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GLOBAL EXISTENCE AND DECAY ESTIMATES FOR NONLINEAR KIRCHHOFF–TYPE EQUATION WITH BOUNDARY DISSIPATION

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Abstract. In this paper, we consider the initial-boundary value problem for nonlinear Kirchhoff-type equation

$$u_{tt}-\varphi(\|\nabla u\|_2^2)\Delta u-a\Delta u_t=b|u|^{\beta-2}u,$$

where $a,b>0$ and $\beta>2$ are constants, φ is a C^1-function such that $\varphi(s)\geqslant\lambda_0>0$ for all $s\geqslant 0$. Under suitable conditions on the initial data, we show the existence and uniqueness of global solution by means of the Galerkin method and the uniform decay rate of the energy by an integral inequality.

1. Introduction

In this paper, we consider the problem

$$\begin{cases} u_{tt}-\varphi(\|\nabla u\|_2^2)\Delta u-a\Delta u_t=b|u|^{\beta-2}u & \text{in}\,\Omega\times(0,\infty),\\ u(x,t)=0 & \text{on}\,\Gamma_1\times(0,\infty),\\ \varphi(\|\nabla u\|_2^2)\frac{\partial u}{\partial\nu}+a\frac{\partial u_t}{\partial\nu}=g(u_t) & \text{on}\,\Gamma_0\times(0,\infty),\\ u(x,0)=u_0,u_t(x,0)=u_1 & \text{in}\,\Omega, \end{cases}\tag{1.1}$$

where Ω is a bounded domain of $R^n(n\geqslant 1)$ with smooth boundary $\Gamma:=\partial\Omega$ such that $\Gamma=\Gamma_0\cup\Gamma_1$ and Γ_0,Γ_1 have positive measures, ν is the unit outward normal on $\partial\Omega$, and $\frac{\partial}{\partial\nu}$ is the outward normal derivative on $\partial\Omega$.

The case of $n=1$, Eq.(1.1) describes the nonlinear vibrations of an elastic string. The original equation is

$$\rho hu_{tt}-a\Delta u_t=\big(p_0+\frac{Eh}{2L}\int_0^L(u_x)^2dx\big)u_{xx}+f$$

for $0<x<L,\ t\geqslant 0$, where $u=u(x,t)$ is the lateral displacement at the space coordinate x and the time t, E the Young modulus, ρ the mass density, h the cross-section area, L the length, p_0 the initial axial tension, a the resistance modulus, and f the

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external force. When $a = f = 0$, the equation is firstly introduced by Kirchhoff [18], and is called the Kirchhoff string after his name.

Physically, the first integro-differential equation in (1.1) occurs in the study of vibrations of damped flexible space structures in a bounded domain in R^n. The term $a\Delta u_t$ is the internal material damping of Kelvin-Voigt type of the structure. In fact, the most common class for the suppression of vibrations of elastic structure is of passive type which absorbs vibration energy. On the other hand, the internal damping mechanism is always present, however small it may be, in real materials so long as the system vibrates (see [11]). The boundary conditions considered here are of mixed Dirichlet and Neumann type. When $g(s)s \leqslant 0$, the term $g(u_t)$ exhibit a boundary dissipative effect, and we may expect certain decay properties of the solutions under suitable assumptions. Moreover, the difficulty increases in the case that the blow-up term $f(u) = b|u|^{\beta-2}u$ appears because semilinear wave equations including blow-up terms may cause certain blow-up phenomena. Our central aim is to show the uniform decay rate of the energy under suitable assumptions on g, β and initial energy.

The *homogeneous Dirichlet boundary value* problems for Kirchhoff-type equations have been considered by many mathematicians (see [1, 14, 27, 28, 30, 33] and [2, 6, 12, 16, 17, 26, 29, 32, 35, 36, 37, 38, 39, 40]). K. Nishihara and Y. Yamada [27] considered the global solvability of the homogeneous Dirichlet boundary value problem for

$$u_{tt} - a\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u + 2\gamma u_t = 0 \text{ in}\,\Omega \times [0,\infty)$$

and showed the global existence, uniqueness and asymptotic decay of solutions provided that the initial datas u_0 $(u_0 \neq 0)$ and u_1 are small and u_1 is much smaller than u_0 in some sense. M. Aassila and A. Benaissa [1] extended the global existence part of [11] to the case where $\varphi(s) > 0$ with $\varphi(\|\nabla u_0\|^2) \neq 0$ and the nonlinear dissipative term $|u_t|^{\alpha-2}u_t$. K. Ono [28] and Ye [33] obtained the global existence of the solution to the homogeneous Dirichlet boundary value problem for

$$u_{tt} - \varphi(\|\nabla u\|_2^2)\Delta u - au_t = b|u|^{\beta-2}u \text{ in}\,\Omega \times (0,\infty),$$

where $a,b > 0$ and $\beta > 2$ are constants, $\varphi(s)$ is a C^1 -class function on $[0,+\infty)$ satisfying

$$\varphi(s) \geqslant m_0,\, s\varphi(s) \geqslant \int_0^s \varphi(\tau)d\tau,\, \forall s \in [0,\infty)$$

with $m_0 \geqslant 1$. Using Galerkin method, K. Ono and K. Nishihara [30] proved the global existence and decay structure of solutions of the homogeneous Dirichlet boundary value problem for

$$u_{tt} - \varphi(\|\nabla u\|_2^2)\Delta u - a\Delta u_t = b|u|^{\beta-2}u \text{ in}\,\Omega \times (0,\infty)$$

without small condition of data. Applying the Banach contraction mapping principle, Li et al. [14] obtained the local existence of the solution to the homogeneous Dirichlet boundary value problem for the higher-order nonlinear Kirchhoff-type equation

$$u_{tt} + M(\|D^m u(t)\|_2^2)(-\Delta)^m u + |u_t|^{q-2}u_t = |u|^{p-2}u,$$

where $p > q \geqslant 2, m \geqslant 1$.

The *mixed Dirichlet and Neumann homogenous boundary value* problems for Kirchhoff-type equations have also been considered, for example [5, 15, 20, 23, 25]. Ganesh C. Gorain [15] studied the uniform stability of two mixed Dirichlet and Neumann homogenous boundary value problems for

$$u_{tt} + 2\delta u_t = \big(a^2 + b\int_\Omega |\nabla u|^2 dx\big)\Delta u \text{ in}\,\Omega \times (0,\infty)$$

and

$$u_{tt} = \big(a^2 + b\int_\Omega |\nabla u|^2 dx\big)\Delta u + 2\lambda \Delta u_t \text{ in}\,\Omega \times (0,\infty).$$

Li [20] and Salim A. Messaoudi et al. [25] investigated global existence and blow-up properties of the solution for the following higher-order Kirchhoff-type equation with Dirichlet and Neumann homogenous boundary conditions

$$u_{tt} + \big(\int_\Omega |D^m u|^2 dx\big)^q (-\Delta u)^m + u_t|u_t|^r = |u|^p u,\ x \in \Omega,\ t > 0.$$

With $m > 1$ is a positive integer and $q, p, r > 0$ are positive constants, Li [20] obtained that the solution exists globally if $p \leqslant r$, while if $p > max\{r, 2q\}$, then for any initial data with negative initial energy, the solution blows up at finite time in L^{p+2} norm. With $m \geqslant 1$ and $q, p, r \geqslant 0$, Salim A. Messaoudi et al. [25] established a blow-up result for certain solutions with positive initial energy.

Besides, Vitillaro [34] considered the following problem

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \Omega \times (0,\infty), \\ u = 0 \text{ on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} + |u_t|^{m-2}u_t = |u|^{p-2}u, \text{ on } \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0, \quad u_t(x,0) = u_1 \text{ in } \Omega \end{cases}$$

and proved local existence of the solution in the energy space when $m > \frac{r}{r+1-p}$ or $n = 1, 2$, where $r = \frac{2(n-1)}{n-2}$, and global existence when $p \leqslant m$ or the initial data was chosen suitably. Cavalcanti and Guesmia [8] considered the following system

$$\begin{cases} u_{tt} - \Delta u + F(x,t,u,\nabla u) = 0 \text{ in } \Omega \times (0,\infty), \\ u = 0 \text{ on } \Gamma_0 \times (0,\infty), \\ u(x,t) = -\displaystyle\int_0^t g(t-s)\frac{\partial u}{\partial \nu} \text{ on } \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0, \quad u_t(x,0) = u_1 \text{ in } \Omega. \end{cases}$$

Cavalcanti et al. [9] studied a problem of the form

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = 0 \,\text{in}\, \Omega \times (0,\infty), \\ u = 0 \text{ on } \Gamma_0 \times (0,\infty), \\ \frac{\partial u}{\partial \nu} - \int_0^t h(t-\tau)\frac{\partial u}{\partial \nu}d\tau + h(u_t) = 0 \text{ on } \Gamma_0 \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \text{ in } \Omega \end{cases} \tag{1.2}$$

for g,h specific functions and established uniform decay rate results under quite restrictive assumptions on both the damping function h and the kernel g. In fact, the function g had to behave exactly like e^{-mt} and the function h had a polynomial behavior near zero. For more general assumptions on g and h, Cavalcanti et al. [10] proved the uniform stability of (1.2) provided that $g(0)$ and $\|g\|_{L^1(0,+\infty)}$ are small enough. They also established explicit decay rate results for some special cases. Lu et al. [24] considered the following wave equation with nonlinear viscoelastic term

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = 0 \,\text{in}\, \Omega \times (0,\infty), \\ u = 0 \text{ on } \Gamma_0 \times (0,\infty), \\ \frac{\partial u}{\partial \nu} - \int_0^t h(t-\tau)\frac{\partial u}{\partial \nu}d\tau + |u_t|^{m-2}u_t = |u|^p u \text{ on } \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \text{ in } \Omega \end{cases}$$

with $m \geqslant 2, p \geqslant 2$. Under some appropriate assumptions on g and with certain initial data, global existence of solutions and a general decay for the energy have been established. Li et al. [21] considered the problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t h(t-\tau)\text{div}\big(a(x)\nabla u(\tau)\big)d\tau + |u|^\gamma u = 0 \,\text{in}\, \Omega \times (0,\infty), \\ u = 0 \text{ on } \Gamma_1 \times (0,\infty), \\ \frac{\partial u}{\partial \nu} - \int_0^t h(t-\tau)\big(a(x)\nabla u(\tau)\big)\cdot \nu d\tau + g(u_t) = 0 \text{ on } \Gamma_0 \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \text{ in } \Omega. \end{cases}$$

They proved the existence and uniqueness of global solution by means of the Galerkin method, and showed the uniform decay rate of the energy under suitable conditions on the initial data and the relaxation function.

Motivated by the above work, we intend to study the global existence of the solution and the decay estimate of the energy for problem (1.1). By using the potential well method, we prove that under some conditions on φ, g and β, the solution exists globally and the general decay rate is obtained. The main contributions of this paper are: (a) the problem considered in this paper is nonlinear equation with mixed inhomogeneous boundary dispassion and this problem is representative; (b)the estimates are precise and the proofs are understood easily; (c)the method to prove the existence of the global solution in [21] can not be applied directly to the case in the problem (1.1).

The rest of this paper is organized as follows. In Section 2, we give the preliminaries and our main results. In Section 3, we prove the existence of a global solution to problem (1.1). Section 4 is devoted to prove the decay result.

2. Preliminaries and main results

In this section, we present some notations and the general hypotheses that will be used throughout the paper, and then we state the main results.

Let $H^m(\Omega)$ denote the Sobolev space with the norm

$$\|u\|_{H^m(\Omega)} = \sum_{|\alpha|\leqslant m} \|D^\alpha u\|^2_{L^2(\Omega)}.$$

For simplicity of notations, hereafter we denote by $\|\cdot\|_p$ the Lebesgue space $L^p(\Omega)$ norm, $\|\cdot\|$ the Lebesgue space $L^2(\Omega)$ norm. Moreover, C, $C_i\,(i=1,2\cdots)$ denote various positive constants and they may be different at each appearance. Throughout this paper, we define

$$V := H^1_{\Gamma_1}(\Omega) = \{u | u \in H^1(\Omega), u = 0 \,\text{on}\, \Gamma_1\},$$

and the following scalar products

$$(u,v) = \int_\Omega u(x)v(x)dx,\ (u,v)_{\Gamma_0} = \int_{\Gamma_0} u(x)v(x)d\Gamma.$$

In order to define the energy functional E of the problem (1.1), we give the following computation. Multiplying the first equation in (1.1) by u_t and integrating the result over Ω and adding Green's formula, we get

$$\begin{aligned}
&\int_\Omega u_t\left(u_{tt} - \varphi(\|\nabla u\|^2)\Delta u - a\Delta u_t - b|u|^{\beta-2}u\right)dx \\
&= \frac{1}{2}\frac{d}{dt}\|u_t\|^2 - \int_\Gamma u_t\varphi(\|\nabla u\|^2)\frac{\partial u}{\partial \nu}d\Gamma + \int_\Omega \varphi(\|\nabla u\|^2)\nabla u\nabla u_t dx - \int_\Gamma a u_t\frac{\partial u_t}{\partial \nu}d\Gamma \\
&\quad + \int_\Omega a\nabla u_t\nabla u_t dx - \frac{b}{\beta}\frac{d}{dt}\|u\|^\beta_\beta \\
&= \frac{1}{2}\frac{d}{dt}\{\|u_t\|^2 + \int_0^{\|\nabla u\|^2}\varphi(s)ds - \frac{2b}{\beta}\|u\|^\beta_\beta\} + a\|\nabla u_t\|^2 - (u_t, g(u_t))_{\Gamma_0} = 0. \qquad (2.1)
\end{aligned}$$

Then (2.1) inspires us to define the energy functional as

$$E(u;t) = \|u_t\|^2 + \int_0^{\|\nabla u\|^2}\varphi(s)ds - \frac{2b}{\beta}\|u\|^\beta_\beta =: \|u_t\|^2 + J(u), \qquad (2.2)$$

where

$$J(u) = \int_0^{\|\nabla u\|^2}\varphi(s)ds - \frac{2b}{\beta}\|u\|^\beta_\beta.$$

Clearly,

$$E(u;0) = \|u_1\|^2 + \int_0^{\|\nabla u_0\|^2} \varphi(s)ds - \frac{2b}{\beta}\|u_0\|_\beta^\beta.$$

In order to establish our results, we apply the potential well theory. Define

$$W = \Big\{u \in V | I(u) = \int_0^{\|\nabla u\|^2} \varphi(s)ds - b\|u\|_\beta^\beta > 0, J(u) < d\Big\} \cup \{0\},$$

where

$$d = \inf\Big\{\sup_{\lambda>0} J(\lambda u), u \in V \setminus \{0\}\Big\}.$$

Remark 1. We call that the constant d is saddle point value of functional J. □

Before stating the general hypotheses, we firstly give the following Lemma.

LEMMA 1. *If* $2 \leqslant q \leqslant \frac{2n}{n-2}(n > 2)$ *or* $2 \leqslant q < \infty(n = 1,2)$ *holds, then there exists a positive constant* C_* *depending on* Ω *and* q *such that*

$$\|u\|_q \leqslant B\|u\|_V \leqslant C_*\|\nabla u\|_{L^2(\Omega)}, \forall u \in V.$$

Proof. In fact, by the Sobolev embedding theorem, we get $V \hookrightarrow L^q(\Omega)$ and $\|u\|_q \leqslant B\|u\|_V$. And by Poincaré inequality, we know that $\|u\|_V$ is equivalent to $\|\nabla u\|_{L^2}$. Then the result follows. □

Now the general hypotheses are as follows. □

(A_1) $\varphi : \varphi \in C^1([0,\infty);R^+)$ is a function satisfying

$$\varphi(s) \geqslant \lambda_0 > 0, s\varphi(s) \geqslant \int_0^s \varphi(\theta)d\theta$$

for all $s \geqslant 0$. For example, $\varphi(s) = \lambda_0 + s^r$, $r \geqslant 1$.

(A_2) g is a non-increasing continuous differentiable function with bounded derivative and satisfies that there exists a positive constant C such that

$$sg(s) \leqslant 0, \ |g(s)| \leqslant C|s|$$

for all $s \in R$.

(A_3) β satisfies

$$2 < \beta \leqslant \frac{2n-2}{n-2}(n > 2) \text{ or } 2 < \beta < \infty(n = 1,2).$$

(A_4) $E(u;0)$ satisfies

$$\rho := \frac{b}{\lambda_0^{\frac{\beta}{2}}} C_*^\beta \cdot \Big(\frac{\beta}{\beta-2}E(u;0)\Big)^{\frac{\beta-2}{2}} < 1 \text{ and } E(u;0) < d,$$

where C_* is the constant in Lemma 1. □

Remark 2. It is clear that assumption $|g(s)| \leqslant C|s|$ implies $g(0)=0$; moreover, as it is supposed in (A_2), if g is differentiable with bounded derivative and $g(0)=0$, then integrating $-M \leqslant g'(s) \leqslant M$ over $[0,s]$, we obtain $-Ms \leqslant g(s) \leqslant Ms$, that is, $|g(s)| \leqslant C|s|$. So it is enough in (A_2) to assume $g(0)=0$.

LEMMA 2. *Let $u \in V$. If $(A_1),(A_3)$ hold, then $d>0$.*

Proof. Since

$$J(u)=\int_0^{\|\nabla u\|^2}\varphi(s)ds-\frac{2b}{\beta}\|u\|_\beta^\beta \geqslant \lambda_0\|\nabla u\|^2-\frac{2b}{\beta}\|u\|_\beta^\beta,$$

we only need to prove

$$\sup_{\lambda>0}\widehat{J}(\lambda u)>0, \forall u \in V\setminus\{0\},$$

where

$$\widehat{J}(u)=\lambda_0\|\nabla u\|^2-\frac{2b}{\beta}\|u\|_\beta^\beta.$$

Let $\frac{d}{d\lambda}\widehat{J}(\lambda u)=0$, then $\lambda_1=\big(\lambda_0\frac{\|\nabla u\|^2}{b\|u\|_\beta^\beta}\big)^{\frac{1}{\beta-2}}$. A simple calculation and using Lemma 1, we get

$$\frac{d^2}{d\lambda^2}\widehat{J}(\lambda u)\Big|_{\lambda=\lambda_1}<0,$$

and

$$\begin{aligned}\widehat{J}(\lambda u)|_{\lambda=\lambda_1}&=\big(1-\frac{2}{\beta}\big)\big(\frac{\lambda_0^\beta}{b}\big)^{2/(\beta-2)}\big(\frac{\|\nabla u\|}{\|u\|_\beta}\big)^{2\beta/(\beta-2)}\\&\geqslant\big(1-\frac{2}{\beta}\big)\big(\frac{\lambda_0^\beta}{b}\big)^{2/(\beta-2)}(C_*)^{-2\beta/(\beta-2)}>0,\end{aligned}$$

where C_* is the constant in Lemma 1. □

LEMMA 3. *Assume Ω is bounded and $\partial\Omega$ is C^1. Then*

$$\|u\|_{L^p(\partial\Omega)} \leqslant C\|u\|_{W^{1,p}(\Omega)}$$

for each $u \in W^{1,p}(\Omega)$, with the constant C depending only on p and Ω.

Proof. The proof can be found in [13]. □

We state our results as follows.

THEOREM 1. *Let* $(u_0,u_1)\in\big(W\cap H^2\big)\times V$. *If the hypotheses* (A_1)-(A_4) *hold, then there exists a unique solution* $u(x,t)$ *of the problem (1.1) satisfying*

$$u\in L^\infty_{loc}(0,\infty;V\cap H^2),\ u_t\in L^\infty_{loc}\big(0,\infty;V\big),\ u_{tt}\in L^2_{loc}\big(0,\infty;L^2(\Omega)\big).$$

Moreover, we have

$$u\in C\big([0,\infty);V\big),\ u_t\in C\big([0,\infty);L^2(\Omega)\big).$$

THEOREM 2. *Let* $(u_0,u_1)\in(W\cap H^2)\times V$. *If the hypotheses* (A_1)-(A_4) *hold, then the global solution of problem (1.1) has the following exponential decay property*

$$E(u;t)\leqslant E(u;0)e^{1-\frac{t}{C}},$$

where $C>0$ *is a constant.*

3. Proof of Theorem 1

In this section, using the Galerkin method(a reference for this method, for instance, the book by professor J. L. Lions [22]), we show the existence and uniqueness of the global solution to the problem (1.1). We choose a basis $\{w_k\}$ $(k=1,2,\cdots)$ in $V\cap H^2(\Omega)$ which is orthonormal in $L^2(\Omega)$ and let V_m the subspace of $V\cap H^2(\Omega)$ generated by the first m vectors.

Define

$$u_m(t)=\sum_{k=1}^{m}d_m^k(t)w_k, \tag{3.1}$$

where $u_m(t)$ is the solution of the following Cauchy problem

$$\begin{aligned}\big(u_m''(t),w_k\big)+\big(\varphi(\|\nabla u_m(t)\|^2)\nabla u_m(t),\nabla w_k\big)+a\big(\nabla u_m'(t),\nabla w_k\big)&\\ =\big(g(u_m'(t)),w_k\big)_{\Gamma_0}+(b|u_m(t)|^{\beta-2}u_m(t),w_k),&\end{aligned} \tag{3.2}$$

with the initial conditions

$$\begin{cases}u_m(0)=\sum_{k=1}^{m}(u_m(0),w_k)w_k\to u_0\,\text{in}\,V\cap H^2(\Omega),\\ u_m'(0)=\sum_{k=1}^{m}(u_m'(0),w_k)w_k\to u_1\,\text{in}\,V.\end{cases} \tag{3.3}$$

Note that we can solve the system (3.2)-(3.3). In fact the problems (3.2)-(3.3) have a unique continuous solution on some interval $[0,T_m)$. The extension of the solution to the whole interval $[0,\infty)$ is a consequence of the estimates which we are going to prove below. □

Step 1. (The first priori estimate) Multiplying ${d_m^k}'(t)$ on both sides of equation (3.2) and summing up the resulting equations from $k=1$ to $k=m$, we have

$$\big(u_m''(t),u_m'(t)\big)+\big(\varphi(\|\nabla u_m(t)\|^2)\nabla u_m(t),\nabla u_m'(t)\big)+a\big(\nabla u_m'(t),\nabla u_m'(t)\big)$$

$$= \big(g(u_m'(t)), u_m'(t)\big)_{\Gamma_0} + (b|u_m(t)|^{\beta-2}u_m(t), u_m'(t)),$$

that is

$$\frac{d}{dt}\Big(\frac{1}{2}\|u_m'(t)\|^2 + \frac{1}{2}\int_0^{\|\nabla u_m(t)\|^2} \varphi(s)ds - \frac{b}{\beta}\|u_m(t)\|_\beta^\beta\Big) + a\|\nabla u_m'(t)\|^2$$
$$= \big(g(u_m'(t)), u_m'(t)\big)_{\Gamma_0}. \quad (3.4)$$

So by the hypothesis (A_2) and $a > 0$, we have $\frac{d}{dt}E(u_m;t) \leqslant 0$, i.e., $E(u_m;t)$ is non-increasing respect to t.

Integrating (3.4) over $(0,t)$ and owing to (A_2),

$$\begin{aligned}
&\frac{1}{2}\|u_m'(t)\|^2 + \frac{1}{2}\int_0^{\|\nabla u_m(t)\|^2} \varphi(s)ds - \frac{b}{\beta}\|u_m(t)\|_\beta^\beta + a\int_0^t \|\nabla u_m'(s)\|^2 ds \\
&= \frac{1}{2}\|u_m'(0)\|^2 + \frac{1}{2}\int_0^{\|\nabla u_m(0)\|^2} \varphi(s)ds - \frac{b}{\beta}\|u_m(0)\|_\beta^\beta + \int_0^t \big(g(u_m'(s)), u_m'(s)\big)_{\Gamma_0} ds \\
&\leqslant \frac{1}{2}\|u_m'(0)\|^2 + \frac{1}{2}\int_0^{\|\nabla u_m(0)\|^2} \varphi(s)ds - \frac{b}{\beta}\|u_m(0)\|_\beta^\beta. \qquad (3.5)
\end{aligned}$$

In order to perform the prior estimation, we give the following lemma.

LEMMA 4. *Assume* (A_1)-(A_4), $\big(u_0, u_1\big) \in (W \cap H^2) \times V$. *Then* $u_m(x,t) \in W$. *That is,*

$$J(u_m) < d,\ I(u_m) = \int_0^{\|\nabla u_m\|^2} \varphi(s)ds - b\|u_m\|_\beta^\beta > 0$$

for each $t \in [0,\infty)$.

Proof. Firstly, we prove $J(u_m) < d$. In fact, since $E(u;0) < d$, there exists a sufficient small constant ε_0 such that $E(u;0) + \varepsilon_0 < d$. For ε_0 mentioned above, by (3.3) and the continuity of $E(u_m;0)$, we have

$$E(u_m;0) \leqslant E(u;0) + \varepsilon_0 < d$$

for sufficient large m. So, owing to the definition of $J(u_m)$ and $E(u_m;t)$ is non-increasing respect to t, for each $t \in [0,\infty)$,

$$J(u_m) = \int_0^{\|\nabla u_m\|^2} \varphi(s)ds - \frac{2b}{\beta}\|u_m\|_\beta^\beta \leqslant E\big(u_m;t\big) \leqslant E(u_m;0) < d. \quad (3.6)$$

Now we prove $I(u_m) > 0$. In fact, fix $T > 0$ arbitrarily. By (3.3) and $I(u_0) > 0$, we have $I(u_m(0)) > 0$ for sufficient large m. Considering again the continuity of u_m respect to t, we have

$$I(u_m(t)) > 0, \text{ for some interval near } t = 0, \quad (3.7)$$

that is, there exists t_m $(t_m < T)$ such that (3.7) holds on $[0,t_m]$.

Note that

$$J(u_m) = \int_0^{\|\nabla u_m\|^2} \varphi(s)ds - \frac{2b}{\beta}\|u_m\|_\beta^\beta = \frac{2}{\beta}I(u_m) + \frac{\beta-2}{\beta}\int_0^{\|\nabla u_m\|^2} \varphi(s)ds$$
$$\geqslant \frac{\beta-2}{\beta}\int_0^{\|\nabla u_m\|^2} \varphi(s)ds,\ t\in[0,t_m].$$

Hence, we have $\forall t \in [0,t_m]$,

$$\lambda_0\|\nabla u_m\|^2 \leqslant \int_0^{\|\nabla u_m\|^2} \varphi(s)ds \leqslant \frac{\beta}{\beta-2}J(u_m) \leqslant \frac{\beta}{\beta-2}E(u_m;t) \leqslant \frac{\beta}{\beta-2}E(u_m;0).$$

And then

$$\begin{aligned} b\|u_m\|_\beta^\beta &\leqslant bC_*^\beta\|\nabla u_m(t)\|^\beta \\ &= \frac{b}{\lambda_0^{\frac{\beta}{2}}}C_*^\beta\left(\lambda_0\|\nabla u_m(t)\|^2\right)^{\frac{\beta}{2}} \\ &\leqslant \frac{b}{\lambda_0^{\frac{\beta}{2}}}C_*^\beta\cdot\left(\frac{\beta}{\beta-2}E(u_m;0)\right)^{\frac{\beta-2}{2}}\int_0^{\|\nabla u_m\|^2}\varphi(s)ds \\ &< \int_0^{\|\nabla u_m\|^2}\varphi(s)ds,\ \forall t\in[0,t_m]. \end{aligned}$$

Therefore $I(u_m) > 0$ on $[0,t_m]$. By repeating this procedure, and using the fact that

$$\lim_{t\to t_m}\frac{b}{\lambda_0^{\frac{\beta}{2}}}C_*^\beta\cdot\left(\frac{\beta}{\beta-2}E(u_m;t)\right)^{\frac{\beta-2}{2}} < 1,$$

for sufficient large m, t_m is extended to T. Owing the arbitrarity of T, we get the conclusion. The proof of Lemma 4 is completed. □

Remark 3. The idea of proof in Lemma 4 can reference Lemma 4.2 in [7]. □

From Lemma 4, we have

$$\begin{aligned} &\frac{1}{2}\|u_m'(t)\|^2 + \frac{1}{2}\int_0^{\|\nabla u_m(t)\|^2}\varphi(s)ds - \frac{b}{\beta}\|u_m(t)\|_\beta^\beta \\ &= \frac{1}{2}\|u_m'(t)\|^2 + \frac{1}{\beta}I\big(u_m(t)\big) + \frac{\beta-2}{2\beta}\int_0^{\|\nabla u_m(t)\|^2}\varphi(s)ds \\ &\geqslant \frac{1}{2}\|u_m'(t)\|^2 + \frac{\beta-2}{2\beta}\int_0^{\|\nabla u_m(t)\|^2}\varphi(s)ds. \end{aligned} \tag{3.8}$$

By (3.3), (3.5) and (3.8), we have

$$\frac{1}{2}\|u_m'(t)\|^2 + \frac{\beta-2}{2\beta}\int_0^{\|\nabla u_m(t)\|^2}\varphi(s)ds + a\int_0^t\|\nabla u_m'(s)\|^2ds$$

$$\leqslant \frac{1}{2}\|u_m'(0)\|^2 + \frac{1}{2}\int_0^{\|\nabla u_m(0)\|^2} \varphi(s)ds - \frac{b}{\beta}\|u_m(0)\|_\beta^\beta \leqslant K_1,$$

that is

$$\frac{1}{2}\|u_m'(t)\|^2 + \frac{\beta-2}{2\beta}\lambda_0\|\nabla u_m(t)\|^2 + a\int_0^t \|\nabla u_m'(s)\|^2 ds \leqslant K_1, \tag{3.9}$$

where K_1 is a constant independent of m. □

Step 2. (The second priori estimate) Multiplying $d_m^{k\,\prime\prime}(t)$ on both sides of equation (3.2) and summing up the resulting equations from $k=1$ to $k=m$, we have

$$\begin{aligned}\big(u_m''(t), u_m''(t)\big) + \big(\varphi(\|\nabla u_m(t)\|^2)\nabla u_m(t), \nabla u_m''(t)\big) + a\big(\nabla u_m'(t), \nabla u_m''(t)\big)\\ = \big(g(u_m'(t)), u_m''(t)\big)_{\Gamma_0} + \big(b|u_m(t)|^{\beta-2}u_m(t), u_m''(t)\big),\end{aligned}$$

that is,

$$\begin{aligned}\|u_m''(t)\|^2 + \big(\varphi(\|\nabla u_m(t)\|^2)\nabla u_m(t), \nabla u_m''(t)\big) + \frac{a}{2}\frac{d}{dt}\|\nabla u_m'(t)\|^2\\ = \big(g(u_m'(t)), u_m''(t)\big)_{\Gamma_0} + \big(b|u_m(t)|^{\beta-2}u_m(t), u_m''(t)\big).\end{aligned} \tag{3.10}$$

On the other hand,

$$\begin{aligned}\big(\varphi(\|\nabla u_m(t)\|^2)\nabla u_m(t), \nabla u_m''(t)\big) = \frac{d}{dt}\big(\varphi(\|\nabla u_m(t)\|^2)\nabla u_m(t), \nabla u_m'(t)\big)\\ - 2\varphi'\big(\|\nabla u_m(t)\|^2\big)\left|\big(\nabla u_m'(t), \nabla u_m(t)\big)\right|^2 - \varphi\big(\|\nabla u_m(t)\|_2^2\big)\|\nabla u_m'(t)\|^2\end{aligned} \tag{3.11}$$

and

$$\big(g(u_m'(t)), u_m''(t)\big)_{\Gamma_0} = \frac{d}{dt}\big(g(u_m'(t)), u_m'(t)\big)_{\Gamma_0} - \big(g'(u_m'(t))u_m''(t), u_m'(t)\big)_{\Gamma_0}. \tag{3.12}$$

Thus (3.10)-(3.12) imply

$$\begin{aligned}&\|u_m''(t)\|^2 + \frac{d}{dt}\big(\varphi(\|\nabla u_m(t)\|^2)\nabla u_m(t), \nabla u_m'(t)\big) + \frac{a}{2}\frac{d}{dt}\|\nabla u_m'(t)\|^2\\ &\quad = 2\varphi'(\|\nabla u_m(t)\|^2)|\big(\nabla u_m(t), \nabla u_m'(t)\big)|^2 + \varphi(\|\nabla u_m(t)\|^2)\|\nabla u_m'(t)\|^2\\ &\qquad + \frac{d}{dt}\big(g(u_m'(t)), u_m'(t)\big)_{\Gamma_0} - \big(g'(u_m'(t))u_m''(t), u_m'(t)\big)_{\Gamma_0} + \big(b|u_m(t)|^{\beta-2}u_m(t), u_m''(t)\big).\end{aligned} \tag{3.13}$$

From (3.9) and the hypothesis (A_1),

$$\begin{aligned}&2\varphi'\big(\|\nabla u_m(t)\|^2\big)|\big(\nabla u_m(t), \nabla u_m'(t)\big)|^2 \leqslant C_1\|\nabla u_m'(t)\|^2;\\ &\varphi\big(\|\nabla u_m(t)\|^2\big)\|\nabla u_m'(t)\|_2^2 \leqslant C_2\|\nabla u_m'(t)\|^2.\end{aligned} \tag{3.14}$$

The hypothesis (A_2) and Lemma 3 imply

$$\begin{aligned}\left|\left(g'(u_m'(t))u_m''(t),u_m'(t)\right)_{\Gamma_0}\right| &\leqslant C_3\left|\left(u_m''(t),u_m'(t)\right)_{\Gamma_0}\right|\\ &\leqslant C_4\|u_m'(t)\|_{\Gamma_0}\|u_m''(t)\|_{\Gamma_0}\\ &\leqslant C_5\|u_m'(t)\|\|u_m''(t)\|\\ &\leqslant \theta(\varepsilon_1)\|u_m'(t)\|^2+\varepsilon_1\|u_m''(t)\|^2. \qquad (3.15)\end{aligned}$$

Since $H^1(\Omega)\hookrightarrow L^{2(\beta-1)}(\Omega)$, from the Hölder inequality, Sobolev-Poincaré inequality, Cauchy inequality and (3.9),

$$\begin{aligned}\left(b|u_m(t)|^{\beta-2}u_m(t),u_m''(t)\right) &\leqslant b\|u_m(t)\|_{2(\beta-1)}^{\beta-2}\|u_m(t)\|_{2(\beta-1)}\|u_m''(t)\|\\ &\leqslant C_6\|\nabla u_m(t)\|\|u_m''(t)\|\\ &\leqslant \theta(\varepsilon_2)\|\nabla u_m(t)\|^2+\varepsilon_2\|u_m''(t)\|^2. \qquad (3.16)\end{aligned}$$

Thus (3.13)-(3.16) imply

$$\begin{aligned}&\|u_m''(t)\|^2+\frac{d}{dt}\left(\varphi(\|\nabla u_m(t)\|^2)\nabla u_m(t),\nabla u_m'(t)\right)+\frac{a}{2}\frac{d}{dt}\|\nabla u_m'(t)\|^2\\ &\leqslant (C_1+C_2)\|\nabla u_m'(t)\|^2+\frac{d}{dt}\left(g(u_m'(t)),u_m'(t)\right)_{\Gamma_0}\\ &\quad+\theta(\varepsilon_1)\|u_m'(t)\|^2+\varepsilon_1\|u_m''(t)\|^2+\theta(\varepsilon_2)\|\nabla u_m(t)\|^2+\varepsilon_2\|u_m''(t)\|^2. \qquad (3.17)\end{aligned}$$

Integrating (3.17) over $(0,t)$ and using (3.9), (A_2),

$$\begin{aligned}&\int_0^t\|u_m''(s)\|^2ds+\left(\varphi(\|\nabla u_m(t)\|^2)\nabla u_m(t),\nabla u_m'(t)\right)+\frac{a}{2}\|\nabla u_m'(t)\|^2\\ &\leqslant\left(\varphi(\|\nabla u_m(0)\|^2)\nabla u_m(0),\nabla u_m'(0)\right)+\frac{a}{2}\|\nabla u_m'(0)\|^2\\ &\quad+(C_1+C_2)\int_0^t\|\nabla u_m'(s)\|^2ds+\left(g(u_m'(t)),u_m'(t)\right)_{\Gamma_0}-\left(g(u_m'(0)),u_m'(0)\right)_{\Gamma_0}\\ &\quad+C_T+(\varepsilon_1+\varepsilon_2)\int_0^t\|u_m''(s)\|^2ds\\ &\leqslant\left(\varphi(\|\nabla u_m(0)\|^2)\nabla u_m(0),\nabla u_m'(0)\right)+\frac{a}{2}\|\nabla u_m'(0)\|^2+(C_1+C_2)\int_0^t\|\nabla u_m'(s)\|^2ds\\ &\quad+C_T-\left(g(u_m'(0)),u_m'(0)\right)_{\Gamma_0}+(\varepsilon_1+\varepsilon_2)\int_0^t\|u_m''(s)\|^2ds,\end{aligned}$$

that is,

$$\begin{aligned}&\int_0^t\|u_m''(s)\|^2ds+\frac{a}{2}\|\nabla u_m'(t)\|^2\\ &\qquad\leqslant-\left(\varphi(\|\nabla u_m(t)\|^2)\nabla u_m(t),\nabla u_m'(t)\right)\\ &\qquad\quad+\left(\varphi(\|\nabla u_m(0)\|^2)\nabla u_m(0),\nabla u_m'(0)\right)+\frac{a}{2}\|\nabla u_m'(0)\|^2\end{aligned}$$

$$
\begin{aligned}
&+(C_1+C_2)\int_0^t \|\nabla u_m'(s)\|^2 ds + C_T \\
&-\big(g(u_m'(0)),u_m'(0)\big)_{\Gamma_0} + \big(\varepsilon_1+\varepsilon_2\big)\int_0^t \|u_m''(s)\|^2 ds
\end{aligned}
\tag{3.18}
$$

for all $t \in [0,T]$ with arbitrary fixed T.

On the other hand, by Cauchy inequality and (A_1), we have

$$
\big|\big(\varphi(\|\nabla u_m(t)\|^2)\nabla u_m(t),\nabla u_m'(t)\big)\big| \leqslant \theta(\varepsilon_3)\|\nabla u_m(t)\|^2 + \varepsilon_3\|\nabla u_m'(t)\|^2. \tag{3.19}
$$

Therefore, from (3.9), (3.18) and (3.19), for sufficient small ε_1, ε_2, ε_3, we have that

$$
\begin{aligned}
&\frac{1}{2}\int_0^t \|u_m''(s)\|^2 ds + \frac{a}{3}\|\nabla u_m'(t)\|^2 \\
&\qquad \leqslant C_7 + \big(\varphi(\|\nabla u_m(0)\|^2)\nabla u_m(0),\nabla u_m'(0)\big) + \frac{a}{2}\|\nabla u_m'(0)\|^2 \\
&\qquad\quad + (C_1+C_2)\int_0^t \|\nabla u_m'(s)\|^2 ds - \big(g(u_m'(0)),u_m'(0)\big)_{\Gamma_0}.
\end{aligned}
\tag{3.20}
$$

By (3.3) and (3.9),

$$
\begin{aligned}
&\big|\big(\varphi(\|\nabla u_m(0)\|^2)\nabla u_m(0),\ \nabla u_m'(0)\big)\big| \leqslant C_8, \\
&\frac{a}{2}\|\nabla u_m'(0)\|^2 \leqslant C_9,\ \int_0^t \|\nabla u_m'(s)\|^2 ds \leqslant C_{10}.
\end{aligned}
\tag{3.21}
$$

Noting (A_2) and using Hölder inequality, Lemma 3,

$$
\big|\big(g(u_m'(0)),u_m'(0)\big)_{\Gamma_0}\big| \leqslant C_{11}\|u_m'(0)\|_{\Gamma_0}^2 \leqslant C_{12}\|u_m'(0)\|^2. \tag{3.22}
$$

Therefore, from (3.20)-(3.22), we have

$$
\frac{1}{2}\int_0^t \|u_m''(s)\|^2 ds + \frac{a}{3}\|\nabla u_m'(t)\|^2 \leqslant K_2, \tag{3.23}
$$

where K_2 is a positive constant independent of m.

Step 3. (Limiting process) By (3.9) and (3.23), we have

$$
\begin{aligned}
&\frac{1}{2}\|u_m'(t)\|^2 + \frac{\beta-2}{2\beta}\lambda_0\|\nabla u_m(t)\|^2 + a\int_0^t \|\nabla u_m'(s)\|^2 ds \\
&\qquad + \frac{1}{2}\int_0^t \|u_m''(s)\|^2 ds + \frac{a}{3}\|\nabla u_m'(t)\|^2 \leqslant K_1 + K_2.
\end{aligned}
\tag{3.24}
$$

Thus,

$$
\begin{cases}
\{u_m\} \text{ is bounded in } L^\infty(0,T;V\cap H^2), \\
\{u_m'\} \text{ is bounded in } L^\infty(0,T;V), \\
\{u_m''\} \text{ is bounded in } L^2\big(0,T;L^2(\Omega)\big).
\end{cases}
\tag{3.25}
$$

Therefore, we can extract a subsequences in $\{u_m\}$ (denote still by the same symbol) such that

$$\begin{cases} u_m \overset{*}{\rightharpoonup} u \ \text{ weak-star in } \ L^\infty(0,T;V\cap H^2), \\ u_m' \overset{*}{\rightharpoonup} u' \ \text{ weak-star in } \ L^\infty(0,T;V), \\ u_m'' \rightharpoonup u'' \ \text{ weakly in } \ L^2\big(0,T;L^2(\Omega)\big), \end{cases} \tag{3.26}$$

which combining with Aubin-Lions compactness lemma (see [21]) imply

$$\begin{cases} u_m \to u \ \text{ strongly in } \ C([0,T];V\cap H^2), \\ u_m' \to u' \ \text{ strongly in } \ C\big([0,T];L^2(\Omega)\big). \end{cases} \tag{3.27}$$

These results are sufficient to pass to the limit in the linear terms of problem (3.2). Next we are going to consider the nonlinear ones. By (3.25) and $H^1(\Omega) \hookrightarrow L^{2(\beta-1)}(\Omega)$ (see [4]), we obtain

$$\{b|u_m|^{\beta-2}u_m\} \ \text{ is bounded in } \ L^\infty(0,T;L^2(\Omega)).$$

Using trace theorem (see [3]) and (A_2), we deduce

$$\{g(u_m')\} \ \text{ is bounded in } \ L^\infty\big(0,T;L^2(\Gamma_0)\big).$$

Therefore, we can extract a subsequences in $\{u_m\}$ (denote still by the same symbol) such that

$$\begin{cases} |u_m|^{\beta-2}u_m \overset{*}{\rightharpoonup} |u|^{\beta-2}u \ \text{ weak-star in } \ L^\infty\big(0,T;L^2(\Omega)\big), \\ g(u_m') \overset{*}{\rightharpoonup} g(u') \ \text{ weak-star in } \ L^\infty\big(0,T;L^2(\Gamma_0)\big). \end{cases} \tag{3.28}$$

By (3.26), (3.28) and letting $m \to \infty$ in (3.2), we see that u satisfies the equation. Now we discuss the initial conditions. Using (3.3), (3.27) and the simple inequality

$$\|u-u_0\|_V \leqslant \|u-u_m\|_V + \|u_m - u_m(0)\|_V + \|u_m(0)-u_0\|_V,$$

we get the first initial condition immediately. In the similar way, we can show the second initial condition.

Step 4. (Uniqueness of the solution) Let u_1, u_2 be two weak solutions of problem (1.1) such that

$$u_i \in L^\infty(0,T;V\cap H^2), u_i' \in L^\infty(0,T;V), u_i'' \in L^2\big(0,T;L^2(\Omega)\big), i=1,2. \tag{3.29}$$

Then $u = u_1 - u_2$ satisfies

$$\begin{cases} (u''(t),w_k) + \big(\varphi(\|\nabla u_1(t)\|^2)\nabla u_1(t) - \varphi(\|\nabla u_2(t)\|^2)\nabla u_2(t), \nabla w_k\big) + a\big(\nabla u'(t),\nabla w_k\big) \\ \ = \big((g(u_1'(t)) - g(u_2'(t)), w_k\big) + b\big(|u_1(t)|^{\beta-2}u_1(t) - |u_2(t)|^{\beta-2}u_2(t), w_k\big), \\ u(0) = u'(0) = 0, \\ u \in L^\infty(0,T;V\cap H^2), u' \in L^\infty(0,T;V), u'' \in L^2\big(0,T;L^2(\Omega)\big). \end{cases} \tag{3.30}$$

Multiplying ${d_m^k}'(t)$ on both sides of the first equation in (3.30) and summing up the resulting equations respect to k, we have

$$\begin{aligned}
&\big(u''(t),u'(t)\big)+\big(\varphi(\|\nabla u_1(t)\|^2)\nabla u_1(t)\\
&\qquad\qquad -\varphi(\|\nabla u_2(t)\|^2)\nabla u_2(t),\nabla u'(t)\big)+a\big(\nabla u'(t),\nabla u'(t)\big)\\
&\quad =\big((g(u_1'(t)-g(u_2'(t)),u'(t)\big)+b\big(|u_1(t)|^{\beta-2}u_1(t)-|u_2(t)|^{\beta-2}u_2(t),u'(t)\big),
\end{aligned}$$

that is

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u'(t)\|^2+a\|\nabla u'(t)\|^2\\
&\quad =-\big(\varphi(\|\nabla u_1(t)\|^2)\nabla u_1(t),\nabla u'(t)\big)+\big(\varphi(\|\nabla u_2(t)\|^2)\nabla u_2(t),\nabla u'(t)\big)\\
&\quad +\big((g(u_1'(t)-g(u_2'(t)),u'(t)\big)+b\big(|u_1(t)|^{\beta-2}u_1(t)-|u_2(t)|^{\beta-2}u_2(t),u'(t)\big). \qquad (3.31)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\frac{d}{dt}\big(\varphi(\|\nabla u_1(t)\|^2)\|\nabla u(t)\|^2\big)\\
&\quad =2\varphi'(\|\nabla u_1(t)\|^2)\big(\nabla u_1(t),\nabla u_1'(t)\big)\|\nabla u(t)\|^2+2\varphi(\|\nabla u_1(t)\|^2)\big(\nabla u(t),\nabla u'(t)\big)\\
&\quad =2\varphi'\big(\|\nabla u_1(t)\|^2\big)\big(\nabla u_1(t),\nabla u_1'(t)\big)\|\nabla u(t)\|^2\\
&\qquad +2\varphi(\|\nabla u_1(t)\|^2)\big(\nabla u_1(t)-\nabla u_2(t),\nabla u'(t)\big). \qquad (3.32)
\end{aligned}$$

Thus (3.31) and (3.32) imply

$$\begin{aligned}
&\frac{d}{dt}\big(\|u'(t)\|^2+\varphi(\|\nabla u_1(t)\|^2)\|\nabla u(t)\|^2\big)+2a\|\nabla u'(t)\|^2\\
&\quad =2\big(\varphi(\|\nabla u_2(t)\|^2)-\varphi(\|\nabla u_1(t)\|^2)\big)\big(\nabla u_2(t),\nabla u'(t)\big)\\
&\qquad +2\varphi'(\|\nabla u_1(t)\|^2)\big(\nabla u_1(t),\nabla u_1'(t)\big)\|\nabla u(t)\|^2+2\big((g(u_1'(t)-g(u_2'(t)),u'(t)\big)\\
&\qquad +2b\big(|u_1(t)|^{\beta-2}u_1(t)-|u_2(t)|^{\beta-2}u_2(t),u'(t)\big). \qquad (3.33)
\end{aligned}$$

The hypotheses (A_1) and (3.29) yield

$$\begin{aligned}
\big|\varphi(\|\nabla u_2(t)\|^2)-\varphi(\|\nabla u_1(t)\|^2)\big| &\leqslant \Big|\int_{\|\nabla u_1(t)\|^2}^{\|\nabla u_2(t)\|^2}\big|\varphi'(s)\big|ds\Big|\\
&\leqslant C_{13}\big|\|\nabla u_1(t)\|^2-\|\nabla u_2(t)\|^2\big|\\
&\leqslant C_{13}\big(\|\nabla u_1(t)\|+\|\nabla u_2(t)\|\big)\|\nabla u(t)\|\\
&\leqslant C_{14}\|\nabla u(t)\|. \qquad (3.34)
\end{aligned}$$

From (3.29) and (3.34), we have

$$\begin{aligned}
2\big(\varphi(\|\nabla u_2(t)\|^2)-\varphi(\|\nabla u_1(t)\|^2)\big)\big(\nabla u_2(t),\nabla u'(t)\big)&\\
\leqslant 2C_{14}\|\nabla u(t)\|\|\nabla u_2(t)\|\|\nabla u'(t)\|&\\
\leqslant \theta(\eta_1)\|\nabla u(t)\|^2+\eta_1\|\nabla u'(t)\|^2.& \qquad (3.35)
\end{aligned}$$

Again from (3.29), we have

$$2\varphi'\left(\|\nabla u_1(t)\|^2\right)\left(\nabla u_1(t),\nabla u_1'(t)\right)\|\nabla u(t)\|^2 \leqslant C_{15}\|\nabla u(t)\|^2. \tag{3.36}$$

Using the hypothesis (A_2), we know

$$\left(g\left(u_1'(t)\right)-g\left(u_2'(t)\right),u'(t)\right)=\left(g'(\xi)\left(u_1'(t)-u_2'(t)\right),u'(t)\right)\leqslant 0, \tag{3.37}$$

where $\xi\in\{\min\{u_1'(t),u_2'(t)\},\max\{u_1'(t),u_2'(t)\}\}$.

From Lemma 1 and Hölder inequality, we get

$$\begin{aligned}
&2b\left|\left(|u_1(t)|^{\beta-2}u_1(t)-|u_2(t)|^{\beta-2}u_2(t),u'(t)\right)\right|\\
&\quad\leqslant 2b\left|\left(\left(|u_1(t)|^{\beta-2}+|u_2(t)|^{\beta-2}\right)u(t),u'(t)\right)\right|\\
&\quad\leqslant 2b\left(\|u_1(t)\|_{2(\beta-1)}^{\beta-2}+\|u_2(t)\|_{2(\beta-1)}^{\beta-2}\right)\|u(t)\|_{2(\beta-1)}\|u'(t)\|\\
&\quad\leqslant C_{16}\|\nabla u(t)\|\|\nabla u'(t)\|\leqslant\theta(\eta_2)\|\nabla u(t)\|^2+\eta_2\|\nabla u'(t)\|^2.
\end{aligned}\tag{3.38}$$

Considering (3.33)-(3.38), for sufficient small η_1, η_2, we have

$$\frac{d}{dt}\left(\|u'(t)\|^2+\varphi\left(\|\nabla u_1(t)\|^2\right)\|\nabla u(t)\|^2\right)+a\|\nabla u'(t)\|^2\leqslant C_{17}\|\nabla u(t)\|^2,$$

which implies

$$\begin{aligned}
&\frac{d}{dt}\left(\|u'(t)\|^2+\varphi\left(\|\nabla u_1(t)\|^2\right)\|\nabla u(t)\|^2\right)\\
&\qquad\qquad\leqslant C_{18}\left(\|u'(t)\|^2+\varphi\left(\|\nabla u_1(t)\|^2\right)\|\nabla u(t)\|^2\right).
\end{aligned}\tag{3.39}$$

Let

$$Z(t)=\|u'(t)\|^2+\varphi\left(\|\nabla u_1(t)\|^2\right)\|\nabla u(t)\|^2.$$

From (3.29) and (3.39), we have

$$\frac{d}{dt}Z(t)\leqslant C_T Z(t)\text{ with }Z(0)=0,$$

which with Gronwall's inequality and Poincaré inequality imply $Z(t)\equiv 0$ i.e. $u_1=u_2$. The proof of Theorem 1 is completed.

4. Proof of Theorem 2

In this section, we prove the exponential energy decay property for Eq.(1.1). Firstly we give some Lemmas as follows.

LEMMA 5. *Let $u(t,x)$ be the solution of problem (1.1). Then $E(u;t)$ is a non-increasing functional for $t>0$ and*

$$\frac{d}{dt}E(u;t)=-2a\|\nabla u_t\|^2+2(u_t,g(u_t))_{\Gamma_0}\leqslant 0.$$

Proof. By (2.1) and the hypothesis (A_2), it is easy to see that

$$\frac{d}{dt}E(u;t) = -2a\|\nabla u_t\|^2 + 2(u_t, g(u_t))_{\Gamma_0} \leqslant 0.$$

Therefore, $E(u;t)$ is a non-increasing functional. □

LEMMA 6. *Let $E(u;t)$ be a nonnegative decreasing function defined on $[0,\infty)$. If*

$$\int_s^{+\infty} E(u;t)dt \leqslant CE(u;s), \forall s \geqslant s_0,$$

for some constants $s_0, C > 0$. Then

$$E(u;t) \leqslant E(u;0)e^{1-\frac{t}{s_0+C}}, \forall t \geqslant 0.$$

Proof. The proof can be found in [9, 10]. □

Now we show the proof of Theorem 2.

Under the conditions in Theorem 2, we have $u(x,t) \in W$ similar to Lemma 4. Thus,

$$J(u) = \int_0^{\|\nabla u\|^2} \varphi(s)ds - \frac{2b}{\beta}\|u\|_\beta^\beta \geqslant \frac{\beta-2}{\beta}\int_0^{\|\nabla u\|^2} \varphi(s)ds \geqslant \frac{\beta-2}{\beta}\lambda_0\|\nabla u\|^2. \quad (4.1)$$

Therefore, we have from (4.1) that

$$\|u_t\|^2 + \frac{\beta-2}{\beta}\lambda_0\|\nabla u\|^2 \leqslant \|u_t\|^2 + J(u) = E(u;t) \leqslant E(u;0) < d. \quad (4.2)$$

Multiplying the first equation in (1.1) by u and integrating the result over $\Omega \times [0,T)$, we obtain

$$0 = \int_s^T \int_\Omega u\big(u_{tt} - \varphi(\|\nabla u\|_2^2)\Delta u - a\Delta u_t - b|u|^{\beta-2}u\big)dxdt, \quad (4.3)$$

where $0 \leqslant s \leqslant T < \infty$. Since

$$\begin{aligned}\int_s^T \int_\Omega uu_{tt}dxdt &= \textstyle\int_\Omega uu_t dx|_s^T - \int_s^T \int_\Omega |u_t|^2 dxdt \\ &= \textstyle\int_\Omega uu_t dx|_s^T - \int_s^T \|u_t\|^2 dxdt. \quad (4.4)\end{aligned}$$

So substituting (4.4) into (4.3), we get

$$\begin{aligned}0 = &\int_s^T \big(\|u_t\|^2 + \varphi(\|\nabla u\|_2^2)\|\nabla u\|^2 - \frac{2b}{\beta}\|u\|_\beta^\beta\big)dt - \int_s^T \int_\Omega \big(2|u_t|^2 - a\nabla u_t \nabla u\big)dxdt \\ &+ \int_\Omega uu_t dx|_s^T + \big(\frac{2}{\beta}-1\big)b\int_s^T \|u\|_\beta^\beta dt - \int_s^T (u, g(u_t))_{\Gamma_0}dt. \quad (4.5)\end{aligned}$$

We derive from (A_1) that

$$\int_0^{\|\nabla u\|^2} \varphi(s)ds \leqslant \varphi\big(\|\nabla u\|^2\big)\|\nabla u\|^2. \tag{4.6}$$

Combining (4.5), (4.6) and noting (2.2), we have

$$\begin{aligned}\int_s^T E(u;t)dt \leqslant &\int_s^T \int_\Omega \big(2|u_t|^2 - a\nabla u_t \nabla u\big)dxdt \\ &- \int_\Omega uu_t dx|_s^T - \big(\frac{2}{\beta}-1\big)b\int_s^T \|u\|_\beta^\beta dt + \int_s^T (u, g(u_t))_{\Gamma_0} dt.\end{aligned} \tag{4.7}$$

Now, we estimate respectively the terms on the right side of (4.7).

We get from Lemma 1, Lemma 5 and (A_2) that

$$\begin{aligned}2\int_s^T \int_\Omega |u_t|^2 dxdt &= 2\int_s^T \|u_t\|^2 dt \leqslant 2C_*^2 \int_s^T \|\nabla u_t\|^2 dt \\ &= -\frac{C_*^2}{a}\big(E(u;T)-E(u;s)\big) + \frac{2C_*^2}{a}\int_s^T (u_t, g(u_t))_{\Gamma_0} dt \\ &\leqslant \frac{C_*^2}{a}E(u;s).\end{aligned} \tag{4.8}$$

It follows from (4.2) that

$$\|\nabla u\|^2 \leqslant \frac{\beta}{(\beta-2)\lambda_0}E(u;t) \leqslant \frac{\beta}{(\beta-2)\lambda_0}E(u;0). \tag{4.9}$$

Applying Young inequality, (4.8) and (4.9), we deduce that

$$\begin{aligned}\Big|-a\int_s^T \int_\Omega \nabla u_t \nabla u dxdt\Big| &\leqslant a\int_s^T \big(\varepsilon_1\|\nabla u\|^2 + \frac{1}{4\varepsilon_1}\|\nabla u_t\|^2\big)dt \\ &\leqslant \frac{a\beta\varepsilon_1}{(\beta-2)\lambda_0}\int_s^T E(u;t)dt + \frac{1}{8\varepsilon_1}E(u;s).\end{aligned} \tag{4.10}$$

From Lemma 1 and (4.2), we have that

$$\begin{aligned}\Big|\int_\Omega uu_t dx|_s^T\Big| &= \Big|\Big(\int_\Omega uu_t dx\Big)_T - \Big(\int_\Omega uu_t dx\Big)_s\Big| \leqslant \Big|\int_\Omega uu_t dx\Big|_T + \Big|\int_\Omega uu_t dx\Big|_s \\ &\leqslant \Big(\frac{1}{2}\|u\|^2 + \frac{1}{2}\|u_t\|^2\Big)_T + \Big(\frac{1}{2}\|u\|^2 + \frac{1}{2}\|u_t\|^2\Big)_s \\ &\leqslant \Big(\frac{\beta C_*^2}{(\beta-2)\lambda_0}\cdot\frac{(\beta-2)\lambda_0}{2\beta}\|\nabla u\|^2 + \frac{1}{2}\|u_t\|^2\Big)_T \\ &\quad + \Big(\frac{\beta C_*^2}{(\beta-2)\lambda_0}\cdot\frac{(\beta-2)\lambda_0}{2\beta}\|\nabla u\|^2 + \frac{1}{2}\|u_t\|^2\Big)_s \\ &\leqslant \max\Big(\frac{2\beta C_*^2}{(\beta-2)\lambda_0}, 1\Big)E(u;T) + \max\Big(\frac{2\beta C_*^2}{(\beta-2)\lambda_0}, 1\Big)E(u;s)\end{aligned}$$

$$\leqslant 2\max\Big(\frac{2\beta C_*^2}{(\beta-2)\lambda_0},1\Big)E(u;s). \tag{4.11}$$

Applying Lemma 1 and (4.9), we arrive at

$$\|u\|_\beta^\beta \leqslant C_*^\beta\|\nabla u\|^\beta = C_*^\beta\|\nabla u\|^{\beta-2}\|\nabla u\|^2 < C_*^\beta\Big(\frac{\beta}{(\beta-2)\lambda_0}E(u;0)\Big)^{\frac{\beta-2}{2}}\|\nabla u\|^2,$$

which with (4.2) imply

$$\begin{aligned} b\Big(1-\frac{2}{\beta}\Big)\|u\|_\beta^\beta &\leqslant bC_*^\beta\Big(\frac{\beta}{(\beta-2)\lambda_0}E(u;0)\Big)^{\frac{\beta-2}{2}}\frac{\beta-2}{\beta}\|\nabla u\|^2 \\ &\leqslant bC_*^\beta\Big(\frac{\beta}{(\beta-2)\lambda_0}E(u;0)\Big)^{\frac{\beta-2}{2}}\frac{\beta-2}{\beta}\frac{\beta}{(\beta-2)\lambda_0}E(u;t) \\ &= \frac{bC_*^\beta}{\lambda_0}\Big(\frac{\beta}{(\beta-2)\lambda_0}E(u;0)\Big)^{\frac{\beta-2}{2}}E(u;t). \end{aligned} \tag{4.12}$$

Using Lemma 3 (for $p=2$) and Poincaré inequality, we have $\|u\|_{\Gamma_0}\leqslant C\|\nabla u\|$, $\forall u\in V$, which with Hölder inequality, Cauchy inequality, (A_2), (4.2), (4.8) and Lemma 1 imply

$$\begin{aligned} \int_s^T (u,g(u_t))_{\Gamma_0}dt &\leqslant \int_s^T \|u\|_{\Gamma_0}\|g(u_t)\|_{\Gamma_0}dt \\ &\leqslant \int_s^T \Big(\varepsilon_2\|u\|_{\Gamma_0}^2+\frac{1}{4\varepsilon_2}\|g(u_t)\|_{\Gamma_0}^2\Big)dt \\ &\leqslant C\int_s^T \Big(\varepsilon_2\|\nabla u\|^2+\frac{1}{4\varepsilon_2}\|\nabla u_t\|^2\Big)dt \\ &\leqslant \frac{C\beta\varepsilon_2}{(\beta-2)\lambda_0}\int_s^T E(u;t)dt+\frac{C}{8a\varepsilon_2}E(u;s). \end{aligned} \tag{4.13}$$

Substituting the estimate (4.8), (4.10)-(4.13) into (4.7), we conclude

$$\begin{aligned} &\Big(1-\frac{bC_*^\beta}{\lambda_0}\big(\frac{\beta}{(\beta-2)\lambda_0}E(u;0)\big)^{\frac{\beta-2}{2}}-\frac{a\beta\varepsilon_1}{(\beta-2)\lambda_0}-\frac{C\beta\varepsilon_2}{(\beta-2)\lambda_0}\Big)\int_s^T E(u;t)dt \\ &\quad\leqslant \frac{C_*^2}{a}E(u;s)+\frac{1}{8\varepsilon_1}E(u;s)+2\max\Big(\frac{2\beta C_*^2}{(\beta-2)\lambda_0},1\Big)E(u;s)+\frac{C}{8a\varepsilon_2}E(u;s). \end{aligned} \tag{4.14}$$

By the hypothesis (A_4), for sufficient small ε_1 and ε_2,

$$\frac{bC_*^\beta}{\lambda_0^{\frac{\beta}{2}}}\Big(\frac{\beta}{\beta-2}E(u;0)\Big)^{\frac{\beta-2}{2}}+\frac{a\beta\varepsilon_1}{(\beta-2)\lambda_0}+\frac{C\beta\varepsilon_2}{(\beta-2)\lambda_0}<1.$$

Then, we have from (4.14) that

$$\int_s^T E(u;t)dt \leqslant CE(u;s). \tag{4.15}$$

Let $T \to +\infty$ in (4.15),

$$\int_s^{+\infty} E(u;t)dt \leqslant CE(u;s). \tag{4.16}$$

Thus, by (4.16) and Lemma 6, $E(u;t) \leqslant E(0)e^{1-\frac{t}{C}}$, $t \in [0,+\infty)$. The proof of Theorem 2 is completed.

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