ON THE DECIDABILITY OF FORMULAE INVOLVING CONTINUOUS ANE CLOSED FUNCTIONS**

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Abstract

The satisfiability problem for a syllogistic embracing 6, E, Boolean set operations, the Kuratowski topological closure operation ~, and continuous and closed maps between topological spaces, along with the operations of point evaluation, set image, and inverse set image, is solved for formulae that meet a particular syntactic non-circularity property.

The unquantified interpreted language £^{2} to be considered has infinitely many sorts of variables, each corresponding to a different topological space. Three kinds of variables are available, namely, *individual, set,* and *map* variables. Individual variables of a given sort are supposed to range over the universe of that sort, whereas set variables range over the subsets of the appropriate universe. Finally, each map variable ranges over the collection of continuous or closed maps between two appropriate topological spaces.

1 Introduction

Once a decision algorithm has been found for a formalized mathematical theory T, it is rewarding to discover that the validity problem for some other theory T*, possibly related to a different branch of mathematics, reduces to the validity problem of T, which one is already able to solve.

Instances of this are common in the literature. We are therefore motivated in undertaking a quest for new problems whose solution is amenable to 'computable set theory', i.e. to the thick cluster of recently attained methods regarding various portions of classical set theory (see [Cantone, 1988] for an extensive bibliography on this subject). Some indications in this direction have been given in [Cantone *et* a/., 1987], where the satisfiability problems of two theories dealing with monotone functions on totally and partially ordered sets have been translated into *MLS,* or "multi-level syllogistic" (cf. [Ferro *et* a/., 1980]), a well-k nown decidable theory.

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Specifically, it was shown there how to reduce to *MLS* the satisfiability problem for propositional combinations of atoms of type

$$x < y\,, \qquad x = y\,, \qquad y = f(x)\,, \qquad \mathrm{up}(f)\,, \qquad \mathrm{down}(f)\,, \qquad \mathrm{strict\_up}(f)\,, \qquad \mathrm{strict\_down}(f)\,,$$

where / stands for a monotone function.

In the same paper a decision test was provided for the satisfiability problem of an unqualified 'extended Tarsia' theory of reals with variables designating continuous real-valued functions, based on Tarski's celebrated result on the real field (see [Tarski, 1951]). Such theory allows real addition and subtraction, multiplication and division, and comparison between variables representing real numbers; also, addition and subtraction of (continuous) functions, function evaluation, positivity, monotonicity and convexity predicates are allowed. More precisely, a decision test was provided for propositional combinations of atoms of the types

$$x = y + z\,, \quad x = y \cdot z\,, \quad x > 0\,, \quad y = f(x)\,, \quad f = g + h\,, \quad f = g\,, \quad f > 0\,, \quad \mathrm{up}(f)\,, \quad \mathrm{down}(f)\,, \quad \mathrm{strict\_up}(f)\,, \quad \mathrm{strict\_down}(f)\,, \quad \mathrm{convex}(f, x, y)\,, \quad \mathrm{concave}(f, x, y)\,,$$

where *x,y,z* are real variables and f,g,h are function variables, and where convex(/, x, *y)* [resp. concave(/, x, *y)]* is true if and only if the function / is *convex* [resp. *concave]* in the real interval *[x,y].*

Additionally, an extension of Tarski's theory with multi-variate continuous functions has been considered in the same paper (see [Bledsoe, 1977, Bledsoe, 1984, Rabin, 1969] for other approaches on related problems).

In this note we give a further application of decision tests for set theory to theories involving elementary topological constructs. More specifically, we extend two-level syllogistic [Ferro and Omodeo, 1978] with the Kuratowski topological closure operation , and with continuous or closed maps between topological spaces, along with the operations of point evaluation, set image, and inverse set image (cf. [Kelley, 1961]).

We will assume that formulae satisfy a syntactic non-circularity property (to be specified below).

The language £$^{2,\circ\circ}$ of our topological two-level syllogistic has countably infinitely many different sorts of variables, each sort corresponding to a different topological space. We will distinguish between *individual*, *set*, and *map* variables. Individual variables of a given sort n are supposed to range over the universe of sort n, whereas set variables range over the set of all subsets of the appropriate universe. Finally, each map variable ranges over the collection of continuous or closed maps between two appropriate topological universes.

As in [Ferro and Omodeo, 1978], the language $\mathcal{L}^{2,\infty}$ does not allow quantification (either universal or existential).

In the following section, we will give in detail the syntax and semantics of $\mathcal{L}^{2,\infty}$. Also, we will specify a subclass of formulae of $\mathcal{L}^{2,\infty}$ for which, in a subsequent section, a decision procedure for the satisfiability problem will be produced.

2 Syntax and semantics of $\mathcal{L}^{2,\infty}$. Acyclicity

$\mathcal{L}^{2,\infty}$ is a two-level multi-sorted language with functions, containing for each natural number n two denumerable sequences of variables, called *individual variables* ($x_0^{(n)}, x_1^{(n)}, \ldots, y_0^{(n)}, y_1^{(n)}, \ldots$, etc.) and *set variables* ($X_0^{(n)}, X_1^{(n)}, \ldots, Y_0^{(n)}, Y_1^{(n)}, \ldots$, etc.); the constants $0^{(n)}$ and $1^{(n)}$ (designating the empty set and a 'universe', i.e. the support of a topological space); the unary operators $'^{(n)}$, $^{-(n)}$ (designating set complementation and topological closure). The usual binary set operators $\cup$, $\cap$, $\setminus$ (designating union, intersection, and set difference) are also available.

In addition, for each pair (n, m) of natural numbers, there are two denumerable sequences, one consisting of *continuous map variables* ($f_1^{(n,m)}, f_2^{(n,m)}, \ldots$) and one of *closed map variables* ($g_1^{(n,m)}, g_2^{(n,m)}, \ldots$).

Finally, the language $\mathcal{L}^{2,\infty}$ contains the usual relators $=$, $\in$, $\subseteq$, the propositional connectives $\neg, \&, \vee, \rightarrow, \leftrightarrow$, and parentheses.

Individual terms and *set terms of sort* n, for each n, are defined as follows. If $t^{(n)}$ is an individual term of sort n, $T_1^{(n)}, T_2^{(n)}$ are set terms of sort n, $f^{(m,n)}, g^{(n,m)}$ are a continuous and a closed map variable of type (m, n) and (n, m) respectively, and $h^{(n,m)}$ stands for a continuous or closed map variable of type (n, m), then $T_1^{(n)} \cup T_2^{(n)}$, $T_1^{(n)} \cap T_2^{(n)}$, $T_1^{(n)} \setminus T_2^{(n)}$, $(T_1^{(n)})'$, $\overline{T_1^{(n)}}$ are set terms of sort n, and $h^{(n,m)}(t^{(n)})$, $(f^{(m,n)})^{-1}[T_1^{(n)}]$, $g^{(n,m)}[T_1^{(n)}]$ are set terms of sort m. Each individual variable of sort n is an individual term of sort n. Each set variable of sort n and the constants $0^{(n)}$, $1^{(n)}$ are set terms of sort n.

Formulae of $\mathcal{L}^{2,\infty}$ are propositional combination of atoms of the forms

$$t_1^{(n)} = t_2^{(n)}, \qquad t_1^{(n)} \in T_1^{(n)},$$
$$T_1^{(n)} = T_2^{(n)}, \qquad T_1^{(n)} \subseteq T_2^{(n)},$$

where $t_1^{(n)}, t_2^{(n)}$ are individual terms of sort n and $T_1^{(n)}, T_2^{(n)}$ are set terms of sort n, for some natural number n.

The semantics of the language $\mathcal{L}^{2,\infty}$ is defined in the most natural way. So, a *topological assignment* M is any interpretation of the variables and constants of $\mathcal{L}^{2,\infty}$ such that $M1^{(n)}$ is a non-empty set endowed with a topology $\tau^{(n)}$; $M0^{(n)}$ is the empty set; $Mx^{(n)}$ is an element of $M1^{(n)}$, for each individual variable $x^{(n)}$; $MX^{(n)}$ is a (possibly empty) subset of $M1^{(n)}$; $\cup$, $\cap$, $\setminus$ are interpreted as binary set union, set intersection, and set difference in the space $M1^{(n)}$; $'^{(n)}$ is interpreted as set complementation in $M1^{(n)}$; $^{-(n)}$ is interpreted as the closure operation within the topological space $(M1^{(n)}, \tau^{(n)})$; $Mf^{(n,m)}$ is a continuous map from the topological space $(M1^{(n)}, \tau^{(n)})$ into the topological space $(M1^{(m)}, \tau^{(m)})$, for each continuous map variable $f^{(n,m)}$ in $\mathcal{L}^{2,\infty}$; $Mg^{(n,m)}$ is a closed map from $(M1^{(n)}, \tau^{(n)})$ into $(M1^{(m)}, \tau^{(m)})$, for each closed map variable $g^{(n,m)}$ in $\mathcal{L}^{2,\infty}$.

A formula p of $\mathcal{L}^{2,\infty}$ is said to be *topologically satisfiable* if there is a topological assignment M such that p^M is true.

p is said to be *topologically valid* if p^M is true for all topological assignments M.

For ease of presentation, we require that all spaces are pairwise disjoint.

To each formula p of $\mathcal{L}^{2,\infty}$, we associate a graph $G_p = (N_p, E_p)$, where $N_p = \{i_0, i_1, \ldots, i_k\}$ is the set of all sorts of individual and set variables present in the formula p. The edge $i_\ell \Rightarrow i_h$ is in E_p if and only if either p contains some occurrence of a continuous map variable of sort (i_h, i_ℓ), or p contains some occurrence of a closed map variable of sort (i_ℓ, i_h). Then we say that a formula p of $\mathcal{L}^{2,\infty}$ is *acyclic* if its associated graph G_p is acyclic.

Notice that an acyclic formula cannot contain any occurrence of map variables of sort (n, n).

In the next section we will give some evidence that the language $\mathcal{L}^{2,\infty}$ is quite expressive. In fact, most of the elementary concepts and properties that can be typically found in the first chapters of introductory textbooks on general topology are expressible in $\mathcal{L}^{2,\infty}$ (and therefore, as will be shown in the following, automatically verifiable).

In the subsequent section, we will sketch a decision procedure to solve the satisfiability problem for the subclass of *acyclic* formulae of $\mathcal{L}^{2,\infty}$. By virtue of the acyclicity assumption we will be able to eliminate map constructs quite smoothly. This fact coupled with a technique for eliminating also the closure operator $^-$ will allow us to reduce the satisfiability problem for acyclic $\mathcal{L}^{2,\infty}$-formulae to the problem of testing ordinary satisfiability of two-level syllogistic formulae. Therefore our decidability result will follow from [Ferro and Omodeo, 1978].

3 Some examples

Several basic notions of general topology can be formulated in the language $\mathcal{L}^{2,\infty}$. Some operators immediately expressible are:

- $Int(A) \equiv_{\text{Def}} (\overline{A'})'$ (the *interior* of A);
- $Ext(A) \equiv_{\text{Def}} Int(A')$ (the *exterior* of A);
- $\partial(A) \equiv_{\text{Def}} \overline{A} \setminus Int(A)$ (the *boundary* of A).

Also, the following topological predicates are expressible in the language $\mathcal{L}^{2,\infty}$:

- $open(A) \equiv_{\mathrm{Def}} A = Int(A)$ (A is an *open* set);
- $closed(A) \equiv_{\mathrm{Def}} A = \overline{A}$ (A is a *closed* set);
- $open\_domain(A) \equiv_{\mathrm{Def}} A = Int(\overline{A})$ (A is an *open domain*; see [Kuratowski, 1922]);
- $neighborhood(A, a) \equiv_{\mathrm{Def}} a \in Int(A)$ (A is a *neighborhood* of a);
- $dense(A) \equiv_{\mathrm{Def}} \overline{A} = 1$ (A is *dense* in the space 1);
- $co\_dense(A) \equiv_{\mathrm{Def}} dense(A')$ (A is *co_dense* in the space 1);
- $nowhere\_dense(A) \equiv_{\mathrm{Def}} co\_dense(\overline{A})$ (A is *nowhere_dense* in the space 1).

Observe that, in general, for each formula $\phi(x_1, \ldots, x_n)$ with free variables $x_1, \ldots, x_n$, the following are equivalent

- $\phi(x_1, \ldots, x_n)$ is satisfiable;
- $(\exists x_1) \ldots (\exists x_n)\phi(x_1, \ldots, x_n)$ is valid;
- $\neg(\forall x_1) \ldots (\forall x_n)\neg\phi(x_1, \ldots, x_n)$ is valid.

Therefore, any satisfiability test for an unquantified theory $\mathcal{T}$ closed with respect to negation provides a validity test for the universal and the existential closures of the formulae in $\mathcal{T}$.

Since the class of acyclic formulae of $\mathcal{L}^{2,\infty}$ is closed under negation, it follows that the validity of the following statements can be recognized by a satisfiability test for acyclic formulae of $\mathcal{L}^{2,\infty}$:

(a) the intersection of two open domains is an open domain;

(b) the union of two open domains need not be an open domain;

(c) if A and B are open domains, then

$$A \subseteq B \leftrightarrow \overline{A} \subseteq \overline{B} ;$$

(d) $\overline{\emptyset} = \emptyset$, and for all A and B

$$A \subseteq \overline{A} , \ \overline{\overline{A}} = \overline{A} , \ \overline{A \cup B} = \overline{A} \cup \overline{B}$$

(*Kuratowski's closure axioms*);

(e) if A is co-dense and B is nowhere dense, then $A \cup B$ is co-dense;

(f) the union of two co-dense sets is not necessarily a co-dense set;

(g) for all subsets A of a topological space U,

$$Int(A) \cup \partial(A) \cup Int(A') = U;$$

(h) for each continuous map f and each set B,

$$f^{-1}[Int(B)] \subseteq Int(f^{-1}[B]) ;$$

(i) for all A,

$$Int(A) \subseteq Ext(Ext(A)) ;$$

(j) every continuous map is *locally* continuous, i.e., for all x, if B is a neighborhood of $f(x)$, then $f^{-1}[B]$ is a neighborhood of x.

It will turn out that the decision test to be described in the next section is "constructive", in the sense that whenever a formula is recognized satisfiable, a model for it can be effectively constructed. In other words, our test can provide counter-examples for unvalid universal formulae. So, for instance, in the case of statement (b) above, our procedure besides proving its validity, can actually exhibit a topological space with two open domains whose union is not an open domain. Analogously for statement (f).

4 The decision method

Given any acyclic formula p, we can assume without loss of generality that (a) the set $N_p = \{i_1, i_2, \ldots, i_k\}$ of all sorts of individual and set variables occurring in p coïncides with $\{1, 2, \ldots, k\}$; (b) if the edge $i \Rightarrow j$ is in G_p, then $i < j$.

In addition, by using a simple normalization procedure, it is easy to see that the problem of topological satisfiability for acyclic formulae of $\mathcal{L}^{2,\infty}$ can be reduced to the problem of testing for topological satisfiability acyclic conjunctions of literals of type

$$\begin{array}{lll} x = y , & x \neq y , & y = f(x) , \\ y = g(x) , & x \in X , & X = 0 , \\ X = 1 , & X = f^{-1}[Y] , & Y = g[X] , \\ X = \overline{Y} , & X = Y \cup Z , & X = Y \setminus Z , \end{array} \tag{1}$$

where x, y stand for individual variables, f and g for a continuous and a closed map variable respectively, and X, Y for set variables. In fact, for example, terms of type T' and $T_1 \cap T_2$ can be replaced by $1 \setminus T$ and $T_1 \setminus (T_1 \setminus T_2)$, respectively.

Any acyclic conjunction of literals of type (1) satisfying properties (a) and (b) above is called a *normalized acyclic formula*.

We describe an effective procedure that given a normalized acyclic formula p of $\mathcal{L}^{2,\infty}$ produces a formula p^{**} in the multi-sorted sublanguage $\mathcal{L}^{**}$ of $\mathcal{L}^{2,\infty}$ in which neither map terms of type $h(t)$, $f^{-1}[T]$, $g[T]$, nor topological terms of type $\overline{T}$ can occur and such that p is topologically satisfiable if and only if p^{**} is (topologically) satisfiable.

Let p be a normalized acyclic formula of $\mathcal{L}^{2,\infty}$ and let k be the largest sort of any variable in p (i.e., $N_p = \{1, 2, \ldots, k\}$).

Below is the procedure for eliminating map and topological terms from p. (Throughout the following procedure, calligraphic capital letters are used to denote sets of variables or sets of sets of variables.)

ELIMINATION PROCEDURE

1. Initialize q to the empty conjunction.
2. Initialize $CLOSED(i)$ to the empty set, for each $i = 1, \ldots, k$.
3. *FOR* $i = 1$ *TO* k *DO*
   3.1. Let $\mathcal{V}^{(i)}$ be the collection of set variables of sort i present in p and in the auxiliary conjunction q.

3.2. For each $\mathcal{S}^{(i)} \subseteq Pow(\mathcal{V}^{(i)})$, introduce a brand new set variable $C_{\mathcal{S}}^{(i)}$ (of sort i); also, for each $X^{(i)}$ in $\mathcal{V}^{(i)}$ introduce the shorthand notation

$$K_X^{(i)} \equiv_{\mathrm{Def}} C_{\{\{X\}\}}^{(i)}.$$

($K_X^{(i)}$ is intended to denote the closure of $X^{(i)}$, whereas the $C_{\mathcal{S}}^{(i)}$'s will be the closed sets of the topology over $1^{(i)}$.)

3.3. For each $\mathcal{S}^{(i)} \subseteq Pow(\mathcal{V}^{(i)})$ add to q the formula

$$C_{\mathcal{S}}^{(i)} = \bigcap_{\mathcal{Q}\in\mathcal{S}} \bigcup_{X\in\mathcal{Q}} K_X^{(i)},$$

where, if $\mathcal{Q} = \emptyset$, $\bigcup_{X\in\mathcal{Q}} K_X^{(i)}$ stands for the constant $0^{(i)}$ and, if $\mathcal{S}^{(i)} = \emptyset$, the right-hand side stands for the constant $1^{(i)}$.

3.4. For each $X^{(i)}$ in $\mathcal{V}^{(i)}$ add to q the formula

$$X^{(i)} \subseteq K_X^{(i)}.$$

3.5. For each $X^{(i)}$ in $\mathcal{V}^{(i)}$ and $\mathcal{S}^{(i)} \subseteq Pow(\mathcal{V}^{(i)})$ add to q the formula

$$X^{(i)} \subseteq C_{\mathcal{S}}^{(i)} \rightarrow K_X^{(i)} \subseteq C_{\mathcal{S}}^{(i)}.$$

3.6. Put

$$\mathcal{K}^{(i)} = \{K_X^{(i)} : X^{(i)} \text{ is in } \mathcal{V}^{(i)}\}$$

and

$$\mathcal{W}^{(i)} = \mathcal{V}^{(i)} \cup \mathcal{K}^{(i)}.$$

3.7. For each $\mathcal{Q}^{(i)} \subseteq \mathcal{W}^{(i)}$ introduce a brand new set variable $P_{\mathcal{Q}}^{(i)}$ of sort i *($P_{\mathcal{Q}}^{(i)}$ is intended to represent a region of the Venn diagram associated with $\mathcal{W}^{(i)}$, as expressed by the following three formulae).*

3.8. For all $\mathcal{Q}_1^{(i)}, \mathcal{Q}_2^{(i)} \subseteq \mathcal{W}^{(i)}$, $\mathcal{Q}_1^{(i)} \neq \mathcal{Q}_2^{(i)}$, add to q the formula

$$P_{\mathcal{Q}_1}^{(i)} \cap P_{\mathcal{Q}_2}^{(i)} = 0^{(i)}.$$

3.9. For all $W^{(i)}$ in $\mathcal{W}^{(i)}$ add to q the formula

$$W^{(i)} = \bigcup_{\substack{\mathcal{Q}\subseteq\mathcal{W}^{(i)} \\ \& \mathcal{Q}\ni W}} P_{\mathcal{Q}}^{(i)}.$$

3.10. For each $\mathcal{S}^{(i)} \subseteq Pow(\mathcal{V}^{(i)})$ add to q the formula

$$C_{\mathcal{S}}^{(i)} = \bigcup_{\substack{\mathcal{Q}\subseteq\mathcal{W}^{(i)} \\ \& (\forall \mathcal{Q}_1\in\mathcal{S})(\exists X\in\mathcal{Q}_1)(K_X\in\mathcal{Q})}} P_{\mathcal{Q}}^{(i)},$$

where, if there is no $\mathcal{Q} \subseteq \mathcal{W}^{(i)}$ such that $(\forall \mathcal{Q}_1 \in \mathcal{S})(\exists X \in \mathcal{Q}_1)(K_X \in \mathcal{Q})$, the right-hand side stands for the constant $0^{(i)}$. *(In view of 3.8 and 3.9, this formula is equivalent to 3.3.)*

3.11. For each $X^{(i)}$ in CLOSED(i), add to q the formula

$$X^{(i)} = K_X^{(i)}.$$

(See comment to step 3.2; see also steps 3.16, 3.19 below.)

3.12. For each $\mathcal{Q}^{(i)} \subseteq \mathcal{W}^{(i)}$ and each continuous map variable $f^{(j,i)}$ of sort (j,i) in p (with $j > i$) introduce a brand new set variable $P_{\mathcal{Q},f}^{(j)}$ of sort j and add to q the formula

$$P_{\mathcal{Q}}^{(i)} = 0^{(i)} \rightarrow P_{\mathcal{Q},f}^{(j)} = 0^{(j)}$$

($P_{\mathcal{Q},f}^{(j)}$ will stand for the set term $(f^{(j,i)})^{(-1)}[P_{\mathcal{Q}}^{(i)}]$, as expressed by the following three formulae).

3.13. For each $\mathcal{Q}_1^{(i)}, \mathcal{Q}_2^{(i)} \subseteq \mathcal{W}^{(i)}$, $\mathcal{Q}_1^{(i)} \neq \mathcal{Q}_2^{(i)}$, and each continuous map variable $f^{(j,i)}$ of sort (j,i) add to q the formula

$$P_{\mathcal{Q}_1,f}^{(j)} \cap P_{\mathcal{Q}_2,f}^{(j)} = 0^{(j)}.$$

3.14. For each $W^{(i)}$ in $\mathcal{W}^{(i)}$ and each continuous map variable $f^{(j,i)}$ of sort (j,i) in p introduce a brand new set variable $W_f^{(j)}$ of sort j and add to q the formula

$$W_f^{(j)} = \bigcup_{\substack{\mathcal{Q}\subseteq\mathcal{W}^{(i)} \\ \& \mathcal{Q}\ni W}} P_{\mathcal{Q},f}^{(j)}$$

($W_f^{(j)}$ will stand for the set term $(f^{(j,i)})^{(-1)}[W^{(i)}]$).

3.15. For each continuous map variable $f^{(j,i)}$ of sort (j,i) in p add to q the formula

$$1^{(j)} = \bigcup_{\mathcal{Q}\subseteq\mathcal{W}^{(i)}} P_{\mathcal{Q},f}^{(j)}.$$

3.16. For each $\mathcal{S}^{(i)} \subseteq Pow(\mathcal{V}^{(i)})$ and each continuous map variable $f^{(j,i)}$ of sort (j,i) in p introduce a brand new set variable $C_{\mathcal{S},f}^{(j)}$ of sort j and add to q the formula

$$C_{\mathcal{S},f}^{(j)} = \bigcup_{\substack{\mathcal{Q}\subseteq\mathcal{W}^{(i)} \\ \& (\forall \mathcal{Q}_1\in\mathcal{S})(\exists X\in\mathcal{Q}_1)(K_X\in\mathcal{Q})}} P_{\mathcal{Q},f}^{(j)},$$

where, if there is no $\mathcal{Q} \subseteq \mathcal{W}^{(i)}$ such that $(\forall \mathcal{Q}_1 \in \mathcal{S})(\exists X \in \mathcal{Q}_1)(K_X \in \mathcal{Q})$, the right-hand side stands for the constant $0^{(j)}$.
Also, add $C_{\mathcal{S},f}^{(j)}$ to $CLOSED(j)$.

($C_{\mathcal{S},f}^{(j)}$ will stand for the set term $(f^{(j,i)})^{(-1)}[C_{\mathcal{S}}^{(i)}]$.)

3.17. For each $\mathcal{Q}^{(i)} \subseteq \mathcal{W}^{(i)}$ and each closed map variable $g^{(i,j)}$ of sort (i,j) in p (with $j > i$) introduce a brand new set variable $P_{\mathcal{Q},g}^{(j)}$ of sort j and add to q the formula

$$P_{\mathcal{Q}}^{(i)} = 0^{(i)} \leftrightarrow P_{\mathcal{Q},g}^{(j)} = 0^{(j)}$$

($P_{\mathcal{Q},g}^{(j)}$ will stand for the set term $g^{(i,j)}[P_{\mathcal{Q}}^{(i)}]$).

3.18. For each $W^{(i)}$ in $\mathcal{W}^{(i)}$ and each closed map variable $g^{(i,j)}$ of sort (i,j) in p introduce a brand new set variable $W_g^{(j)}$ of sort j and add to q the formula

$$W_g^{(j)} = \bigcup_{\substack{Q \subseteq \mathcal{W}^{(i)} \\ \& Q \ni W}} P_{Q,g}^{(j)}$$

($W_g^{(j)}$ will stand for the set term $g^{(i,j)}[W^{(i)}]$).

3.19. For each $\mathcal{S}^{(i)} \subseteq Pow(\mathcal{V}^{(i)})$ and each closed map variable $g^{(i,j)}$ of sort (i,j) in p introduce a brand new set variable $C_{\mathcal{S},g}^{(j)}$ of sort j and add to q the formula

$$C_{\mathcal{S},g}^{(j)} = \bigcup_{\substack{Q \subseteq \mathcal{W}^{(i)} \\ \& (\forall Q_1 \in \mathcal{S})(\exists X \in Q_1)(K_X \in Q)}} P_{Q,g}^{(j)},$$

where, if there is no $Q \subseteq \mathcal{W}^{(i)}$ such that $(\forall Q_1 \in \mathcal{S})(\exists X \in Q_1)(K_X \in Q)$, the right-hand side stands for the constant $0^{(j)}$.

Also, add $C_{\mathcal{S},g}^{(j)}$ to *CLOSED*(j).

($C_{\mathcal{S},g}^{(j)}$ will stand for the set term $(g^{(i,j)})[C_{\mathcal{S}}^{(i)}]$.)

4. *END FOR.*

5. For each literal $y^{(j)} = g^{(i,j)}(x^{(i)})$ in p and each $Q \subseteq \mathcal{W}^{(i)}$, $1 \le i < j \le k$, add to q the formula

$$x^{(i)} \in P_Q^{(i)} \rightarrow y^{(j)} \in P_{Q,g}^{(j)}.$$

6. For each literal $y^{(j)} = f^{(i,j)}(x^{(i)})$ in p and each $Q \subseteq \mathcal{W}^{(i)}$, $1 \le i < j \le k$, add to q the formula

$$y^{(j)} \in P_Q^{(j)} \rightarrow x^{(i)} \in P_{Q,f}^{(i)}.$$

7. For each pair of literals $y_1^{(j)} = h^{(i,j)}(x_1^{(i)})$, $y_2^{(j)} = h^{(i,j)}(x_2^{(i)})$ in p, with $h^{(i,j)}$ a continuous or closed map variable, $1 \le i, j \le k$, add to q the formula

$$x_1^{(i)} = x_2^{(i)} \rightarrow y_1^{(j)} = y_2^{(j)}.$$

8. Let p^* be obtained from p by substituting

   each term $\overline{Y^{(i)}}$ in p by $K_Y^{(i)}$,

   each term $(f^{(j,i)})^{-1}[Y^{(i)}]$ by $Y_f^{(j)}$,

   each term $g^{(i,j)}[X^{(i)}]$ by $X_g^{(j)}$,

   and by dropping from p all literals of type $y = h(x)$, with x, y individual variables and h a map variable.

9. Let p^{**} be the conjunction $p^* \ \& \ q$.

10. Return p^{**}.

Our satisfiability result is an immediate consequence of the following lemmas.

LEMMA 4.1 *p is topologically satisfiable if and only if p^{**} is satisfiable.*

Proof. Clearly, any topological model M for p can be naturally extended to an ordinary model M^{**} for p^{**}, by putting, for instance,

$$M^{**}W_f^{(i)} =_{\text{Def}} (Mf^{(i,j)})^{-1}[M^{**}W^{(j)}] ,$$

$$M^{**}K_X^{(i)} =_{\text{Def}} \overline{M^{**}X^{(i)}} ,$$

$$M^{**}C_{\mathcal{S}}^{(i)} =_{\text{Def}} \bigcap_{Q \in \mathcal{S}} \bigcup_{X \in Q} M^{**}K_X^{(i)} ,$$

etc. .

At this point, it is an easy matter to show that M^{**} is indeed a model for p^{**}.

Conversely, given a model M^{**} for p^{**}, we can assume that

$$|M^{**}P_Q^{(i)}| \ge |M^{**}P_{Q,g}^{(j)}|, \qquad (2)$$

for all $Q \subseteq \mathcal{W}^{(i)}$ and for all closed map variables $g^{(i,j)}$ of sort (i,j) occurring in p, $1 \le i < j \le k$ (otherwise a suitable "enlargement" procedure could be used to force (2)).

By (2), one can then define $M^{**}g^{(i,j)}$ as any map G from $M^{**}1^{(i)}$ into $M^{**}1^{(j)}$ such that

$$G[M^{**}P_Q^{(i)}] = M^{**}P_{Q,g}^{(j)} ,$$

for all $Q^{(i)} \subseteq \mathcal{W}^{(i)}$.

Likewise, $M^{**}f^{(i,j)}$ can be defined as any map F from $M^{**}1^{(i)}$ into $M^{**}1^{(j)}$ such that

$$F^{-1}[M^{**}P_Q^{(j)}] = M^{**}P_{Q,f}^{(i)} ,$$

for all $Q^{(j)} \subseteq \mathcal{W}^{(j)}$.

In addition, it can easily be verified that

$$\tau^{(i)} = \{M^{**}1^{(i)} \setminus M^{**}C_{\mathcal{S}}^{(i)} \ : \ \mathcal{S}^{(i)} \subseteq Pow(\mathcal{V}^{(i)})\}$$

is a topology over $M^{**}1^{(i)}$, for $1 \le i \le k$.

By inspecting the Elimination Procedure, it is easy to see that the assignment M^{**} so extended is a topological model for p. For instance, if $X^{(i)} = \overline{Y^{(i)}}$ is a literal in p, then by step 8 of the Elimination Procedure, p^* contains the literal $X^{(i)} = K_Y^{(i)}$. Hence, $M^{**}X^{(i)} = M^{**}K_Y^{(i)}$, and since $M^{**}K_Y^{(i)} = M^{**}C_{\{\{Y\}\}}^{(i)}$ is closed, one only needs to show that if $M^{**}Y^{(i)} \subseteq M^{**}C_{\mathcal{S}}^{(i)}$ then $M^{**}X^{(i)} \subseteq M^{**}C_{\mathcal{S}}^{(i)}$, for all $\mathcal{S}^{(i)} \subseteq Pow(\mathcal{V}^{(i)})$. But this follows at once from clauses added to q at step 3.5 of the Elimination Procedure.

This concludes the proof of the lemma. ■

So far, the topological satisfiability problem for the class of acyclic formulae of $\mathcal{L}^{2,\infty}$ has been reduced to the ordinary satisfiability problem for the class of formulae in the sublanguage $\mathcal{L}^{**}$. To further reduce this latter problem to the satisfiability of the two-level syllogistic language (*2LS* in short) considered in [Ferro and Omodeo, 1978], the following simple procedure can be used [1].

Let p^{**} be a formula of $\mathcal{L}^{**}$ and assume that p^{**} involves individual and set variables of sorts in $\{1, 2, \ldots, k\}$ only. Let 1 and 0 be the constants of *2LS* language (denoting the universe and the empty set, respectively).

1′. For each $1 \le i \le k$, introduce a brand new *2LS* set variable U_i (which will denote the constant $1^{(i)}$).

2′. For each set variable $X^{(i)}$ [resp. individual variable $x^{(i)}$] of sort i in p^{**}, introduce a brand new *2LS* set [resp. individual] variable X_i [resp. x_i] of *2LS*.

[1] see also [Cantone *et al.*, 1989] for a reduction of the satisfiability problem for *2LS* to the same problem for the multilevel syllogistic language *MLS*.

3'. Let p be the $2LS$-formula resulting by substituting in p^{**}

each occurrence of $1^{(i)}$ by U_i, for $1 \leq i \leq k$,
each occurrence of $0^{(i)}$ by 0, for $1 \leq i \leq k$,
each occurrence of $X^{(i)}$ by X_i, for $1 \leq i \leq k$,
each occurrence of $x^{(i)}$ by x_i, for $1 \leq i \leq k$,

and by adding the conjunctions

$$(\&_{1 \leq i \leq k} U_i \neq 0) \; \& \; (\&_{1 \leq i < j \leq k} U_i \cap U_j = 0)$$

(which expresses that the universes U_i's are non-empty and pairwise disjoint) and

$$(\&_{X^{(i)} \text{ in } p^{**}} X_i \subseteq U_i) \; \& \; (\&_{x^{(i)} \text{ in } p^{**}} x_i \in U_i)$$

(which expresses that the variables X_i and x_i are of sort i, $1 \leq i \leq k$).

Then we have

LEMMA 4.2 *p^{**} is satisfiable if and only if p is satisfiable.* □

Thus, by [Ferro and Omodeo, 1978], we obtain

THEOREM 4.3 *The class of acyclic formulae in the language $\mathcal{L}^{2,\infty}$ has a solvable satisfiability problem.* □

Slight modifications of the Elimination Procedure allow one to add the map predicates *injective* and *surjective,* without disrupting the decidability of the satisfiability problem for acyclic formulae.

Facing slightly harder technicalities, one could also allow in the language $C^{2,\infty}$ the predicate

"x is an accumulation point of X "

and the operator *Der(X)* (where *Der(X)* stands for the set of all accumulation points of *X,* i.e. the *derived set of X)* without undermining decidability.

An interesting open problem is if one could relax the requirement of acyclicity, without losing decidability.

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