \cdots\right)\frac{\partial}{\partial x}\left(v_0 + pv_1 + p^2v_2 + \cdots\right)\mathrm{d}\xi \\
&- p\int_0^t -\left(v_0 + pv_1 + p^2v_2 + \cdots\right)\frac{\partial}{\partial y}\left(v_0 + pv_1 + p^2v_2 + \cdots\right)\mathrm{d}\xi \\
&- p\int_0^t -\left(w_0 + pw_1 + p^2w_2 + \cdots\right)\frac{\partial}{\partial z}\left(v_0 + pv_1 + p^2v_2 + \cdots\right)\mathrm{d}\xi \\
&- p\int_0^t \frac{1}{R}\Delta\left(v_0 + pv_1 + p^2v_2 + \cdots\right)\mathrm{d}\xi,
\end{aligned} \tag{41}$$

$$\begin{aligned}
w_0 + pw_1 + p^2w_2 + \cdots = w_0 &- p\int_0^t -\left(u_0 + pu_1 + p^2u_2 + \cdots\right)\frac{\partial}{\partial x}\left(w_0 + pw_1 + p^2w_2 + \cdots\right)\mathrm{d}\xi \\
&- p\int_0^t -\left(v_0 + pv_1 + p^2v_2 + \cdots\right)\frac{\partial}{\partial y}\left(w_0 + pw_1 + p^2w_2 + \cdots\right)\mathrm{d}\xi \\
&- p\int_0^t -\left(w_0 + pw_1 + p^2w_2 + \cdots\right)\frac{\partial}{\partial z}\left(w_0 + pw_1 + p^2w_2 + \cdots\right)\mathrm{d}\xi \\
&- p\int_0^t \frac{1}{R}\Delta\left(w_0 + pw_1 + p^2w_2 + \cdots\right)\mathrm{d}\xi.
\end{aligned} \tag{42}$$

by comparing the coefficients of like power of p, we get

$$p^0: \begin{cases} u_0(x,y,z) = u_0, \\ v_0(x,y,z) = v_0, \quad \text{from initial conditions} \\ w_0(x,y,z) = w_0. \end{cases}$$

$$p^1: \begin{cases} u_1(x,y,z,t) = \int_0^t \left(u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} + w_0\frac{\partial u_0}{\partial z} - \frac{1}{R}\Delta u_0\right)\mathrm{d}\xi, \\ v_1(x,y,z,t) = \int_0^t \left(u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y} + w_0\frac{\partial v_0}{\partial z} - \frac{1}{R}\Delta v_0\right)\mathrm{d}\xi, \\ w_1(x,y,z,t) = \int_0^t \left(u_0\frac{\partial w_0}{\partial x} + v_0\frac{\partial w_0}{\partial y} + w_0\frac{\partial w_0}{\partial z} - \frac{1}{R}\Delta w_0\right)\mathrm{d}\xi. \end{cases} \tag{43}$$

$$p^2: \begin{cases} u_2(x,y,z,t) = \int_0^t \left(u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x} + v_1\frac{\partial u_0}{\partial y} + v_0\frac{\partial u_1}{\partial y} + w_1\frac{\partial u_0}{\partial z} + w_0\frac{\partial u_1}{\partial z} - \frac{1}{R}\Delta u_1\right)\mathrm{d}\xi, \\ v_2(x,y,z,t) = \int_0^t \left(u_0\frac{\partial v_1}{\partial x} + u_1\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_1}{\partial y} + v_1\frac{\partial v_0}{\partial y} + w_1\frac{\partial v_0}{\partial z} + w_0\frac{\partial v_1}{\partial z} - \frac{1}{R}\Delta v_1\right)\mathrm{d}\xi, \\ w_2(x,y,z,t) = \int_0^t \left(u_1\frac{\partial w_0}{\partial x} + u_0\frac{\partial w_1}{\partial x} + v_1\frac{\partial w_0}{\partial y} + v_0\frac{\partial w_1}{\partial y} + w_1\frac{\partial w_0}{\partial z} + w_0\frac{\partial w_1}{\partial z} - \frac{1}{R}\Delta w_1\right)\mathrm{d}\xi. \end{cases}$$

$\vdots$

The approximate solutions are given by

$$\begin{aligned} u(x,y,z,t) &= u_0 + u_1 + u_2 + \cdots, \\ v(x,y,z,t) &= v_0 + v_1 + v_2 + \cdots, \\ w(x,y,z,t) &= w_0 + w_1 + w_2 + \cdots. \end{aligned} \tag{44}$$

6.2. Example

To solve (1) by VIHPM, we follow the methodology discussed in subsection (6.1). The accuracy of VIHPM for the three-dimensional coupled Burgers' Equations agrees very well with the exact solution and absolute errors are very small for the current choice of *x*, *y*, *z* and *t*. The result are shown in Tables 13-15 for $R = 100$, $x = 0.1$, $y = 0.02$ and $z = 0.03$.

7. Conclusion

In this work, these previous methods mentioned above have been successfully used for finding the solution of three-dimensional coupled Burgers' Equations. The numerical

Table 13. The AEs of $u(x,y,z,t)$ by VIHPM for example 6.2.

t	$u^*(x,y,z,t)$	$u(x,y,z,t)$	$\lvert u^* - u \rvert$
0	−0.01819168001	−0.01819168001	0
0.002	−0.01819166033	−0.01819167942	1.909×10^{-8}
0.004	−0.01819164064	−0.01819167883	3.819×10^{-8}
0.006	−0.01819162103	−0.01819167823	5.720×10^{-8}
0.008	−0.01819160143	−0.01819167764	7.621×10^{-8}
0.010	−0.01819158189	−0.01819167705	9.516×10^{-8}

Table 14. The AEs of $v(x,y,z,t)$ by VIHPM for example 6.2.

t	$v^*(x,y,z,t)$	$v(x,y,z,t)$	$\lvert v^* - v \rvert$
0	−0.00005443257070	−0.00005443257070	0
0.002	−0.00005432382028	−0.00005442930491	1.0548463×10^{-7}
0.004	−0.00005421528708	−0.00005442603912	2.1075204×10^{-7}
0.006	−0.00005410697072	−0.00005442277329	3.1580257×10^{-7}
0.008	−0.00005399887076	−0.00005441950745	4.2063669×10^{-7}
0.010	−0.00005389098678	−0.00005441624161	5.2525483×10^{-7}

Table 15. The AEs of $w(x,y,z,t)$ by VIHPM for example 6.2.

t	$w^*(x,y,z,t)$	$w(x,y,z,t)$	$\lvert w^* - w \rvert$
0	−0.00003628233106	−0.00003628233106	0
0.002	−0.00003620984286	−0.00003628015423	7.031137×10^{-8}
0.004	−0.00003613749944	−0.00003627797737	1.4047793×10^{-7}
0.006	−0.00003606530058	−0.00003627580049	2.1049991×10^{-7}
0.008	−0.00003599324596	−0.00003627362357	2.8037761×10^{-7}
0.010	−0.00003592133528	−0.00003627144664	3.5011136×10^{-7}

results are obtained for approximation and compared with the exact solutions and the results show that we achieve an excellent approximation to the actual solution of the equations by using only two iterations. The results show that these methods are powerful mathematical tools to solving a three-dimensional coupled Burgers' Equation. In our work, we use the Maple to calculate approximate solutions in our systems by using those very efficient methods.

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