Euclidean and Mereological Qualitative Spaces: a Study of SCC and DCC

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Abstract

We determine the implicit assumptions and the structure of the Single and Double Cross Calculi within Euclidean geometry, and use these results to guide the construction of analogous calculi in mereogeometry. The systems thus obtained have strong semantic and deductive similarities with the Euclidean-based Cross Calculi although they rely on a different geometry. This fact suggests that putting too much emphasis on usual classification of qualitative spaces may hide important commonalities among spaces living in different classes.

Keywords: Single Cross Calculus, Double Cross Calculus, Mereology, Qualitative Spaces.

1 Introduction

In the symbolic approach to Artificial Intelligence (AI) the expression 'qualitative spaces' points to formal systems suitable to reasoning with imprecise, incomplete and subjective information. Important examples in the area of spatial knowledge representation, the subject of this paper, can be found in the overviews like [Cohn and Renz, 2007; Vieu, 1997]. Our interest is general but in this work we concentrate mainly on the representation of orientation and directional information and put at the center of our analysis two well known systems: the Single Cross Calculus (SCC) and the Double Cross Calculus (DCC) of Freksa [1992].

The quantitative vs. qualitative distinction of spaces for spatial information has three motivations. One, already anticipated, refers to (i) the imprecise knowledge a (human or artificial) agent may have of the space in which she lives, (ii) the local perspective she has of the space and (iii) her limited reasoning capacity. Given these assumptions, the goal in this area is to find formal frameworks suitable for managing the information available with particular emphasis on systems motivated by or at least compatible with *cognitive* arguments [Cohn and Renz, 2007].

On the other hand, computational issues motivate the interpretation of the term 'quantitative' as pointing to (explicit or implicit) *metric* systems [Pratt, 1999] like the Real space $\Re^n$ or the Euclidean space $\mathcal{E}^n$. These two systems are quantitative in the sense that they both provide a complete characterization of spatial information and, in particular, of the notion of distance.

Another standard classification in space representation distinguishes formalisms based on the notion of point and those based on the notion of extended region, usually identified by open regular sets. These are sets that equal the topological interior of their own closure, that is, they satisfy equation $A = [A]^{\circ}$. A classical example of the first type of systems is Euclidean geometry [Tarski, 1959]. An example of a region-based space is classical extensional mereology [Simons, 1987].[1]

Both point-based and region-based qualitative spaces usually come in families. A family is a claster of theories that exploit the same intuitive notions by extending a core theory or by providing definitionally equivalent versions of it. Often the relationships within a family reduce to some form of refinement like in the case of RCC-5, RCC-8, RCC-15 and RCC-23 [Cohn *et al.*, 1997] or similarity like the Star Calculus and Opra_m [Renz and Mitra, 2004; Dylla and Wallgrün, 2007]. Only a few works relate or at least classify large classes of spaces, e.g., orientation calculi [Dylla and Wallgrün, 2007] and mereogeometry [Borgo and Masolo, 2009]. The open challenge today is to find a coherent landscape suitable to relate region-based and point-based spaces. A proper understanding of the classification of qualitative spaces cannot be reached without a comprehensive comparison of these two important classes. The purpose of this work is to show that, at a closer look, even spaces that rely on disparate geometries can end up formalizing the "same" notions, a result suggesting that new relationships may be undisclosed by a systematic comparison of systems in the two classes.

The classification problem we are tackling has its roots in the wider goal of providing optimal and reliable information exchanges between different representation schemata as needed in confederated database, in human-computer interaction and in multi-agent systems. An important byproduct of this research is the possibility to generalize (and likely extend) well known techniques for qualitative reasoning. This

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[1] Points and extended regions can coexist in the same formal system. A further distinction that we do not pursue here, separates spaces made out of atomic regions and spaces with only atomless regions, see the work of Masolo and Vieu [1999].

generalization may open the way to extend families of algorithms today confined within a specific class of spaces.

2 Cross Calculi: the Euclidean Grounds

SCC and DCC of Freksa [1992] are Euclidean systems in the sense that they are point-based and their interpretation is constrained to the standard two-dimensional Euclidean space ($\mathcal{E}^2$). To distinguish these systems from those based on mereology that we will study in next section, we will refer to the systems of Freksa as $\mathrm{SCC}_\mathcal{E}$ and $\mathrm{DCC}_\mathcal{E}$.

The formalizations of $\mathrm{SCC}_\mathcal{E}$ and $\mathrm{DCC}_\mathcal{E}$ rely on a relational language and a very constrained class of models. They are introduced for a restricted type of inferences, namely, inferences on entities' location within a class of partitions of the space. In order to compare $\mathrm{SCC}_\mathcal{E}$ and $\mathrm{DCC}_\mathcal{E}$ to other formal systems, we first extend the language as needed to make explicit the overall expressive power that underlies these approaches. The formalization is given to clarify the formal bases of the languages and does not contradict the qualitative status of these systems as we will see.

The Single and the Double Cross Calculus use points and orthogonal lines to partition the space in a finite number of locations or geometrical loci (these are points, lines and extended regions). Given distinct points x, y, z, we write $\mathsf{Ortho}(x, y, z)$ for "the line through points x and y is orthogonal to the line through points y and z". Note that the standard alignment relation (Align) can be defined by

- $\mathsf{Align}(x, y, z) \triangleq \exists w(\mathsf{Ortho}(x, y, w) \wedge \mathsf{Ortho}(w, y, z)) \wedge x \neq z$

and betweenness (Btw) by

- $\mathsf{Btw}(x, y, z) \triangleq \mathsf{Align}(x, y, z) \wedge \exists u(\mathsf{Ortho}(x, u, z) \wedge \mathsf{Ortho}(x, y, u) \wedge \mathsf{Ortho}(u, y, z))$

Thus, the *implicit* language[2] of $\mathrm{SCC}_\mathcal{E}$ and of $\mathrm{DCC}_\mathcal{E}$ consists of the relation Ortho and three constants a, b, c satisfying $\mathsf{Ortho}(a, b, c)$. In $\mathrm{SCC}_\mathcal{E}$ the role of a and c is functional to the space orientation while b identifies the center of reference: a allows to distinguish back/front, c left/right (Fig. 1). $\mathrm{DCC}_\mathcal{E}$ adds the orthogonal line through a refining $\mathrm{SCC}_\mathcal{E}$'s partition of the space. We write $\{a, b\}_\mathcal{E}$ for the $\mathrm{SCC}_\mathcal{E}$ frame of Fig. 1 and, when indicated, for the $\mathrm{DCC}_\mathcal{E}$ frame as well.

Scott [1956] showed that a system that can identify right-angles, e.g. by including the Ortho relation, is equivalent to Euclidean geometry. Thus, $\mathrm{SCC}_\mathcal{E}$ and $\mathrm{DCC}_\mathcal{E}$ rely on a set of relations and properties that are proper of a quantitative system. They are however qualitative spaces in the sense that their *explicit* languages are much weaker, they lack the expressivity to fully represent even a property like alignment. The proper language of $\mathrm{SCC}_\mathcal{E}$ consists only of the nine relations that identify the relative position of an entity in the provided partition of the space: *being straight-front of* b (i.e., being in region S_0 of Fig. 1), *being right-front of* b (being in region S_1), *being right-neutral of* b (being in region S_2), and so on. Analogously, the language of $\mathrm{DCC}_\mathcal{E}$ has fifteen relations. Note that $\mathrm{SCC}_\mathcal{E}$ and $\mathrm{DCC}_\mathcal{E}$ cannot talk about lines, angles or distance except indirectly via these relations. Furthermore, the orthogonality condition on the lines is not formally forced by the theory itself, it is imposed as a semantic constraint on the models.

[2] 'Implicit' because it is the language needed to express the syntactic and semantic constraints on which these systems rely. In contrast, we use 'explicit' for the actual languages of $\mathrm{SCC}_\mathcal{E}$ and $\mathrm{DCC}_\mathcal{E}$.

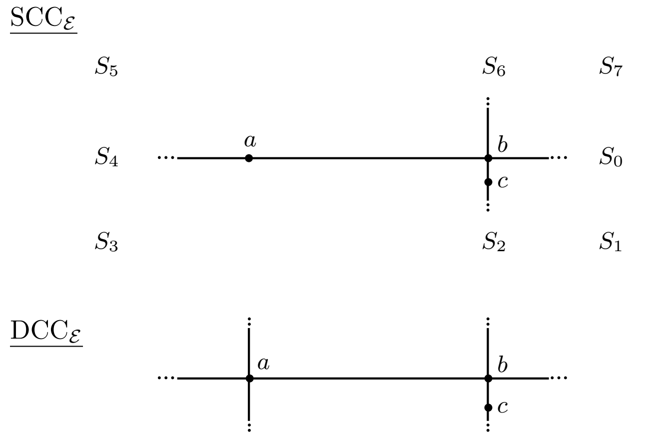

Figure 1: $\mathrm{SCC}_\mathcal{E}$ (above) and $\mathrm{DCC}_\mathcal{E}$ (below). The lines intersect at right angles. For $\mathrm{SCC}_\mathcal{E}$ the locations (sets of points) that partition the space are: $b, S_0, S_1, \ldots, S_7$. b is a single point, S_0, S_2, S_4, S_6 are half-lines (open at the origin which is b), S_1, S_3, S_5, S_7 are extended locations.

The deductive power of $\mathrm{SCC}_\mathcal{E}$ and $\mathrm{DCC}_\mathcal{E}$ is then relative to the identification of an entity's location especially when information is gathered from more than one reference frame. The inferences that can be drawn from the combination of two frames centered at the same point b are collected in so called *composition tables*. An example of the information one finds in a composition table is: if c is right-front of b in $\{a, b\}_\mathcal{E}$ and p is right-neutral of b in $\{c, b\}_\mathcal{E}$, then p is left-front of b in $\{a, b\}_\mathcal{E}$.

3 Cross Calculi: the Mereological Grounds

In this section we look for a mereological system in the two-dimensional space with the right expressive power to model the Single and Double Cross Calculi.

We begin with a system, equivalent to one presented by Cohn in [1995], that we call *mereoconvexity* or $\mathcal{M}$+Cv, where $\mathcal{M}$ is classical extensional mereology with sum, product and complement operators, and Cv is the convexity predicate. At first sight our choice might seem arbitrary since, one argues, $\mathrm{SCC}_\mathcal{E}$ and $\mathrm{DCC}_\mathcal{E}$ are first of all about orthogonally intersecting lines and $\mathcal{M}$+Cv seems to be unrelated to this. In the mereological perspective, a line is the boundary of a half-plane, thus what we should look for is a mereological primitive for half-plane regions and an operator to split them in congruent halves in order to get right angles. We prove here that convex mereology is indeed such a system. There are alternative theories in mereology, as we will see, but $\mathcal{M}$+Cv is preferred for its historical role in the knowledge representation and reasoning community and for the cognitive immediacy of the convexity property.

Our first goal is to show that "being a half plane" (HP) and "being a stripe of finite width" (Str) are alternative predicates that can be used to formalize $\mathcal{M}$+Cv. The fact that these predicates can be defined in mereology by the convexity predicate (Cv) is postponed to the next section for presentation clarity. Here we show that the predicates HP and Str can each define Cv. The formal proof that these definitions do the job in the domain of open regular regions of $\Re^2$ is routine [Borgo and Masolo, 2009].

- $\mathsf{Cv}(x) \triangleq \forall y(\mathsf{PP}(x,y) \rightarrow \exists z(\mathsf{HP}(z) \wedge \mathsf{P}(x,z) \wedge \neg\mathsf{P}(y,z)))$

  x is convex if for each region y that properly contains x, there is a half plane that contains x but not y.

For the other case, we first define the predicates Fnt ("being finite") and FCv ("being finite and convex") by saying that x is finite if it is included in the intersection of two stripes (provided their intersection is not a stripe itself), and if x is finite then it is also convex if for each region y that properly contains x, there is a stripe that contains x but not y.

- $\mathsf{Fnt}(x) \triangleq \exists y,z\,(\mathsf{Str}(y) \wedge \mathsf{Str}(z) \wedge \mathsf{O}(y,z) \wedge \neg\mathsf{Str}(y \cdot z) \wedge \mathsf{P}(x, y \cdot z))$
- $\mathsf{FCv}(x) \triangleq \mathsf{Fnt}(x) \wedge \forall y(\mathsf{PP}(x,y) \rightarrow \exists z(\mathsf{Str}(z) \wedge \mathsf{P}(x,z) \wedge \neg\mathsf{P}(y,z)))$

Finally, we rely on a well-known property of convex sets: a set x is convex if its intersection with any finite convex set is convex as well.

- $\mathsf{Cv}(x) \triangleq \forall y((\mathsf{O}(x,y) \wedge \mathsf{FCv}(y)) \rightarrow \mathsf{FCv}(x \cdot y))$

Lemma 1 *$\mathcal{M}$+Cv is a subtheory of $\mathcal{M}$+HP and of $\mathcal{M}$+Str.*

3.1 The Cross Calculus Grounds in Mereoconvexity

In this part of the paper we concentrate on the mereological counterpart of $\mathrm{SCC}_{\mathcal{E}}$. The constructions and the proofs for $\mathrm{DCC}_{\mathcal{E}}$ are more involved since there are more relations to model but all the needed elements are developed in this paper.

The definitions of HP (half-plane) and C (connection) given below, as well as the definition of Fnt (finite region) given in the previous section, are new formalizations in the mereoconvexity language of notions indirectly captured by I. Pratt in [1999]. Pratt carried out his work in the restricted domain of finite polygons and provided a formalization that is constrained to that domain. Although some of his results hold in the full domain of regular regions, one obtains them only by analytical arguments on the natural model(s), in the sense used by Borgo and Masolo in [2009], or more precisely by considering the limit of approximating polygons. This technique leaves us in the dark regarding, for instance, how to define specific relations in the object language. In particular, Pratt's definition of connection works in the polygon domain and does not generalize to standard mereological universes like that of open regular regions D_{O}. This is unsatisfactory if the aim is to compare theories' expressiveness. Thus, we have to recast every notion and result directly in the object language interpreted in the mereological domain D_{O} (this choice ensures generality because definitions in D_{O} are easily adapted to a variety of important domains as shown by Borgo and Masolo in [2009]). In this way, the simplicity and the cognitive values of these notions become evident and so the expressivity of the language. Our exploitation of the mereological viewpoint, as opposed to relying on the reconstruction of point-based notions (like line and parallelism), shows that mereological properties and notions are indeed easy to grasp both formally and cognitively.

- The universe is the region that contains any other region[3]
  $\mathsf{U}(x) \triangleq \forall y.\mathsf{P}(y,x)$ (the universe)
- A cut is a partition of the universe in two convex regions
  $\mathsf{Cut}(x,y) \triangleq \mathsf{Cv}(x) \wedge \mathsf{Cv}(y) \wedge x + y = \mathsf{U} \wedge \neg\mathsf{O}xy$ (complementary half-planes)
- A half-plane is any region forming a cut
  $\mathsf{HP}(x) \triangleq \exists y.\mathsf{Cut}(x,y)$ (half-plane)
- A stripe is a convex region which is not a half-plane but can be added to one to obtain a strictly bigger half-plane
  $\mathsf{Str}(y) \triangleq \mathsf{Cv}(y) \wedge \neg\mathsf{HP}(y) \wedge \exists x.\mathsf{HP}(x) \wedge \neg\mathsf{P}(y,x) \wedge \mathsf{HP}(x+y)$ (stripe)

From these definitions, we have

Lemma 2 *$\mathcal{M}$+HP and $\mathcal{M}$+Str are subtheories of $\mathcal{M}$+Cv.*

Theorem 3 *$\mathcal{M}$ + HP, $\mathcal{M}$ + Str and $\mathcal{M}$+Cv are equivalent theories.*

(The proofs are done via geometrical arguments on the models in $\Re^2$. The theorem is thus independent of axiomatization provided the predicates are interpreted as intended.)

- A stripe extends a half-plane if it does not overlap the half-plane and their sum is a half-plane
  $\mathsf{ExtHP}(y,x) \triangleq \mathsf{Str}(y) \wedge \mathsf{HP}(x) \wedge \neg\mathsf{O}(x,y) \wedge \mathsf{HP}(x+y)$ (y is a stripe that extends half-plane x)

We can now define the topological relation of connection C. Informally, two regions are connected if the closure of their interpretations in the Euclidean space share at least a point. In mereoconvexity one can identify a point by a pair of intersecting (non-parallel) half-planes. Since two such planes individuate four distinct cone-like regions, say a, b, c, and d, to verify that two regions x and y are connected, it suffices to check the existence of intersecting half-planes such that each convex region that overlaps all a, b, c, and d must also overlap x and y. Here is the formalization:

- $\mathsf{C}(x,y) \triangleq \exists u,v(u \neq v \wedge \mathsf{HP}(u) \wedge \mathsf{HP}(v) \wedge \mathsf{O}(u,v) \wedge \neg\mathsf{Str}(u \cdot v) \wedge \forall z[(\mathsf{Cv}(z) \wedge O(z, u \setminus v) \wedge \mathsf{O}(z, v \setminus u) \wedge \mathsf{O}(z, u \cdot v) \wedge \mathsf{O}(z, \mathsf{U} \setminus (u + v))) \rightarrow (\mathsf{O}(z,x) \wedge \mathsf{O}(z,y))])$

Using these relations, $\mathcal{M}$+Cv can partition the mereoconvexity space in finite regions in a way that resembles $\mathrm{SCC}_{\mathcal{E}}$ on the Euclidean space. By matching Euclidean lines with mereological stripes, the following relation $\mathrm{SpDiv}_{\mathsf{Cv}}$ identifies the suitable partitions of the mereological space. Informally, x_i is the region playing the role R_i in Fig. 2 and corresponding to S_i in Fig. 1. The formulation constrains the x_i's to be convex, to be all connected to x_B, to cover the whole

[3] In general the existence of the universe U must be forced with an axiom but in our case this is not necessary because of the complement operator.

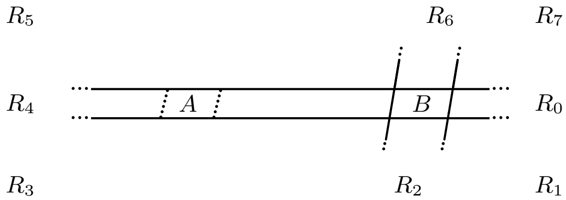

Figure 2: SCC frame in $\mathcal{M}$+Cv. The regions $R0 + B + R4$ and $R2 + B + R6$ are stripes. A is part of $R4$. Stripes may be of different width and do not need to intersect at right angles.

space without overlapping and to form two pairs of stripe and half-plane each satisfying ExtHP. Let $I = \{B, 0, 1, \ldots, 7\}$,

- $\mathrm{SpDiv}_{\mathsf{Cv}}(x_B, x_0, x_1, \ldots, x_7) \triangleq$
$$\sum_{i \in I} x_i = \mathsf{U} \wedge \left(\bigwedge_{i,j \in I, i \neq j} \neg \mathsf{O}(x_i, x_j)\right) \wedge \\ \left(\bigwedge_{i \in I} \mathsf{Cv}(x_i)\right) \wedge \left(\bigwedge_{i \in I} \mathsf{C}(x_B, x_i)\right) \wedge \\ \mathsf{ExtHP}(x_0 + x_B + x_4, x_1 + x_2 + x_3) \wedge \\ \mathsf{ExtHP}(x_2 + x_B + x_6, x_3 + x_4 + x_5)$$

We write $\{A, B\}_{\mathsf{Cv}}$ for a SCC reference frame in $\mathcal{M}$+Cv.

3.2 The Double Cross Grounds in Mereogeometry

Combining the results of Scott [1956] and Pratt [1999], $\mathrm{SpDiv}_{\mathsf{Cv}}$ cannot force stripes to be congruent or to meet at right angles. However, what really makes $\mathrm{SpDiv}_{\mathsf{Cv}}$ (and any other relation in $\mathcal{M}$+Cv) unsuitable to match $\mathrm{SCC}_{\mathcal{E}}$ is the set of regions that are generated in the intersection of two frames. In Fig. 3 we see that an entity located in D (formally, a region for which *identical location of* D holds) can be *straight-front*, *left-neutral*, *right-neutral*, *straight-back*, *identical location of* B or any combination of these (indeed, D overlaps all the stripes of $\{A, B\}_{\mathsf{Cv}}$). More generally, since inferences in these frames are affected even by small changes in the B, D overlapping area, one would think possible to standardize how B and D overlap. However, this seems not possible since the two regions, although somehow constrained in shape, may be of considerably different size. This comparison proves that the inference power of mereological SCC, or $\mathrm{SCC}_{\mathcal{M}}$, in $\{A, B\}_{\mathsf{Cv}}$ is much weaker with respect to that of $\mathrm{SCC}_{\mathcal{E}}$ in $\{a, b\}_{\mathcal{E}}$.

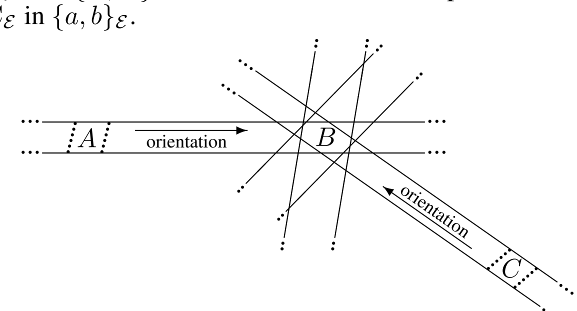

Figure 3: SCC: the interaction between $\{A, B\}_{\mathsf{Cv}}$ and $\{C, D\}_{\mathsf{Cv}}$ is problematic even for B and D congruent (region D is not labeled).

We overcome the frame interaction problem by requiring region B to be a maximal sphere in the area covered by the stripes, Fig. 4. In $\mathrm{DCC}_{\mathcal{M}}$ this constraint holds for region A as well, Fig. 5. We also posit that the stripes must be of the same width (congruent) and intersect at right angles. Fig. 6 is then the counterpart in $\mathrm{SCC}_{\mathcal{M}}$ of Fig. 3.

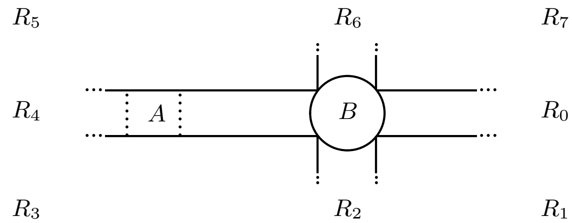

Figure 4: $\mathrm{SCC}_{\mathcal{M}}$ reference frame (labeling as in Fig. 2).

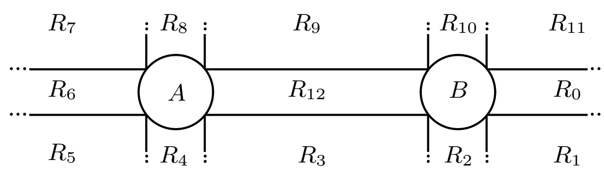

Figure 5: $\mathrm{DCC}_{\mathcal{M}}$ reference frame.

Let Sph be the predicate "being a sphere". Since mereology augmented with Sph is a full mereogeometry, cf. [Borgo and Masolo, 2009], then $\mathcal{M}$+Sph is a mereological equivalent to Euclidean geometry. In particular, congruence and right angles are formally definable in $\mathcal{M}$+Sph.

Our conclusion, further motivated below, that a full mereology is needed to formalize SCC in mereology should not come as a surprise: after all to define $\mathrm{SCC}_{\mathcal{E}}$ we had to resort to (full) Euclidean geometry.

Now we will face two remaining issues: the explicit language of $\mathrm{SCC}_{\mathcal{M}}$ (and $\mathrm{DCC}_{\mathcal{M}}$) and the formal relationship between $\mathrm{SCC}_{\mathcal{E}}$ and $\mathrm{SCC}_{\mathcal{M}}$.

4 Where is the entity located?

Now that we have captured the structure of $\mathrm{SCC}_{\mathcal{M}}$ in mereogeometry, we have all the formal elements to translate formulas of $\mathrm{SCC}_{\mathcal{E}}$ into formulas of $\mathrm{SCC}_{\mathcal{M}}$ and vice versa.

Before looking at the composition table of these systems, we push the analysis further to study the notion of *entity location* in SCC and DCC. The $\mathrm{SCC}_{\mathcal{E}}$ assumption that entities

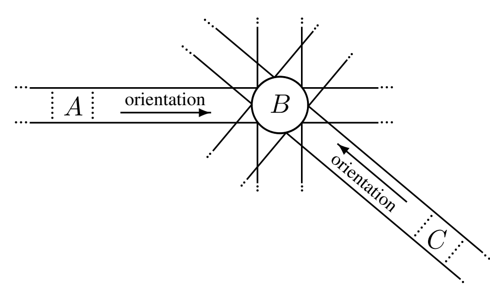

Figure 6: $\mathrm{SCC}_{\mathcal{M}}$: example of frame interaction (cf. Fig. 3).

are represented by points requires the pre-semantic individuation of a single point which functions as "the entity location". This is not so in $\mathrm{SCC}_{\mathcal{M}}$ where the entity location is the actual region the entity occupies. Can we make explicit the relation between the two ways to localize entities? There are two strategies to enrich the $\mathrm{SCC}_{\mathcal{M}}$ expressivity on this topic: to rely on atoms or to introduce a location relation.

The first solution requires us to work in an atomic mereogeometry and to isolate a fragment of the system that behaves like $\mathrm{SCC}_{\mathcal{E}}$ if we substitute atoms for points. Although worth consideration, we do not follow this approach since it would force us to take the partitioning of the space provided by atoms as the underlying grid of the system and to put on a subsidiary level the mereological structure. Also, little is known about full atomic mereogeometries.

The second solution consists in introducing a new relation to formally relate arbitrary regions and points or, in other terms, to formalize a notion of "pointed region" in $\mathrm{SCC}_{\mathcal{M}}$. At first, this technique seems unrealistic for the lack of any suitable notion of point in mereogeometry. However, taking advantage of the fact that $\mathrm{SCC}_{\mathcal{M}}$ works with *finite* partitions of the space only, it turns out that all we need is a relationship that, given a region and any finite partition of it, consistently picks out one and only one of the subregions. Informally, consider an extended[4] entity A located in region A_r of $\mathcal{E}^2$, and denoted in $\mathrm{SCC}_{\mathcal{E}}$ by constant a. Let point p of $\mathcal{E}^2$ be the interpretation of a, thus $p \in A_r$. What we need in $\mathrm{SCC}_{\mathcal{M}}$ is a relation, that we dub Pivot, such that if the open regular regions $B_1, \ldots, B_n$ partition A_r in mereological terms, then $\mathsf{Pivot}(A_r, B_i)$ holds for one and only one i such that $p \in [B_i]$ (we need the closure operator since the B_i are open).[5]

Generally speaking, the association that relates an entity to a single point of $\mathrm{SCC}_{\mathcal{E}}$ lays outside the scope of $\mathrm{SCC}_{\mathcal{E}}$ itself. Instead $\mathrm{SCC}_{\mathcal{M}}$ takes the extension of the entities at face value. We use the Pivot relation to make explicit the relationship between a region and its $\mathrm{SCC}_{\mathcal{E}}$ representative and, in this way, fill the gap between the two entity representation views. The precise formalization of the Pivot relation depends on the way we choose representatives in $\mathrm{SCC}_{\mathcal{E}}$ (or $\mathrm{DCC}_{\mathcal{E}}$ for what it matters): indeed there are several *location relations* like the geometric orthocentric relation, the barycenter relation (assuming the domain is that of material entities), and the relation associating a building to its main entrance (assuming this is the domain at stake).

We now augment the signature of $\mathrm{SCC}_{\mathcal{M}}$ with the binary Pivot relation and provide a minimal axiomatization with the assumption that the relation is further formalized according to the chosen location relation in $\mathrm{SCC}_{\mathcal{E}}$. We call the resulting language $\mathrm{SCC}_{\mathcal{M}}$+Pivot and read $\mathsf{Pivot}(X, Y)$ as "the pivot of X is located in Y" or, equivalently, "X_{Pivot} is in Y".

Axioms:

A1 $\mathsf{Pivot}(X, X)$

The pivot of a region is located in the region itself.

A2 $(X = Y + Z \wedge \neg \mathsf{O}(Y, Z)) \rightarrow (\mathsf{Pivot}(X, Y) \leftrightarrow \neg \mathsf{Pivot}(X, Z))$

For any partition of X in finite regions, the X_{Pivot} is in one and only one of the partitioning regions.

A3 $(\mathsf{Pivot}(X, Y) \wedge \mathsf{Pivot}(X, Z)) \rightarrow \mathsf{Pivot}(X, Y \cdot Z)$

If the X_{Pivot} is in Y and also in Z, then it is in their overlapping region (which must exist).

Theorem 4

1. $\mathsf{Pivot}(X, Y) \rightarrow \mathsf{O}(X, Y)$

2. $(\mathsf{Pivot}(X, Y) \wedge \mathsf{P}(Y, Z) \wedge \mathsf{P}(Z, X)) \rightarrow \mathsf{Pivot}(X, Z)$

Note that relation Pivot is not transitive in general.

We need to be clear on the expressiveness $\mathrm{SCC}_{\mathcal{M}}$+Pivot. Pivot does not force the domain to contain points, nor allows to construct points out of regions. Nonetheless, if we assume that regions have a point-wise "core", no matter how one splits a region in two parts, Pivot will identify the part that contains the core. In other words, we can get as closely as we wish to the point-wise representative of a region but always using extended regions only.

5 $\mathrm{SCC}_{\mathcal{E}}$ and $\mathrm{SCC}_{\mathcal{M}}$: two of a kind

In sections 2 and 3.2 we gave the explicit languages of $\mathrm{SCC}_{\mathcal{E}}$ and of $\mathrm{SCC}_{\mathcal{M}}$. The interpretation of the primitives of $\mathrm{SCC}_{\mathcal{E}}$ is standard in Euclidean geometry, that of $\mathrm{SCC}_{\mathcal{M}}$ is standard in mereogeometry [Borgo and Masolo, 2009]. The relationship between the two (explicit) SCC theories is given by a systematic translation of formulas in the language of $\mathrm{SCC}_{\mathcal{E}}$ into formulas of $\mathrm{SCC}_{\mathcal{M}}$ and vice versa. Then, relation Pivot allows us to verify that the translation is truth-preserving. However, since the focus is the study of the relationship and inferential power in SCC frames, we now look at formulas in the implicit languages of $\mathrm{SCC}_{\mathcal{E}}$ and $\mathrm{SCC}_{\mathcal{M}}$.

Let us use lowercase variables for points in $\mathrm{SCC}_{\mathcal{E}}$ and uppercase variables for regions in $\mathrm{SCC}_{\mathcal{M}}$, we write $t^{I_{\mathcal{E}}}$ for the point in $\mathcal{E}^2$ that is the interpretation of the term t of $\mathrm{SCC}_{\mathcal{E}}$ and $T^{I_{\mathcal{M}}}$ for the open regular region in $\mathcal{E}^2$ that is the interpretation of the term T of $\mathrm{SCC}_{\mathcal{M}}$. Given a frame $\{a, b\}_{\mathcal{E}}$ and a frame $\{A, B\}_{\mathcal{M}}$ with $a^{I_{\mathcal{E}}}$ the center of ball $A^{I_{\mathcal{M}}}$ and $b^{I_{\mathcal{E}}}$ the center of ball $B^{I_{\mathcal{M}}}$, an atomic formula $Rel(b, x)$ in $\mathrm{SCC}_{\mathcal{E}}$ is translated to formula $Rel'(B, X)$ in $\mathrm{SCC}_{\mathcal{M}}$ (and vice versa) with Rel, Rel' as follows:

$Rel(b, x)$	$Rel'(B, X)$
x is identical-location of b	X_{Pivot} is in B
x is straight-front(back) of b	X_{Pivot} is in R_0 (R_4)
x is right(left)-front of b	X_{Pivot} is in R_1 (R_7)
x is right(left)-neutral of b	X_{Pivot} is in R_2 (R_6)
x is right(left)-back of b	X_{Pivot} is in R_3 (R_5)

6 Results

After an analysis of the formal aspects on which SCC an DCC rely, we have analyzed and defined the elements needed to model these calculi in mereology. We now draw some formal results to support the claim that the $\mathrm{SCC}_{\mathcal{M}}$ system is the mereological counterpart of $\mathrm{SCC}_{\mathcal{E}}$.

The first result connects the models for these calculi. These models are all in $\mathcal{E}^2$ where regions are interpreted as regular

[4]If the entity to model is itself point-wise, $\mathrm{SCC}_{\mathcal{E}}$ takes its point location and $\mathrm{SCC}_{\mathcal{M}}$ an arbitrary region with that point location as "core", see below.

[5]To simplify the notation, here we do not distinguish the constants in the language of $\mathrm{SCC}_{\mathcal{M}}$ from their interpretations in $\mathcal{E}^2$.

sets of points. What can we say about the models for $\mathrm{SCC}_{\mathcal{E}}$ and those of $\mathrm{SCC}_{\mathcal{M}}$? Let $\{A, B\}_{\mathcal{M}}$ be a $\mathrm{SCC}_{\mathcal{M}}$ frame, we write $\{A, B\}^{w}_{\mathcal{M}}$ to indicate that w is the radious of region B (this determines the width of all the stripes and, at least in $\mathrm{DCC}_{\mathcal{M}}$, the size of region A as well). Then, $\{A, B\}^{w/n}_{\mathcal{M}}$ is the frame $\{A', B'\}_{\mathcal{M}}$ in which B and B' have same center and the radious of B' is $1/n$ of that of B. Also, for any ball $X \subseteq \mathcal{E}^2$, we write c_X for the center of X, then

Theorem 5 $\lim_{w\to 0} \{A, B\}^{w}_{\mathcal{M}} = \{c_A, c_B\}_{\mathcal{E}}$

Note that the limit, for w going to 0, of the stripes of $\{A, B\}_{\mathcal{M}}$ is not the empty region since $\mathcal{E}^2$ is compact and in the frames each stripe contains the corresponding line (a quasi-closed set) of the Euclidean frame $\{c_A, c_B\}_{\mathcal{E}}$, from which the theorem. Also, the theorem can be generalized to include the regions themselves (not just those that form the frame) by assuming: $\lim_{w\to 0} X = X_{\mathsf{Pivot}}$ (i.e. $x^{I_{\mathcal{E}}}$). The reader should notice that the limit of a region R_i as part of the frame is different from the limit of the same region as element of the space, e.g., for R_0 the first limit gives a half-line, the latter a point. Analogously, for region B the two limits can lead to different points.

The next result shows that $\mathrm{SCC}_{\mathcal{E}}$ and $\mathrm{SCC}_{\mathcal{M}}$ have a common kernel at the deduction level. The two systems are really the same calculus modulo some domain constrains. For the sake of simplicity, we state it using constants $R_0, \ldots, R_7$ of Fig. 4 and, similarly, constants $R'_0, \ldots, R'_7$ for the partition relative to $\{C, B\}_{\mathcal{M}}$. We write $\{A, B\}_{\mathcal{M}} \approx_B \{C, B\}_{\mathcal{M}}$ whenever the two frames generate the same partition of the space perhaps with A and C belonging to different stripes. Also, let $\hat{R} = R_0 + R_2 + R_4 + R_6$, then

Theorem 6 *The composition table of* $\mathrm{SCC}_{\mathcal{E}}$ *is the composition table of* $\mathrm{SCC}_{\mathcal{M}}$ *when regions are limited as follows:*

- *If* $\{A, B\}_{\mathcal{M}} \approx_B \{C, B\}_{\mathcal{M}}$*, then* X *satisfies* $\mathsf{P}(X, B) \vee \left(\bigvee_{0\le i\le 7} \mathsf{P}(X, R'_i)\right)$
- *otherwise* X *satisfies* $\mathsf{P}(X, B) \vee \mathsf{P}(X, R'_1) \vee \mathsf{P}(X, R'_3) \vee \mathsf{P}(X, R'_5) \vee \mathsf{P}(X, R'_7) \vee \mathsf{P}(X, R'_0 \setminus \hat{R}) \vee \mathsf{P}(X, R'_2 \setminus \hat{R}) \vee \mathsf{P}(X, R'_4 \setminus \hat{R}) \vee \mathsf{P}(X, R'_6 \setminus \hat{R})$

In short, the composition tables are the same if we consider only regions that are entirely contained in one component of the $\{C, B\}_{\mathcal{M}}$ partition and do not overlap any of the eight finite quasi-triangular regions around B generated by frames which are not in the same $\approx_B$ class, cf. Fig. 6.

A final statement in mathematical fashion sums up the overall picture we unveiled. For any regular region $X \subseteq \mathcal{E}^2$, we write $\mathsf{Pnt_X}$ to denote the point of X for which the following holds: $\mathsf{Pnt_X} \in Y$ iff $\mathsf{Pivot}(X, Y)$. Pnt is the *punctualization* relation and is the Euclidean counterpart of the mereological Pivot relation: to each element (region or point) to model in $\mathrm{SCC}_{\mathcal{E}}$, Pnt associates a unique point in the $\mathrm{SCC}_{\mathcal{E}}$ domain. Then,

$$\frac{\mathrm{SCC}_{\mathcal{E}}\text{+}\mathsf{Pnt}}{\text{Euclidean Geometry}} = \frac{\mathrm{SCC}_{\mathcal{M}}\text{+}\mathsf{Pivot}}{\text{Mereogeometry}}$$

Mutatis mutandis, these results hold for $\mathrm{DCC}_{\mathcal{E}}$ and $\mathrm{DCC}_{\mathcal{M}}$.

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