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New Approach for General Convergence of the Adomian Decomposition Method

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Abstract: Convergence of Adomian decomposition method (ADM) is ofa great importance when applied to different types of linear and nonlinear equations.In this paper, a general proof of convergence of ADM is introduced. In this work, a reliable approach for convergence of the Adomian method is discussed. Convergence analysis is reliable enough to estimate the maximum absolute truncated error of the Adomianseries solution.

Key words: Convergence · Adomiandecomposition method · Adomian polynomials

INTRODUCTION

At the beginning of 1980s,Adomian proposed a new method to solve some functional equations [1, 2, 3, 4]. In Adomian's decomposition method(ADM), the given equation is decomposed into linear and nonlinear parts. The highest-order derivative operator involved in the linear operatoris applied on both sides of the given equation.The initial and/or boundary conditions and the non-homogeneous term involving the independent variable alone is identified as initial approximation. In ADM, the unknown function is decomposed into a series whose components are to be determined and the nonlinear termis decomposed in terms of special polynomials called Adomian's polynomials. To determine the solution, the successive terms of the series solution have been obtained by recurrence relation using Adomian's polynomials. Large classes of ordinary and partial linear and nonlinear differential equationscan be solved by the Adomian decomposition method [1, 2, 3, 4]. A reliable modification of Adomian decomposition method has been done by Wazwaz [5]. ADM and its modifications [5, 6] have been efficiently used to solve linear and nonlinear differential equations and the theoretical treatment of the convergence of Adomian decomposition method has been considered in [7, 8, 9].

In this paper, a simple proof of convergence of the Adomian's technique is presented. The Adomian Decomposition Method (ADM) solves successfully different types of linear and nonlinear equations in deterministic or stochastic fields [1, 2, 3, 4]. Convergence of ADM is of a great importance to mathematicians nowadays. Using ADM, the solution is obtained as an infinite series in which each term can be easily obtained using the preceding terms that converges rapidly towards the accurate solution. In nonlinear equations, the nonlinear term expressed in a special type of Adomian polynomials, see [4].

Recent work by Cherruault *et al*. [7, 8, 9, 10, 11] on the mathematical framework has provided the rigorous basis for the accuracy and rapid rate of convergence of ADM. Also, rigorous examination on convergence of ADM has been done by Lionel Gabet [12].

It is indeed one of the most notable features of the comparison of perturbation and decomposition that the result of perturbation converges slowly while decomposition converges rapidly, so few terms are generally sufficient. (When this is not the case, one can use Padé approximants or other acceleration technique or the method of asymptotic decomposition). As mentioned before, the convergence of the decomposition series have been investigated by several authors. The theoretical treatment of convergence of the decomposition method has been considered by Cherruault [7, 8, 9, 10, 11]. Cherruault and Adomian and Bougoffa *et al*. [8, 13] proposed a new convergence proof of Adomian's technique based on properties of convergent series. They obtained some results about the speed of convergence of this method providing us to solve linear and nonlinear functional equations. The

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convergence of this series has also been established in [12]. In [7] a proof of convergence is established using fixed point theorem. In [14] the hypotheses for proving convergence are less restrictive. A new condition for obtaining convergence of the decomposition series is included in ref. [15]. The convergence of Adomian decomposition method for initial-value problems have been discussed by Abdelrazec and Pelinovsky [16]. The convergence analysis of the decomposition method for the (1+1)-Parabolic problem in non-uniform media have been described by Rodrigues and Rocha [17].

In this paper, the contribution of the work can be systematically summarized as:

- New sufficient condition has been introduced that guarantees existence of a unique solution to the problem eq. (2.2) (see Theorem 1).
- Based on this, general convergence of ADM has been proved (see Theorem 2).
- The maximumabsolute error of the Adomian truncated series solution eq. (2.8) is estimated in Section 4.

The Principle of Standard Adomian Decomposition Method

Beginning with a Deterministic Equation:

$$F\,u(t) = g(t) \tag{2.1}$$

where F represents a general non-linear ordinary differential operator involving both linear and nonlinear terms, the linear operator is decomposed into $L + R$, where L is easily invertible and R is the remainder of the linear operator. L is taken as the highest order derivative, avoiding the difficult integrations which result when complicated Green's functions are involved. Thus, the equation (2.1) can be written as:

$$Lu + Ru + Nu = g \tag{2.2}$$

whereNu represents the non-linear terms. In this paper, $u(t)$ is assumed to be bounded for all $t \in I = [0,T]$ and the nonlinear term Nu is Lipschitzian viz. $|Nu - Nv| \le L_1|u - v|$ where L_1 is Lipschitz constant.

Solving for Lu,

$$Lu = g - Ru - Nu \tag{2.3}$$

Because L is invertible, operating with its inverse L^{-1}yields.

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu \tag{2.4}$$

An equivalent expression is;

$$u = \Phi + L^{-1}g - L^{-1}Ru - L^{-1}Nu \tag{2.5}$$

where Φ is the integration constant and satisfies $L\Phi = 0$ and $L^{-1}(.) \equiv \underbrace{\int_0^t \dots \int_0^t}_{k\ fold} (.)\,dt$. For initial-value problems we conveniently define L^{-1} for $L \equiv \frac{d^k}{dt^k}$ as the k-fold definite integration operator from 0 to t. For the operator $L \equiv \frac{d^2}{dt^2}$, for example, we have $L^{-1}Lu = u - u(0) - tu'(0)$ and therefore.

$$u = u(0) + tu'(0) + L^{-1}g - L^{-1}Ru - L^{-1}Nu \tag{2.6}$$

For boundary value problems (and, if desired, for initial value problems as well), indefinite integrations are used and the constants are evaluated from the given conditions. Solving for u yields.

$$u = A + Bt + L^{-1}g - L^{-1}Ru - L^{-1}Nu \tag{2.7}$$

The Adomian decomposition method [1, 2, 3, 4] assumes an infinite series solution for unknown function u given by

$$u = \sum_{n=0}^{\infty} u_n \tag{2.8}$$

and the nonlinear term Nu, assumed to be an analytic function $f(u)$, is decomposed as follows:

$$Nu = f(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n) \tag{2.9}$$

where A_n is the appropriate Adomian's polynomial which is generated according to algorithm determined in [1-4]. These A_n polynomials depend, of course, on the particular nonlinearity and these A^n Adomian polynomials are calculated by the general formula.

$$A_n(u_0,u_1,...,u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right]_{\lambda=0}, n \geq 0 \quad (2.10)$$

This formula is easy to set computer code to get as many polynomials as we need in calculation of the numerical as well as explicit solutions.
Substituting eq. (2.8) and eq. (2.9) into eq. (2.5) yields.

$$\sum_{n=0}^{\infty} u_n = \Phi + L^{-1}g - L^{-1}R\sum_{n=0}^{\infty}u_n - L^{-1}\sum_{n=0}^{\infty}A_n \quad (2.11)$$

Each term of series (2.8) is given by the recurrence relation.

$$u_0 = \Phi + L^{-1}g,$$
$$u_n = -L^{-1}Ru_{n-1}, -L^{-1}A_{n-1}, n \geq 1 \quad (2.12)$$

It is worth noting that once the zeroth component u_0 is defined, then the remaining components u_n, $n \geq 1$ can be completely determined; each term is computed by using the previous term. As a result, the components u_0, u_1,... are identified and the series solutions thus entirely determined. However, in many cases the exact solution in a closed form may be obtained.

The decomposition series (2.8) solutions generally converge very rapidly in real physical problems [4]. The rapidity of this convergence means that few terms are required for the analysis.

The practical solutions will be the m-term approximations ϕ_m given by.

$$\phi_m = \sum_{i=0}^{m-1} u_i, \ m \geq 1 \quad (2.13)$$

with,

$$\lim_{m\to\infty} \phi_m = u \quad (2.14)$$

Convergence Analysis: In this section the sufficient condition that guarantees existence of a unique solution is introduced in Theorem 1, convergence of the series solution (2.8) is proved in Theorem 2.

Theorem 1 (Uniqueness theorem). Eq. (2.2) has a unique solution whenever $0 < \alpha < 1$, where, $a = \frac{(L_1+L_2)t^k}{k!}$.

Proof: Let $X = (C[I], \|.\|)$ be the Banach space of all continuous functions on $I = [0,T]$ with the norm $\|u(t)\| = \max_{t\in I}|u(t)|$. We define a mapping $F : X \to X$ where,

$$F(u(t)) = \varphi(t) + L^{-1}g(t) - L^{-1}Ru(t) - L^{-1}Nu(t)$$

Let, $u, \tilde{u} \in X$, we have

$$\|Fu - F\tilde{u}\| = \max_{t\in I}\left|-L^{-1}Ru - L^{-1}Nu + L^{-1}R\tilde{u} + L^{-1}N\tilde{u}\right|$$
$$= \max_{t\in I}\left|-L^{-1}(Ru - R\tilde{u}) - L^{-1}(Nu - N\tilde{u})\right|$$
$$= \max_{t\in I}\left|L^{-1}(Ru - R\tilde{u}) + L^{-1}(Nu - N\tilde{u})\right|$$
$$\leq \max_{t\in I}\left(\left|L^{-1}(Ru - R\tilde{u})\right| + \left|L^{-1}(Nu - N\tilde{u})\right|\right)$$

Now suppose R_u is also Lipschitzian with $|Ru - R\tilde{u}| \leq L_2|u - \tilde{u}|$ where L_2 is Lipschitz constant.

Therefore,

$$\|Fu - F\tilde{u}\| \leq \max_{t\in I}\left(L^{-1}\left|(Ru - R\tilde{u})\right| + L^{-1}\left|(Nu - N\tilde{u})\right|\right)$$
$$\leq \max_{t\in I}\left(L_2L^{-1}|u - \tilde{u}| + L_1L^{-1}|u - \tilde{u}|\right)$$
$$\leq (L_1 + L_2)\|u - \tilde{u}\|\frac{t^k}{k!}$$
$$= \alpha \ \|u - \tilde{u}\|$$

where $\alpha = \frac{(L_1+L_2)t^k}{k!}$.

Under the condition $0 < \alpha < 1$, the mapping is contraction. Therefore, by Banach fixed point theorem for contraction, there existsa unique solution to eq. (2.2).

Theorem 2 (Convergence Theorem): The solution (2.8) of eq. (2.2) using ADM converges if $0 < \alpha < 1$ and $|u_1| < \infty$.

Proof: Let S_n be the n^{th} partial sum, i.e., $S_n = \sum_{i=0}^{n} u_i(t)$. We shall prove that $\{S_n\}$ is a Cauchy sequence in Banachspace X.

$$\|S_{n+p} - S_n\| = \max_{t\in I}|S_{n+p} - S_n| = \max_{t\in I}\left|\sum_{i=n+1}^{n+p} u_i(t)\right|, \quad p = 1,2,3,...$$

We know that,

$$N(u_0 + u_1 + ... + u_n) = \sum_{i=0}^{n-1} A_i + A_n$$

Expanding function Ru about u_0, we have,

$$Ru = \sum_{k=0}^{\infty}\frac{(u-u_0)^k}{k!}\frac{d^kRu}{dt^k}\bigg|_{t=t_0} = \sum_{k=0}^{\infty}\bar{A}_k \ (u_0,u_1,...,u_k)$$

Therefore, it can be arranged as;

$$\bar{A}_0 = R(u_0) = R(S_0)$$
$$\bar{A}_0 + \bar{A}_1 = R(u_0 + u_1) = R(S_1)$$

and so on.

So,

$$\overline{A}_n + \sum_{i=0}^{n-1} \overline{A}_n = R(S_n)$$

This implies

$$\sum_{i=0}^{n-1} \overline{A}_n = R(S_n) - \overline{A}_n \quad (2.15)$$

Similarly,

$$N(S_n) = \sum_{r=0}^{n-1} A_r + A_n \quad (2.16)$$

Now, since R and N are assumed to Lipschitzian operator, we have,

$$\left\|S_{n+p} - S_n\right\| = \max_{t\in I}\left| -L^{-1}\left(\sum_{i=n+1}^{n+p} Ru_{i-1}\right) - L^{-1}\left(\sum_{i=n+1}^{n+p} A_{i-1}\right)\right|$$

$$= \max_{t\in I}\left| -L^{-1}\left(\sum_{i=n}^{n+p-1} Ru_i\right) - L^{-1}\left(\sum_{i=n}^{n+p-1} A_i\right)\right|$$

$$= \max_{t\in I}\left| L^{-1}\left(R(S_{n+p-1}) - R(S_{n-1})\right) + L^{-1}\left(N(S_{n+p-1}) - N(S_{n-1})\right)\right|$$

$$\leq \max_{t\in I}\left| L^{-1}\left(R(S_{n+p-1}) - R(S_{n-1})\right)\right| + \max_{t\in I}\left| L^{-1}\left(N(S_{n+p-1}) - N(S_{n-1})\right)\right|$$

$$\leq \max_{t\in I} L^{-1}\left(\left|R(S_{n+p-1}) - R(S_{n-1})\right|\right) + \max_{t\in I} L^{-1}\left(\left|N(S_{n+p-1}) - N(S_{n-1})\right|\right)$$

$$\leq L_2 \max_{t\in I} L^{-1}\left(\left|S_{n+p-1} - S_{n-1}\right|\right) + L_1 \max_{t\in I} L^{-1}\left(\left|S_{n+p-1} - S_{n-1}\right|\right)$$

$$= \frac{(L_1 + L_2)t^k}{k!}\left\|S_{n+p-1} - S_{n-1}\right\|$$

Therefore,

$$\left\|S_{n+p} - S_n\right\| \leq \alpha \left\|S_{n+p-1} - S_{n-1}\right\|, \text{ where } \alpha = \frac{(L_1 + L_2)t^k}{k!}$$

Similarly, we have,

$$\left\|S_{n+p-1} - S_{n-1}\right\| \leq \alpha \left\|S_{n+p-2} - S_{n-2}\right\|$$

So,

$$\left\|S_{n+p} - S_n\right\| \leq \alpha^2 \left\|S_{n+p-2} - S_{n-2}\right\|$$

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$$\leq \alpha^n \left\|S_p - S_0\right\|$$

If we set $p = 1$,

$$\left\|S_{n+1} - S_n\right\| \leq \alpha^n \left\|S_1 - S_0\right\| \leq \alpha^n \left\|u_1\right\|$$

Nowfor $n > m$, where $n, m \in N$.

$$\left\|S_n - S_m\right\| \leq \left\|S_{m+1} - S_m\right\| + \left\|S_{m+2} - S_{m+1}\right\| + \ldots + \left\|S_n - S_{n-1}\right\|$$

$$\leq \left(\alpha^m + \alpha^{m+1} + \ldots + \alpha^{n-1}\right)\left\|u_1\right\|$$

$$\leq \alpha^m \left(\frac{1-\alpha^{n-m}}{1-\alpha}\right)\left\|u_1\right\|$$

Since $0 < \alpha < 1$, so $1 - \alpha^{n-m} < 1$,

Then we have,

$$\left\|S_n - S_m\right\| \leq \frac{\alpha^m}{1-\alpha}\left\|u_1\right\| \quad (2.17)$$

Since, $u(t)$ is bounded, $|u_1| < \infty$.

So, as $n \to \infty$, $\|S_n - S_m\| \to 0$. Hence $\{S_n\}$ is a Cauchy sequence in X. Therefore, the series in eq. (2.8) converges.

Error Estimation:

From eq. (2.17) we have

$$\left\|S_n - S_m\right\| \leq \frac{\alpha^m}{1-\alpha}\max_{t\in I}\left|u_1(t)\right| \quad (2.18)$$

Since, $\{S_n\}$ is convergent, as $n \to \infty$, $S_n \to u(t)$.

Therefore from eq. (2.18), we get,

$$\left\|u(t) - S_m\right\| \leq \max_{t\in I}\frac{\alpha^m}{1-\alpha}\left|u_1(t)\right|$$

Finally, the maximum absolute truncation error in the interval I is,

$$\max_{t\in I}\left|u(t) - \sum_{i=0}^{m} u_i(t)\right| \leq \max_{t\in I}\frac{\alpha^m}{1-\alpha}\left|u_1(t)\right| = \frac{\alpha^m}{1-\alpha}\left\|u_1(t)\right\|$$

CONCLUSION

A new generalproof for convergence of Adomiandecomposition is introduced. The contraction mapping principle can be employed successfully to prove the convergence of ADM. The convergence study is reliable enough to estimate the maximum absolute truncated error of the Adomian series solution. New sufficient condition has been introduced for obtaining convergence of the decomposition series. In this paper, a new proof of convergence has been presented which is more general than earlier proofs obtained by learned researchers.

REFERENCES

1. Adomian, G., 1983. Stochastic System.Academic Press.
2. Adomian, G., 1986. Nonlinear Stochastic Operator Equations. Academic Press, San Diego.
3. Adomian, G., 1989. Nonlinear Stochastic Systems: Theory and Applications to Physics.Kluwer.
4. Adomian, G., 1995. Solving Frontier Problems of Physics: The Decomposition Method. Kluwer.
5. Wazwaz, A.M., 1999. A reliable modification of Adomian decomposition method. Applied Mathematics and Computation, 102: 77-86.
6. Wazwaz, A.M. and S.M. El-Sayed, 2001. A new modification of the Adomian decomposition method for linear and nonlinear operators. Applied Mathematics and Computation, 122: 393-405.
7. Cherruault, Y., 1989. Convergence of Adomian's method. Kybernetics, 18: 31-38.
8. Cherruault, Y. and G. Adomian, 1993. Decomposition methods: a new proof of convergence. Mathematical and Computer Modelling, 18: 103-106.
9. Abbaoui, K. and Y. Cherruault, 1994. Convergence of Adomian's Method Applied to Differential Equations. Computers and Mathematics with Applications, 28(5): 103-109.
10. Abbaoui, K. and Y. Cherruault, 1995. New ideas for proving convergence of decomposition methods. Computers and Mathematics with Applications, 29(7): 103-108.
11. Himoun, N., K. Abbaoui and Y. Cherruault, 1999. New Results of Convergence of Adomian's Method. Kybernetes, 28(4-5): 423-429.
12. Gabet, L., 1992. Equissed'unethéoriedécompositionnelleet application aux equations aux dérivéespartialles, Dissertation.EcoleCentrale de Paris, Paris (in French).
13. Bougoffa, L., R.C. Rach and S. EI-Manouni, 2013. A convergence analysis of the Adomian decomposition method for an abstract Cauchy problem of a system of first-order nonlinear differential equations. International Journal of Computer Mathematics, 90(2): 360-375.
14. Cherruault, Y., G. Saccomandi and B. Some, 1992. New Results For Convergence of Adomian's Method Applied to Integral Equations. Mathematical and Computer Modelling, 16(2): 85-93.
15. Abbaoui, K., M.J. Pujol, Y. Cherruault, N. Himoun and P. Grimalt, 2001.A new formulation of Adomian method: convergence result. Kybernetes, 30(9/10): 1183-1191.
16. Abdelrazec, A. and D. Pelinovsky, 2011. Convergence of the Adomian decomposition method for initial-value problems, Numer. Methods Partial Differential Equation, 27(4): 749-766.
17. Rodrigues, M.M. and E.M. Rocha, 2010. The convergence Analysis of the Decomposition method for the (1+1)-Parabolic problem in Nonuniform media. Acta Appl. Math., 112: 299-308.