Scientific Research Publishing

Application of the Generalized Simplest Equation Method to the Burgers Equation

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Abstract

We successfully constructed wide classes of exact solutions for the Burgers equation by using the generalized simplest equation method. This method yielded a Bäcklund transformation between the Burgers equation and a related constraint equation. By dealing with the constraint equation, we obtained the traveling wave solutions and non-traveling wave solutions of the Burgers equation.

Keywords

The Generalized Simplest Equation Method, Exact Solution, the Burgers Equation

1. Introduction

The investigation of exact solutions to nonlinear evolution equations (NLEEs) plays an important role in the study of nonlinear physical phenomena. In the past several decades, many methods for obtaining exact solutions of NLEEs have been presented. Such as the inverse scattering method [1], the Hirota bilinear method [2], the Bäcklund transformation method [3], the homogeneous balance method [4], the Tanh-function method [5], the F-expansion method [6], the Exp-function method [7], the (G'/G)-expansion method [8] [9], the simplest equation method [10] [11] and other algebraic methods [12] [13] [14]. These methods are very effective for obtaining exact solutions of NLEEs, especially, for the traveling wave solutions.

Particularly, Ma *et al.* introduced the idea of transforming complicated partial differential equations (PDEs) into simple PDEs in the literature [15] and all such expansions, such as the extended tanh-function method and the simplest equation method, are concluded into the transformed rational function method. This method provides a more systematical and convenient handling of the solution

process of NLEEs, unifying the tanh-function type methods, the homogeneous balance method, the exp-function method, the mapping method and the F-expansion type methods [16]. In addition, by using the multiple exp-function method, multiple wave solutions can be generated definitely based on Fourier theory [17]. Other interesting solutions methods include many new bilinear techniques [18] and the invariant subspace method [19]. But up to now, a unified method that can be used to deal with all types of NLEEs has not been discovered. Hence, developing new method and finding more general exact solutions of NLEEs have drawn a lot of interests of a diverse group of scientists.

For obtaining more general form exact solutions of NLEEs, recently we introduced a new direct method called the generalized simplest equation method to look for exact solutions of NLEEs [20]. In the method, we chose a solution expression with a variable coefficient and a variable coefficient ordinary differential auxiliary equation. This method can yield a Bäcklund transformation between NLEEs and a related constraint equation. By dealing with the constraint equation, we can derive an infinite number of exact solutions for NLEEs. These solutions include the traveling wave solutions, non-traveling wave solutions, multi-soliton solutions, rational solutions and other types of solutions. This method has its own advantages [20] which are different from other methods, and it can derive new exact solutions of NLEEs which are not obtained by using other methods [5]-[14]. In this paper, we will use this method to construct more exact solutions of the Burgers equation.

2. The Exact Solutions of the Burgers Equation

Let us consider the Burgers equation

$$u_t + uu_x - cu_{xx} = 0, \tag{1}$$

where c is constant.

2.1. The General Procedure of the Method

First we show the general procedure of the generalized simplest equation method. The method is consisted of the following four steps.

Step 1. We introduce the solutions of Equation (1) that can be written as the following form

$$u(x,t) = a_0(x,t) + a_1(t,x)\phi(\xi), \tag{2}$$

where $a_0(t,x)$, $a_1(t,x)$ will be determined later and the function $\phi = \phi(\xi)$ satisfies a first order ordinary differential Equation (ODE) with variable coefficients:

$$\xi^2\left(\phi' + \phi^2\right) + \delta\xi\phi + v = 0, \tag{3}$$

where δ, v are constants.

Step 2. We substitute (2) along with Equation (3) into Equation (1), and collect all terms with the same order of ϕ. As a result, the left-hand sides of Equation (1) are converted into another polynomials in ϕ. Let each coefficient of

powers of ϕ to be zero to get a set of over-determined PDEs with respect to $a_0(t,x)$, $a_1(t,x)$ and $\xi(t,x)$.

Step 3. Using Mathematica to solve PDEs that have obtained in Step 2, we obtain the following solutions of $a_0(t,x)$, $a_1(t,x)$:

$$a_0(t,x) = \left(c\delta\xi_x^2 - c\xi\xi_{xx} - \xi\xi_t\right)/\xi\xi_x, a_1(t,x) = 2c\xi_x, \qquad (4)$$

where the function $\xi = \xi(t,x)$ satisfies the constraint equation of ξ as follow

$$c^2 s\left(\xi_x^2 - \xi\xi_{xx}\right)\xi_x^4 - \xi^3\left[\xi_x^2\left(c^2\xi_{xxxx} + 2c\xi_{xxt} + \xi_{tt}\right) - 2\left(2c\xi_{xx} + \xi_t\right)\left(\xi_{xt} + c\xi_{xxx}\right)\xi_x \right.$$
$$\left. + \left(\xi_t^2 + 3c^2\xi_{xx}^2 + 4c\xi_t\xi_{xx}\right)\xi_{xx}\right] = 0. \qquad (5)$$

If consider the case of $\xi_x \neq 0$, we can get a more general form of solutions. Any solution of the constraint Equation (5) can be derived a set of coefficients (4) which result in three types of exact solutions (2) of Equation (1), as follows:

$$u_i(t,x) = a_0(t,x) + a_1(t,x)\phi_i(\xi), \qquad (6)$$

where $\phi_i(\xi)$ are three types of general solutions of Equation (3) as follows:

$$\phi_1(\xi) = \varepsilon \cdot \frac{\left[(1-\delta)c_1 - 2\eta c_2\right]\sin\left(\eta\ln|\xi|\right) + \left[2\eta c_1 + (1-\delta)c_2\right]\cos\left(\eta\ln|\xi|\right)}{2\xi\left[c_1\sin\left(\eta\ln|\xi|\right) + c_2\cos\left(\eta\ln|\xi|\right)\right]}, \qquad (7)$$
$$i=1, \ s>1,$$

$$\phi_2(\xi) = \varepsilon \cdot \frac{c_1(2\eta - \delta + 1)|\xi|^{2\eta} - c_2(2\eta + \delta - 1)}{2|\xi|\left(c_1|\xi|^{2\eta} + c_2\right)}, \quad i=2, \ s<1, \qquad (8)$$

$$\phi_3(\xi) = \varepsilon \cdot \frac{(1-\delta)(c_1 + c_2\ln|\xi|) + 2c_2}{2\xi(c_1 + c_2\ln|\xi|)}, \quad i=3, \ s=1, \qquad (9)$$

where c_1, c_2 are constants and

$$s = 2\delta - \delta^2 + 4v, \quad \eta = |1-s|^{1/2}/2, \qquad (10)$$

and if $\xi > 0$, then $\varepsilon = 1$; if $\xi < 0$, then $\varepsilon = -1$.

The solution expression (6) has established a Bäcklund transformation between Equation (1) and the constraint Equation (5). By using the Bäcklund transformation, it is easy for us to obtain infinite number of exact solutions for the Burgers equation. Therefore, if we get the solution of the constraint Equation (5), the more general form exact solutions of Equation (1) will be obtained.

Step 4. In the following, we will solve the constraint Equation (5) and obtain the exact traveling wave and non-traveling wave solutions of Equation (1).

2.2. The Exact Traveling Wave Solutions

In this section, in order to find the exact traveling wave solutions of Equation (1), we define $\xi(t,x) = \xi(x - Vt)$, then Equation (5) becomes the following form

$$\xi(z)^3\left[4\xi'(z)\xi''(z)\xi^{(3)}(z) - \xi'(z)^2\xi^{(4)}(z) - 3\xi''(z)^3\right]$$
$$+ s\left[\xi'(z)^2 - \xi(z)\xi''(z)\right]\xi'(z)^4 = 0 \qquad (11)$$

where $z = x - Vt$. We have given twelve kinds of exact solutions for Equation

(11) in literature [20]. We substitute these solutions $\xi = \xi(x - Vt)$ into (7)-(9), (4) and assemble them in (6), we can get twelve kinds of exact traveling wave solutions of Equation (1). In the following, we only give three kinds of exact travelling wave solution of Equation (1).

Case 1: If $s > 1$, we only give two kinds of exact travelling wave solutions as follows.

Subcase 1.1:

When $d_2^2 + d_3^2 - d_1^2 > 0$, $A = \sqrt{d_2^2 + d_3^2 - d_1^2}$, $A_1 = d_2 + \exp\left[d_3(z + d_4)\delta_0\right]$,

$$\xi(z) = \begin{cases} \varepsilon \exp\left[-\dfrac{2\varepsilon}{\delta_0}\arctan\left(\dfrac{A\tan\left(\dfrac{A}{2}\delta_0(z + d_4\varepsilon)\right) + d_3\varepsilon}{d_1 - d_2}\right)\right], & \text{when } d_1 \neq d_2 \\[4mm] \varepsilon \exp\left[\dfrac{2}{\delta_0}\arccos\left(\dfrac{\pm d_3}{\sqrt{d_2^2 + d_3^2 + \exp(d_3\delta_0(z + d_4))(d_2 + A_1)}}\right)\right], & \\ & \text{when } d_1 = d_2 \end{cases} \qquad (12)$$

(I) If $d_1 \neq d_2$, then we get $a_0(t,x)$, $a_1(t,x)$ as follows by substituting $\xi(z)$ into (4).

$$a_0(t,x) = \mathrm{sech}B_5^2\left[B_3 V + 2A^2 c\varepsilon B_4 + B_2 V\cosh(2B_5)\right.$$
$$\left. + cAB_2\delta_0\sinh(2B_5)\right]\Big/2\left[(d_1 - d_2)^2 + A^2\tanh B_5^2\right] \qquad (13)$$

$$a_1(t,x) = -\frac{2A^2 c(d_1 - d_2)\exp\left(-\dfrac{B_1}{\delta_0}\right)\mathrm{sech}\left(d_3\varepsilon + \dfrac{1}{2}\delta_0 B_0\right)^2}{(d_1 - d_2)^2 + A^2\tanh B_5^2} \qquad (14)$$

where B_0, B_1, B_2, B_3, B_4, B_5 can be expressed as follows:

$$\begin{cases} B_0 = -tV + x + d_4\varepsilon, \ B_1 = -(2\varepsilon/\delta_0)\arctan\left[A\tanh B_5/(d_1 - d_2)\right], \\ B_2 = A^2 + (d_1 - d_2)^2, \ B_3 = -A^2 + (d_1 - d_2)^2, \\ B_4 = d_1 - d_2 - d_1\delta + d_2\delta, \ B_5 = d_3\varepsilon + A\delta_0 B_0/2. \end{cases} \qquad (15)$$

(II) If $d_1 = d_2$, then we get $a_0(t,x)$, $a_1(t,x)$ as follows by substituting $\xi(z)$ into Equation (4).

$$a_0(t,x) = \frac{C_4 + 2\left[-C_0 V + C_2(-V + C_1)\right]d_3^2 \mp d_3 C_5 C_6}{2(C_2 + C_0)(C_2 + C_0 - d_3^2)} \qquad (16)$$

$$a_1(t,x) = \pm\frac{2c\varepsilon d_3^2(A_1 + d_2)\exp\left[C_3 - \dfrac{1}{\delta_0}2\arccos\left(\dfrac{\pm d_3}{\sqrt{C_2 + C_0}}\right)\right]}{C_6(C_2 + C_0)^{3/2}} \qquad (17)$$

where C_0, C_1, C_2, C_3, C_4, C_5, C_6 can be expressed as follows:

$$\begin{cases} C_0 = \exp C_3, \ C_1 = c\left(A_1 + d_2\right)d_3\delta_0, \\ C_2 = d_2^2 + d_3^2, \ C_3 = d_3\delta_0\left(A_1 + d_2\right)\left(d_4 - tV + x\right), \\ C_4 = -C_2C_0\left(-4V + C_1\right) - 2C_2^2\left(-V + C_1\right) + C_0\left(2V + C_1\right), \\ C_5 = 2cd_3C_0\left(A_1 + d_2\right)\sqrt{C_2 + C_0}\left(\delta - 1\right), \ C_6 = \sqrt{1 - d_3^2\big/\left(C_2 + C_0\right)}. \end{cases} \quad (18)$$

Subcase 1.2:

When $d_2^2 + d_3^2 - d_1^2 < 0$, $A^2 = d_1^2 - d_2^2 - d_3^2$, $d_1 \neq d_2$,

$$\xi(z) = \varepsilon \exp\left[\frac{2}{\delta_0} \arctan\left(\frac{A\tan\left(-\frac{A}{2}\delta_0\varepsilon\left(z + d_4\right)\right) + d_3}{d_2 - d_1} \right) \right] \quad (19)$$

By substituting $\xi(z)$ into (4), we get $a_0(t,x)$, $a_1(t,x)$ as follows,

$$\begin{aligned} a_0(t,x) = \frac{1}{2\left(D_0 - A^2\tanh D_2^2\right)} &\Big[2\left(A^2 + D_0\right)V \\ &- A\mathrm{sech}D_2^2\left(2AV + D_1 + c\left(A^2 + D_0^2\right)\delta_0\varepsilon\sinh D_3\right) \Big], \end{aligned} \quad (20)$$

$$a_1(t,x) = -\frac{2A^2c\left(d_1 - d_2\right)\exp\dfrac{2\arctan D_4}{\delta_0}\mathrm{sech}D_2^2}{D_0 + A^2\tanh D_2^2} \quad (21)$$

where D_0, D_1, D_2, D_3, D_4, D_5 can be expressed as follows:

$$\begin{cases} D_0 = \left(d_1 - d_2\right)^2, \ D_1 = 2Ac\left(d_1 - d_2\right)\left(\delta - 1\right)\varepsilon, \\ D_2 = d_3 - \dfrac{1}{2}AD_5\delta_0\varepsilon, \ D_3 = 2d_3 - AD_5\delta_0\varepsilon, \\ D_4 = A\tan D_2\big/\left(d_1 - d_2\right), \ D_5 = d_4 - tV + x. \end{cases} \quad (22)$$

Case 2: If $s = 1$, we only give one kinds of exact travelling wave solution as follows. When $4d_1d_3 - d_2^2 < 0$, $d_3 \neq 0$, $A^2 = d_2^2 - 4d_1d_3$,

$$\xi(z) = \varepsilon \exp\left[-\frac{\varepsilon A\tanh\left(\dfrac{1}{2}A\left(z + d_4\right)\right) + d_2}{2d_3} \right] \quad (23)$$

By substituting $\xi(z)$ into (4), we get $a_0(t,x)$, $a_1(t,x)$ as follows,

$$a_0(t,x) = \frac{4d_3V + Ac\,\mathrm{sech}\left(\dfrac{1}{2}AE_0\right)^2\left[E_2 + 2d_3\sinh\left(AE_0\right)\right]}{4d_3} \quad (24)$$

$$a_1(t,x) = \frac{A^2c\exp E_3}{d_3\left[1 + \cosh\left(AE_0\right)\right]} \quad (25)$$

where E_0, E_1, E_2, E_3 can be expressed as follows:

$$\begin{cases} E_0 = d_4 - tV + x, \ E_1 = AE_0/2, \\ E_2 = A\left(-1 + \delta\right)\varepsilon, \ E_3 = \left(d_2 + A\varepsilon\tanh E_1\right)/2d_3. \end{cases} \quad (26)$$

In all of above cases, by substituting $a_0(t,x)$, $a_1(t,x)$ and the general solu-

tions of ODE (3) into the solution expression (6) respectively, we can obtain the exact traveling wave solutions of Equation (1).

The characteristics of these solutions are multi-layer composite solutions, *i.e.*, they are constructed by compounding several elementary functions. For example, when $s > 1$, the solutions are composed of five elementary functions exp, arctan, tanh, tan, arccos etc. These solutions cannot be obtained by other methods [5] [6] [7] [8]. While $s < 1$ or $s = 1$, the solutions will included rational or irrational solutions and soliton solutions if we select the appropriate parameters. This makes it possible for system of exact solutions for Equation (1) becomes more rich.

2.3. The Exact Non-Traveling Wave Solutions

In fact, Equation (5) can be written as

$$\xi^3 \left[P\left(P + 2c\xi_{xx}\right)\xi_{xx} - 2P_x\left(P + c\xi_{xx}\right)\xi_x + \left(P_t - aP_x + cP_{xx}\right)\xi_x^2 \right] \\ -c^2 s\xi_x^4 \left(\xi_x^2 - \xi\xi_{xx}\right) = 0 \tag{27}$$

where $P = \xi_t + c\xi_{xx} - a\xi_x$ for arbitrary constant a. We consider the following two cases for getting most general form exact solutions.

Case I. For $\xi\xi_{xx} - \xi_x^2 = 0$ and $s = 2\delta - \delta^2 + 4v$.

In this case, we got $\xi(t, x) = d_4 \exp\left[\left(d_2 x + d_3\right)/\left(t + d_1\right)\right]$, which results in the exact solutions of Equation (1) given by (6) with (4) as follows

$$u(t, x) = \frac{cd_2^2(\delta - 1) + d_3 + d_2 x + 2cd_2^2 \xi(t, x)\phi_i(\xi)}{d_2(d_1 + t)} \tag{28}$$

where d_i are arbitrary constants, $\phi_1(\xi)$, $\phi_2(\xi)$ and $\phi_3(\xi)$ are given in (7)-(9).

Case II. For $s = 2\delta - \delta^2 + 4v = 0$.

In this case, Equation (5) or (27) allows the solutions $\xi(t, x)$ of the linear heat equation

$$\xi_t = a\xi_x - c\xi_{xx} \tag{29}$$

with a conductive term $a\xi_x$ for arbitrary constant a.

It is fascinating that a transformation in the form of (6) between Equation (1) and (29) has been found. We will obtain infinite number of exact solutions of Equation (1) by this transformation. In addition, this connection offers us the integrability of Equation (1). For instance, we can make an investigation of the rational solutions, multi-soliton solutions and other forms of solutions to the Burgers equation by using this transformation.

In the Case II, according to the solution expression (6) with $i = 2$, substituting any solution of Equation (29) into (4), we get the exact solutions of Equation (1). For example, corresponding to the solutions of Equation (29), we obtain four exact solutions $(u_j, v_j)(j = 1, 2, 3, 4)$ of Equation (1) as follows.

1. $\xi_1(t, x) = A \exp\left[\left(x + at\right)^2 \big/ 4ct\right]\big/ \sqrt{t} + B, \ (t, x) \in R^2$:

$$u_1(t, x) = \left[Ah_1(t, x)\left[at(\delta - 2) + x\delta\right] - 2aBt^{3/2} \right] \big/ 2\left[Ath_1(t, x) + Bt^{3/2} \right] \\ + A(at + x)h_1(t, x)\phi_2(\xi_1)\big/ t^{3/2} \tag{30}$$

2. $\xi_2(t,x) = A\exp(ck^2 t)\cos[k(x+at)+B]+C,\ (t,x)\in R^2:$

$$u_2(t,x) = -\frac{aC + aAh_2(t,x)\cos\eta_1 + Ack\delta h_2(t,x)\sin\eta_1}{2h_2(t,x)\cos\eta_1 + C} \qquad (31)$$
$$- 2ckAh_2(t,x)\sin\eta_1\phi_2(\xi_2).$$

3. $\xi_3(t,x) = A\exp[-k(x+at)]\cos[kx+k(a+2ck)t+B]+C,\ (t,x)\in R^2:$

$$u_3(t,x) = -\frac{aC + A[(a+ck\delta)\cos\eta_2 + ck\delta\sin\eta_2]h_3(t,x)}{C + Ah_3(t,x)\cos\eta_2} \qquad (32)$$
$$- 2ckAh_3(t,x)h_4(t,x)\phi_2(\xi_3).$$

4. $\xi_4(t,x) = c_0\,erfc\left(\dfrac{x+at}{2\sqrt{-ct}}\right),\ t>0,\ x+at>0:$

$$u_4(t,x) = -a + \frac{\delta\sqrt{-ct}}{\sqrt{\pi}th_1(t,x)erfc(\eta_3)} - \frac{2cc_0}{\sqrt{-ct}\pi h_1(t,x)}\phi_2(\xi_4). \qquad (33)$$

In all of above cases, $\phi_2(\xi)$ which is given in (8), and A, B, C, c_0 are costants,

$$\begin{cases} \eta_1 = B + k(x+at), \\ \eta_2 = B + kx + kt(a+2kc), \\ \eta_3 = (x+at)/(2\sqrt{-ct}). \end{cases} \qquad (34)$$

$$\begin{cases} h_1(t,x) = \exp[(x+at)^2/4ct],\ h_2(t,x) = \exp(ck^2 t), \\ h_3(t,x) = \exp[-k(at+x)], \quad h_4(t,x) = \sin\eta_2 + \cos\eta_2. \end{cases} \qquad (35)$$

By the same manner, from the rest two solutions of Equation (29) which are given in literature [20], we should obtain additional exact solutions in different styles to the Burgers equation. We have also obtained the multi-soliton solutions and the rational solutions of the Burgers equation by using the method. Due to the lack of space, we omit the reasoning and the solution expressions.

3. Conclusions

In this paper, we obtained abundant exact solutions of the Burgers equation by using the generalized simplest equation method. We found a Bäcklund transformation between the Burgers equation and the constraint equation in the process of calculation. By dealing with the constraint equation, we obtained the exact solutions of Burgers equation. These solutions include the traveling wave solutions and non-traveling wave solutions. The obtained results illustrate that our method is effective and it is meaningful for further research.

When $s=0$, we obtained a connection between the constraint Equation (5) and the linear heat equation; therefore more exact solutions of the Burgers equation are constructed. But we found new constraint equation which is different from Equation (5) to other NLEEs in the study of this method. Can we construct new exact solutions to NLEEs because of these new constraint equations? It is an excellent topic for further research.

Acknowledgements

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