A Four-Valued Fuzzy Propositional Logic

Umberto Straccia
Istituto di Elaborazione della Informazione
Via S. Maria, 46 56126 Pisa (PI) ITALY

Abstract

It is generally accepted that knowledge based systems would be smarter and more robust if they can manage inconsistent, incomplete or imprecise knowledge. This paper is about a four-valued fuzzy propositional logic, which is the result of the combination of a four-valued logic and a fuzzy propositional logic. Besides the nice computational properties, the logic enables us also to deal both with inconsistency and imprecise predicates in a simple way.

1 Introduction

The management of uncertainty in inference systems is an important issue due to the imperfect nature of real world information. There are several fields in which this information has to do with *vague concepts, i.e.* concepts without clear definition. The key fact about vague concepts is that while they are not well defined, propositions involving them may be quite well defined. For instance, the boundaries of the Mount Everest are ill defined, whereas the proposition stating that the Mount Everest is the highest mountain of the world is definite, and its definiteness is not compromised by the ill-definitess of it's exact boundaries. Propositions of this kind are called *fuzzy propositions.* Each fuzzy proposition may have a degree of truth between [0,1]. On the other hand, there exists propositions which are true or false, but due to the lack of precision of the available information we can in general only estimate to what extend it is possible or necessary that they are true. This kind of propositions are called *uncertain propositions.* For example, the concept triangle is well defined, but we can only estimate to what extend it is possible that *e.g.* a shape in a picture is a triangle if the segments are not exactly bounded. Certainly, any combination of the two is possible, *e.g.* uncertain fuzzy propositions are fuzzy propositions for which the available reference information is not precise.

In this paper we will concentrate our attention to (certain) fuzzy propositions. In particular, fuzzy proposition we will handle are of the form *[A* > n] (where A is a proposition and *n* € [0,1]) and have intended meaning "it is *certain* that the degree of truth of A is at *least* n". But, rather mapping *[A > n]* as usual into *true* or *false* (as *e.g.* in [Chen and Kundu, 1996]), we will give to it a four-valued semantics. This will be done by mapping *[A > n]* into an element of *2(t,f)* where *{t},* {/}, 0 and *{t, /}* stand for the four truth values *true, false, unknown* and *contradiction,* respectively, as in [Levesque. 1984]. A first consequence of this semantics is that in certain "useful" circumstances the deduction process is tractable from a computational point of view. A second consequence is that the semantics enables us to deal with inconsistencies as the four-valued logic we will adopt is known to be paraconsistent (see, *e.g.* [Wagner, 1991]).

Our four-valued fuzzy semantics has been shown to be useful in the area of content-based retrieval of multimedia data [Meghini *et* a/., 1997]. In this context the (semantic) content of *e.g.* an image region r is described by means of fuzzy propositions like "r represents the Mount Everest with degree > 0.8". Since images (or any other media) are the subjective work of their authors, contradictions could arise among their content representations (possibly together with domain knowledge), which typically may not be the subject of a belief revision process.

This paper is organised as follows. In the next section we will briefly resume some aspects of the four-valued logic we are based on and in Section 3 we will extend it to the fuzzy case. In Section 4 we will extend our logic by allowing a sort of conditional reasoning[1]. Calculi for deciding entailment will be given for all logics presented and Section 5 concludes.

2 Four-valued propositions

The four-valued logic we will base our work on is essentially [Belnap, 1977; Levesque, 1984]. Let *C* be the language of propositional logic, with connectives A. V and ¬. We will use metavariable *A, B, C,...* and *p, q, r___* for propositions and propositional letters, respectively[2]. *Negation Normal Forms* (NNF) and *Conjunctive Normal Forms* (CNF) are defined as usual.

A *four-valued interpretation X* maps a proposition into an element of 2(t,f) and has to satisfy the following equa-

[1] Notice that in our basic logic modus ponens is not a valid rule of inference.

[2] All metavariables could have an optional subscript.

tions: $t \in (A \wedge B)^{\mathcal{I}}$ iff $t \in A^{\mathcal{I}}$ and $t \in B^{\mathcal{I}}$; $f \in (A \wedge B)^{\mathcal{I}}$ iff $f \in A^{\mathcal{I}}$ or $f \in B^{\mathcal{I}}$; $t \in (A \vee B)^{\mathcal{I}}$ iff $t \in A^{\mathcal{I}}$ or $t \in B^{\mathcal{I}}$; $f \in (A \vee B)^{\mathcal{I}}$ iff $f \in A^{\mathcal{I}}$ and $f \in B^{\mathcal{I}}$; $t \in (\neg A)^{\mathcal{I}}$ iff $f \in A^{\mathcal{I}}$ and $f \in (\neg A)^{\mathcal{I}}$ iff $t \in A^{\mathcal{I}}$. It is worth noting that a two-valued interpretation is just a four-valued interpretation $\mathcal{I}$ such that $A^{\mathcal{I}} \in \{\{t\},\{f\}\}$, for each A. We might characterise the distinction between two-valued and four-valued semantics as the distinction between *implicit* and *explicit* falsehood: in a two-valued logic a formula is (implicitly) false in an interpretation iff it is not true, while in a four-valued logic this need not be the case. Our truth conditions are always given in terms of belongings $\in$ (and never in terms of non belongings $\notin$) of truth values to interpretations.

Let $\mathcal{I}$ be an interpretation, let A, B be two propositions and let Σ be a set of propositions, called *Knowledge Base* (KB): $\mathcal{I}$ *satisfies* (*is a model of*) A iff $t \in A^{\mathcal{I}}$; A and B are *equivalent* (written $A \equiv_4 B$) iff they have the same models; $\mathcal{I}$ *satisfies* (*is a model of*) Σ iff $\mathcal{I}$ is a model of A, for all $A \in \Sigma$; Σ *entails* A (written $\Sigma \models_4 A$) iff all models of Σ are models of A. Without loss of generality, we can restrict our attention to propositions in NNF only, as $\neg\neg A \equiv_4 A$, $\neg(A \wedge B) \equiv_4 \neg A \vee \neg B$ and $\neg(A \vee B) \equiv_4 \neg A \wedge \neg B$ hold. For ease of notation, we will often omit braces, thus writing *e.g.* $A, B \models_4 C$ in place of $\{A, B\} \models_4 C$ and $\models_4 A$ in place of $\emptyset \models_4 A$.

Relation 1 *The following relations can easily be verified:* 1. $A \equiv_4 B$ *does not imply* $\neg A \equiv_4 \neg B$ *(and viceversa),* 2. $A \wedge B \models_4 A$, 3. $A \models_4 B$ *and* $B \models_4 C$ *implies* $A \models_4 C$, $A \models_4 A \vee B$, $A \wedge (\neg A \vee B) \not\models_4 B$ *and* $\Sigma \models_4 A$ *implies* $\Sigma \models_2 A$, *where* $\models_2$ *is the classical two-valued entailment relation. Note that there are no tautologies,* i.e. *there is no* A *such that* $\models_4 A$. *Moreover, every KB is satisfiable. Hence,* $A \wedge \neg A \not\models_4 B$, *as there is a model* $\mathcal{I}$ $(A^{\mathcal{I}} = \{t, f\}, B^{\mathcal{I}} = \emptyset)$ *of* $A \wedge \neg A$ *not satisfying* B. ■

Certainly, not allowing modus ponens is penalizing. But, we will include this form of inference in the extended language $\mathcal{L}^+$ described in Section 4.

2.1 Deciding entailment in $\mathcal{L}$

Effectively deciding whether $\Sigma \models_4 A$ requires a calculus. A well known algorithm for deciding entailment in $\mathcal{L}$ is Levesque's algorithm [Levesque, 1984]: in order to check whether $A \models_4 B$, we put A and B into an equivalent CNF (say C and D) and verify whether for each conjunct D_j of D there is a conjunct C_i of C such that $C_i \subseteq D_j$, where C_i and D_j are clauses. Hence, entailment between two propositions C and D in CNF can be verified in time $O(|C||D|)$, whereas checking whether $A \models_4 B$ is a coNP-complete problem in the general case. We propose an alternative calculus which (*i*) does not require any transformation into CNF, (*ii*) has the same polynomial complexity for the CNF case and (*iii*) is easy extensible to the treatment of conditional reasoning which will be the topic of Section 4. The calculus we have developed is one inspired on the calculus **KE** [D'Agostino and Mondadori, 1994]. The calculus, a semantic tableaux, is based on *signed propositions of type* α ("conjunctive propositions") and of *type* β ("disjunctive propositions") and on their *components* which are defined as usual [Smullyan, 1968][3]:

α	α_1	α_2
$\mathrm{T}A \wedge B$	$\mathrm{T}A$	$\mathrm{T}B$
$\mathrm{NT}A \vee B$	$\mathrm{NT}A$	$\mathrm{NT}B$

β	β_1	β_2
$\mathrm{T}A \vee B$	$\mathrm{T}A$	$\mathrm{T}B$
$\mathrm{NT}A \wedge B$	$\mathrm{NT}A$	$\mathrm{NT}B$

$\mathrm{T}A$ and $\mathrm{NT}A$ are called *conjugated signed propositions* and with β_i^c we indicate the *conjugate* of β_i. An interpretation $\mathcal{I}$ *satisfies* $\mathrm{T}A$ iff $\mathcal{I}$ satisfies A, whereas $\mathcal{I}$ *satisfies* $\mathrm{NT}A$ iff $\mathcal{I}$ does not satisfy A. A set of signed propositions is *satisfiable* iff each element of it is satisfiable. Therefore, $\Sigma \models_4 A$ iff $\mathrm{T}\Sigma \cup \{\mathrm{NT}A\}$ is not satisfiable, where $\mathrm{T}\Sigma = \{\mathrm{T}A : A \in \Sigma\}$. The calculus is based on the rules:

$$(A)\ \frac{\alpha}{\alpha_1, \alpha_2} \qquad (PB)\ \frac{}{\mathrm{T}A \mid \mathrm{NT}A}$$

$$(B1)\ \frac{\beta, \beta_1^c}{\beta_2} \qquad (B2)\ \frac{\beta, \beta_2^c}{\beta_1}$$

An instance of *e.g.* rule $(B1)$ is $\frac{\mathrm{T}A \vee B, \mathrm{NT}A}{\mathrm{T}B}$. Notice that the only branching rule is (PB) (called *Principle of Bivalence*). As usual, a deduction is represented as a tree, called *deduction tree*. A branch ϕ in a deduction tree is closed iff for some proposition A, both $\mathrm{T}A$ and $\mathrm{NT}A$ are in ϕ. With S^{ϕ} we indicate the set of signed propositions occurring in ϕ. A set of signed propositions S has a *refutation* iff in each deduction tree all branches ϕ are closed. Furthermore, we will restricted the proof procedure to the so-called *canonical* form [D'Agostino and Mondadori, 1994, p. 299]: a proposition is *AB-analysed* in a branch ϕ if either (*i*) it is of type α and both α_1 and α_2 occur in ϕ; or (*ii*) it is of type β and (*iia*) if β_1^c occurs in ϕ then β_2 occurs in ϕ, (*iib*) if β_2^c occurs in ϕ then β_1 occurs in ϕ. A branch is *AB-completed* if all the propositions in it are AB-analysed. A proposition of type β is *fulfilled* in a branch ϕ if either β_1 or β_2 occurs in ϕ. We say that a branch ϕ is *completed* if it is AB-completed and, every proposition of type β occurring in ϕ is fulfilled. A deduction tree is *completed* if all its branches are completed. The procedure $Sat(S)$ below determines whether S is satisfiable or not.

Algorithm 1 ($Sat(S)$)
$Sat(S)$ starts from the root labelled S and applies the rules until the resulting tree is either closed or completed. If the tree is closed, $Sat(S)$ returns false, otherwise true. At each step of the construction the following steps are performed:

1. select a branch ϕ which is not yet completed;
2. expand ϕ by means of the rules (A), $(B1)$ and $(B2)$ until it becomes AB-completed, generating branch ϕ';
3. if ϕ' is neither closed nor completed then
   (a) select a proposition of type β which is not yet fulfilled in the branch;
   (c) apply rule (PB) with β_1 and β_1^c as PB-formulae and go to step 1.

   otherwise, go to step 1. ■

The following proposition can be shown.

[3] T and NT play the role of "True" and "Not True", respectively. In classical calculi NT may be replaced with F ("False").

Proposition 1 *Let S be a set of signed propositions in $\mathcal{L}$. Then $Sat(S)$ iff S is satisfiable.* ⊣

Example 1 It can easily verified that a canonical proof of $p \vee (q \wedge r) \models_4 (p \vee q) \wedge (p \vee r)$ starts with $S = \{\mathtt{T}p \vee (q \wedge r), \mathtt{NT}(p \vee q) \wedge (p \vee r)\}$ and generates two branches ϕ_1 and ϕ_2, by using $\beta_1 = \mathtt{NT}(p \vee q)$ and $\beta_1^c = \mathtt{T}(p \vee q)$ as PB-formulae, such that $S^{\phi_1} = S \cup \{\mathtt{NT}p \vee q, \mathtt{NT}p, \mathtt{NT}q, \mathtt{T}q \wedge r, \mathtt{T}q, \mathtt{T}r\}$ and $S^{\phi_2} = S \cup \{\mathtt{T}p \vee q, \mathtt{NT}p \vee r, \mathtt{NT}p, \mathtt{NT}r, \mathtt{T}q \wedge r, \mathtt{T}q, \mathtt{T}r\}$. Both ϕ_1 and ϕ_2 are closed. ∎

If Σ and A are in CNF[4], then rule (PB) is *not* needed, *i.e.* we can eliminate Step 3. from *Sat*. Hence, any deduction tree will have one branch. As a consequence, by observing that $\Sigma \models_4 A_1 \wedge \ldots A_n$ iff for each $1 \leq i \leq n$ $\Sigma \models_4 A_i$, it can be shown that

Proposition 2 *If Σ and A are in CNF then checking $\Sigma \models_4 A$ can be done in time $O(|\Sigma||A|)$ using Sat.* ⊣

Two-valued soundness and completeness is obtained by extending signed propositions as usual: (i) $\mathtt{T}\neg A$ is of type α and $\mathtt{NT}A$ is its α_1 and α_2 component; (ii) $\mathtt{NT}\neg A$ is of type α and $\mathtt{T}A$ is its α_1 and α_2 component. Just notice that in this case *Sat* is exactly the canonical procedure for **KE** [D'Agostino and Mondadori, 1994].

3 Fuzzy propositions

Now, we extend our propositional language $\mathcal{L}$ to the fuzzy case. A *fuzzy valuation* is a function mapping propositions into $[0,1]$. Consistently with our approach of distinguishing explicit from implicit falsehood (*i.e.* distinguishing $f \in A^{\mathcal{I}}$ from $t \notin A^{\mathcal{I}}$) we will use two fuzzy valuations, $|\cdot|^t$ and $|\cdot|^f$: $|A|^t$ will naturally be interpreted as the *degree of truth* of A, whereas $|A|^f$ will analogously be interpreted as the *degree of falsity* of A. Classical "two-valued" fuzzy propositions $|\cdot|^t$ and $|\cdot|^f$ are such that $|A|^f = 1-|A|^t$, for each A. In our case, instead, we might well have $|A|^t = 0.6$ and $|A|^f = 0.8$. This is a natural consequence of our four-valued approach.

A (certain) *fuzzy proposition* is an expression of type $[A \geq n]$, where A is a proposition in $\mathcal{L}$ and $n \in [0,1]$; $\mathcal{L}^f$ is just the set of fuzzy propositions. For instance, $[\mathtt{ItsCold} \geq 0.7]$ is a fuzzy proposition meaning that it is certain that the degree of truth of ItsCold is at least 0.7, while $[\mathtt{ItsCold} \geq 1]$ means that it is definitely cold. On the other hand $[\neg\mathtt{ItsCold} \geq 0.7]$ means that it is certain that it is likely to be not cold, while $[\neg\mathtt{ItsCold} \geq 1]$ may be interpreted as saying that it is definitely not cold. $[A \geq n]$ is in CNF whenever A is.

A *fuzzy interpretation* $\mathcal{I}$ is a triple $\mathcal{I} = ((\cdot)^{\mathcal{I}}, |\cdot|^t, |\cdot|^f)$, where $|\cdot|^t$ and $|\cdot|^f$ are fuzzy valuations and $(\cdot)^{\mathcal{I}}$ maps each fuzzy proposition into an element of $2^{\{t,f\}}$. Additionally, $(\cdot)^{\mathcal{I}}, |\cdot|^t$ and $|\cdot|^f$ have to satisfy the following equations: $|A \wedge B|^t = \min\{|A|^t, |B|^t\}$; $|A \wedge B|^f = \max\{|A|^f, |B|^f\}$; $|A \vee B|^t = \max\{|A|^t, |B|^t\}$; $|A \vee B|^f = \min\{|A|^f, |B|^f\}$; $|\neg A|^t = |A|^f$; $|\neg A|^f = |A|^t$; $t \in [A \geq n]^{\mathcal{I}}$ iff $|A|^t \geq n$; and $f \in [A \geq n]^{\mathcal{I}}$ iff $|A|^f \geq n$. It is easy to see that, *e.g.* $|A \wedge B|^t = |\neg A \vee \neg B|^f$. Similarly for $|A \vee B|^t$.

It is worth noting that there is a simple connection between the four-valued semantics given in Section 2 and the fuzzy counterpart. In fact, the above conditions can be reformulated as *e.g.* $t \in [A \wedge B \geq n]^{\mathcal{I}}$ iff $t \in [A \geq n]^{\mathcal{I}}$ and $t \in [B \geq n]^{\mathcal{I}}$; and $f \in [A \wedge B \geq n]^{\mathcal{I}}$ iff $f \in [A \geq n]^{\mathcal{I}}$ or $f \in [B \geq n]^{\mathcal{I}}$. If $|A|^f = 1 - |A|^t$ and $[A \geq n]^{\mathcal{I}} \in \{\{t\},\{f\}\}$, classical "two-valued" fuzzy logic is obtained.

Fuzzy satisfiability, fuzzy equivalence and fuzzy entailment are defined as the natural extensions of the non fuzzy case. We will use the relation $\approx_{(\cdot)}$ in place of $\models_{(\cdot)}$ whenever we refer to the fuzzy case (*e.g.* $\Sigma \approx_4 [A \geq n]$). Since $\approx_4 [A \geq 0]$, we will not consider those $[A \geq n]$ for $n = 0$. Given a KB Σ and a proposition A, we define the *maximal degree of truth* of A with respect to Σ (written $Maxdeg(\Sigma, A)$) to be $\sup\{n > 0 : \Sigma \approx_4 [A \geq n]\}$ ($\sup \emptyset = 0$). Notice that $\Sigma \approx_4 [A \geq n]$ iff $Maxdeg(\Sigma, A) \geq n$.

There is a strict relation between fuzzy propositions and propositions. Given a KB Σ, let $\overline{\Sigma}$ be the (crisp) KB $\{A : [A \geq n] \in \Sigma\}$.

Proposition 3 *Let Σ be a KB and let $[A \geq n]$ be a fuzzy proposition. If $\Sigma \approx_4 [A \geq n]$ then $\overline{\Sigma} \models_4 A$.* ⊣

Proposition 3 states that there cannot be fuzzy entailment without entailment. Hence, $\{[A \geq 0.7], [\neg A \geq 0.5]\} \not\approx_4 [B \geq n]$, for all $n > 0$. In fact, consider an interpretation $\mathcal{I}$ such that $|A|^t = 0.7$, $|A|^f = 0.5$, $|B|^f = 0$ and $|B|^t = \frac{n}{2}$.

Example 2 Let Σ be the set $\Sigma = \{[p \geq 0.1], [p \wedge q \geq 0.5] [q \vee r \geq 0.6]\}$. Let A be $p \vee r$. One may check that $\Sigma \approx_4 [A \geq 0.5]$ and $Maxdeg(\Sigma, A) = 0.5$. $\overline{\Sigma} \models_4 A$ is easily verified, thereby confirming Proposition 3. ∎

3.1 Deciding fuzzy entailment in $\mathcal{L}^f$

The calculus is a straightforward extension of the procedure *Sat*. In fact, just consider the following *fuzzy signed propositions* and the obvious extension of the definition of satisfiability:

α	α_1	α_2
$\mathtt{T}[A \wedge B \geq n]$ $\mathtt{NT}[A \vee B \geq n]$	$\mathtt{T}[A \geq n]$ $\mathtt{NT}[A \geq n]$	$\mathtt{T}[B \geq n]$ $\mathtt{NT}[B \geq n]$
β	β_1	β_2
$\mathtt{T}[A \vee B \geq n]$ $\mathtt{NT}[A \wedge B \geq n]$	$\mathtt{T}[A \geq n]$ $\mathtt{NT}[A \geq n]$	$\mathtt{T}[B \geq n]$ $\mathtt{NT}[B \geq n]$

By considering *Sat* extended to the fuzzy case, where $\mathtt{T}[A \geq n]$ and $\mathtt{NT}[A \geq m]$ are *conjugated* whenever $n \geq m$, we obtain

Proposition 4 *Let S be a set of signed fuzzy propositions in $\mathcal{L}^f$. Then $Sat(S)$ iff S is satisfiable.* ⊣

Example 3 The application of *Sat* to Σ and $[A \geq 0.5]$, as in Example 2, starts with $S = \mathtt{T}\Sigma \cup \{\mathtt{NT}[A \geq 0.5]\}$ and generates a closed deduction tree with unique branch ϕ such that $S^\phi = S \cup \{\mathtt{T}[p \geq 0.5], \mathtt{T}[q \geq 0.6], \mathtt{NT}[p \geq 0.5], \mathtt{NT}[r \geq 0.5]\}$. Hence, $\Sigma \approx_4 [A \geq 0.5]$. ∎

[4] A set is in CNF iff each component of it is.

Using Proposition 4, fuzzy entailment and entailment may be shown to be in the same complexity class.

Proposition 5 *Checking $\Sigma \approx_4 [A \geq n]$ is a coNP-complete problem, as is $\Sigma \models_4 A$. Given Σ, $[A \geq n]$ in CNF, checking $\Sigma \approx_4 [A \geq n]$ can be done in time $O(|\Sigma||A|)$.* ⊣

One can notice that, any successful refutation of $T\Sigma \cup \{NT[A \geq n]\}$ does not rely on those $[B \geq m] \in \Sigma$ such that $m < n$. Hence, if we let Σ be a KB and consider the set $\Sigma^n = \{[A \geq m] \in \Sigma : m \geq n\}$, then

Proposition 6 *Let Σ be a $\mathcal{L}^f$ KB. Then $\Sigma \approx_4 [A \geq n]$ iff $\overline{\Sigma^n} \models_4 A$.* ⊣

As a consequence, *fuzzy entailment inherits all the properties of entailment* seen in Section 2 (Relation 1):

Proposition 7 *Let Σ be a $\mathcal{L}^f$ KB. If $\overline{\Sigma} \models_4 A$ then there is a $n > 0$ such that $\Sigma \approx_4 [A \geq n]$.* ⊣

which completes Proposition 3. Just note that Proposition 6 does not hold for $\approx_2$. In fact, consider $\Sigma_1 = \{[p \geq 0.2], [\neg p \geq 0.3]\}$ and $\Sigma_2 = \{[p \geq 0.2], [\neg p \geq 0.9]\}$. It can easily be verified that $\overline{\Sigma_1^{0.1}} \models_2 q$ and $\Sigma_1 \not\approx_2 [q \geq 0.1]$, whereas $\Sigma_2 \approx_2 [q \geq 0.3]$ and $\overline{\Sigma_2^{0.3}} \not\models_2 q$.

Proposition 6 gives us a way for computing $Maxdeg(\Sigma, A)$ in the style of the method proposed in [Hollunder, 1994]. This is important, as computing $Maxdeg(\Sigma, A)$, is in fact the way to answer a query of type "to which degree is A (at least) true, given the facts in Σ ?" The method, which requires an algorithm for computing (crisp) entailment (*e.g.* Sat), is based on the observation that $Maxdeg(\Sigma, A) \in \{0\} \cup N_\Sigma$, where $N_\Sigma = \{n : [A \geq n] \in \Sigma\}$, and that $\Sigma^m \supseteq \Sigma^n$ if $n \geq m$.

Algorithm 2
Let Σ be a KB and A a proposition. Set $Min = 0$, $Max = 2$.

1. *Pick $n \in N_\Sigma$ such that $Min < n < Max$. If there is no such n, then set $Maxdeg(\Sigma, A) := Min$ and exit.*
2. *Check if $\overline{\Sigma^n} \models_4 A$. If so, then set $Min = n$ and go to Step 1. If not so, then set $Max = n$ and go to Step 1.*

By a binary search on N_Σ the value of $Maxdeg(\Sigma, A)$ can be determined in $O(\log |N_\Sigma|)$ entailment tests. Hence, if Σ and A are in CNF, the complexity of determining $Maxdeg(\Sigma, A)$ is $O(|A||\Sigma| \log |\Sigma|)$.

Example 4 Consider Example 2: $N_\Sigma = \{0.1, 0.5, 0.6\}$. By binary search, let $n := 0.5$. $\overline{\Sigma^{0.5}} = \{p \wedge q, q \vee r\} \models_4 A$ holds. Thus, $Min := 0.5$; pick $n := 0.6$. Now, $\overline{\Sigma^{0.6}} = \{q \vee r\} \not\models_4 A$ holds. Thus, $Max := 0.6$. Since there is no $Min < n < Max$ such that $n \in N_\Sigma$, the procedure stops. Hence, $Maxdeg(\Sigma, A) = 0.5$ as expected. ∎

A drawback, which Algorithm 2 inherits is that checking entailment several times is generally not feasible from a practical point of view, as it could be exponential in time and N_Σ can be $O(\Sigma)$. In Section 4.1 we will present a method where computing $Maxdeg(\Sigma, A)$ "corresponds" to performing the entailment test only *once*.

3.2 Relations to Possibilistic Logic

There is a strict connection between our logic and (necessity-valued) possibilistic logic [Dubois and Prade, 1986], which allows the expression of uncertain propositions. In possibilistic logics, the expressions are of type $(A, \mathrm{P}n)$ and $(A, \mathrm{N}n)$. A weight $\mathrm{P}n$ (resp. $\mathrm{N}n$) attached to A models to what extent A is *possible* (resp. *necessarily*) true. The semantics is given in terms of fuzzy sets of interpretations, *i.e.* to each propositional interpretation $\mathcal{I}$ a weight $\pi(\mathcal{I}) \in [0, 1]$ is assigned. The possibility and necessity of a proposition is then given by $\Pi(A) = \max\{\pi(\mathcal{I}) : \mathcal{I} \text{ satisfies } A\}$ and $N(A) = 1 - \Pi(\neg A)$. An interpretation satisfies an expression of type $(A, \mathrm{P}n)$ (resp. $(A, \mathrm{N}n)$) if $\Pi(A) \geq n$ (resp. $N(A) \geq n$).

A closer look to Proposition 6 reveals that it is similar to Hollunder's Theorem 3.4 in [Hollunder, 1994]:

Theorem 1 (Hollunder) *Let Σ be a set of possibilistic propositions and $n > 0$. Then $\Sigma \models_2^{pos} (A, \mathrm{N}n)$ iff $\Phi_n \models_2 A$, where $\models_2^{pos}$ is the possibilistic entailment relation and $\Phi_n = \{A : (A, \mathrm{N}m) \in \Sigma$ and $m \geq n\}$.* ⊣

As a consequence, let Σ be a $\mathcal{L}^f$ KB and let $\hat{\Sigma}$ be $\{(A, \mathrm{N}n) : [A \geq n] \in \Sigma\}$. Since $\models_4 \subset \models_2$, from Proposition 6 and Theorem 1 it follows that

Proposition 8 *Let Σ be a $\mathcal{L}^f$ KB. If $\Sigma \approx_4 [A \geq n]$ then $\hat{\Sigma} \models_2^{pos} (A, \mathrm{N}n)$.* ⊣

The converse of Proposition 8 is not true, *i.e.* $\hat{\Sigma} \models_2^{pos} (A, \mathrm{N}n)$ does not imply $\Sigma \approx_4 [A \geq n]$. For instance, $[p \geq 0.6], [\neg p \geq 0.7] \not\approx_4 [q \geq 0.6]$, whereas $(p, \mathrm{N}0.6), (\neg p, \mathrm{N}0.7) \models_2^{pos} (q, \mathrm{N}0.6)$ and, thus, $\approx_4 \subset \models_2^{pos}$ holds. Neither Proposition 8 nor the converse of it holds for $\approx_2$. We can confirm this by considering Σ_1 and Σ_2 of the previous section. Therefore, $\Sigma_2 \approx_2 [q \geq 0.3]$ and $\hat{\Sigma}_2 \not\models_2^{pos} (q, \mathrm{N}0.3)$, whereas $\Sigma_1 \not\approx_2 [q \geq 0.1]$ and $\hat{\Sigma}_1 \models_2^{pos} (q, \mathrm{N}0.1)$.

4 Conditionals

In Section 2 we have seen that modus pones is not a valid inference rule in $\mathcal{L}$ and, thus, in $\mathcal{L}^f$. In order to deal with conditional reasoning we introduce a new connective $\Rightarrow$. Let $\mathcal{L}_+$ be $\mathcal{L}$ plus the set of propositions involving connective $\Rightarrow$. From a semantic point of view, an interpretation $\mathcal{I}$ has also to satisfy the following conditions: $t \in (A \Rightarrow B)^{\mathcal{I}}$ iff $t \in A^{\mathcal{I}}$ implies $t \in B^{\mathcal{I}}$, whereas $f \in (A \Rightarrow B)^{\mathcal{I}}$ iff $t \in A^{\mathcal{I}}$ and $f \in B^{\mathcal{I}}$. Notice, that now $A, A \Rightarrow B \models_4 B$ holds. Moreover, $\neg(A \Rightarrow B) \equiv_4 A \wedge \neg B$, $A \Rightarrow B \not\equiv_4 \neg B \Rightarrow \neg A$ (no contraposition) and $\models_4 A \Rightarrow (B \Rightarrow A)$ hold. Hence, there are tautologies in $\mathcal{L}_+$. As for $\mathcal{L}$, every $\mathcal{L}_+$ KB is satisfiable. A complete calculus with respect to $\mathcal{L}_+$ is obtained by extending the definition of signed propositions of type α and type β to the cases $\mathrm{NT}A \Rightarrow B$ and $\mathrm{T}A \Rightarrow B$, respectively, in a similar way as in [D'Agostino and Mondadori, 1994]. Furthermore, we extend Sat with Step 3b: if β_1 is of type $\mathrm{T}A \Rightarrow B$ then let $S' = (S^{\phi'} \cup \{\mathrm{NT}A\}) \setminus \{\mathrm{T}A \Rightarrow B\}$ and if not $Sat(S')$ then expand ϕ' by means of one children node labelled $\mathrm{T}B$ and go to step 1.

Proposition 9 *Let S be a set of signed propositions in $\mathcal{L}_+$. Then $Sat(S)$ (with Step 3b) iff S is satisfiable.* $\dashv$

Checking whether $\Sigma \models_4 A$ is a coNP-complete problem in $\mathcal{L}_+$. But, if we restrict $\mathcal{L}_+$ to $\overline{\mathcal{L}}_+$ we obtain a tractable logic. $\overline{\mathcal{L}}_+$ is defined inductively as follows: $\overline{\mathcal{L}}_+$ is the minimal set such that (i) every proposition in $\mathcal{L}$ in CNF is in $\overline{\mathcal{L}}_+$; (ii) if $A, A_1, \ldots, A_n$ and $B, B_1, \ldots, B_m$ are literals in $\mathcal{L}$, then both $A_1 \wedge \ldots \wedge A_n \Rightarrow B$ and $A \Rightarrow B_1 \vee \ldots \vee B_m$ are in $\overline{\mathcal{L}}_+$. By considering that Step 3. can be eliminated (rule (PB) is *not* necessary), we have

Proposition 10 *Let Σ be a $\overline{\mathcal{L}}_+$ KB and let A be in $\overline{\mathcal{L}}_+$. Checking if $\Sigma \models_4 A$ can be done in time $O(|\Sigma||A|)$.* $\dashv$

Now, let $\mathcal{L}_+^f$ be the extension of $\mathcal{L}_+$ to the fuzzy case. An interpretation $\mathcal{I}$ has now also to satisfy the semantic clauses $|A \Rightarrow B|^t = \min\{1, \frac{|B|^t}{|A|^t}\}$ (Gödel implication) and $|A \Rightarrow B|^f = \min\{|A|^t, |B|^f\}$. It is worth noting that $|A \Rightarrow B|^f = |A \wedge \neg B|^t$, whereas $|A \Rightarrow B|^t \neq |\neg B \Rightarrow \neg A|^t$ (no contraposition). The clause for $|A \Rightarrow B|^t$ models a sort of conditional $Cond(B|A) = \frac{\|B \cap A\|}{\|A\|}$. It is easily verified that the above conditions are equivalent to: $t \in [A \Rightarrow B \geq n]^{\mathcal{I}}$ iff $\forall m \in [0,1]$, if $t \in [A \geq m]^{\mathcal{I}}$ then $t \in [B \geq n \cdot m]^{\mathcal{I}}$; $f \in [A \Rightarrow B \geq n]^{\mathcal{I}}$ iff $t \in [A \geq n]^{\mathcal{I}}$ and $f \in [B \geq n]^{\mathcal{I}}$, which are similar to the non fuzzy case. The semantics for $\mathcal{L}_+^f$ enables thus a simple form of modus ponens: $[A \geq m], [A \Rightarrow B \geq n] \approx_4 [B \geq n \cdot m]$. Just notice that if $|\cdot|^t, |\cdot|^f \in \{0,1\}$ then classical two-valued $\Rightarrow$ is obtained.

Example 5 Let $\Sigma = \{[J \Rightarrow S \wedge A \geq 0.9], [B \Rightarrow T \geq 0.4], [G \Rightarrow A \geq 0.9], [A \Rightarrow T \geq 0.8], [K \Rightarrow C \geq 0.7], [C \Rightarrow T \geq 0.2], [S \Rightarrow T \geq 0.5]\}$, where J, S, A, B, T, G, K and C stand for *Jon*, *Student*, *Adult*, *Boy*, *Tall*, *Gil*, *Karl* and *Child*, respectively. Then $\Sigma \cup \{[G \geq 0.8]\} \approx_4 [T \geq 0.576]$, $\Sigma \cup \{[J \geq 0.7]\} \approx_4 [T \geq 0.504]$ and $\Sigma \cup \{[G \vee K \geq 0.8]\} \approx_4 [T \geq 0.112]$ (the values are maximal). ∎

Note that $\models_4 [A \Rightarrow (B \Rightarrow A) \geq 1]$ holds, whereas if C is $(p \Rightarrow q) \vee ((p \Rightarrow q) \Rightarrow q)$ then $\models_4 C$ and $\not\approx_4 [C \geq n]$, for all $n > 0$. In fact, let $\mathcal{I}$ be an interpretation such that $|p|^t = \frac{n}{2}$ and $|q|^t = \frac{n^2}{3}$. $|C|^t = \frac{2n}{3} < n$ holds. As a consequence, Proposition 7 is not valid in $\mathcal{L}_+^f$, whereas Proposition 3 and Proposition 8 remain valid[5].

4.1 Deciding entailment in $\mathcal{L}_+^f$

Unfortunately, finding a calculus for entailment in $\mathcal{L}_+^f$ is not as easy as for $\mathcal{L}^f$, since Algorithm 2 does not work in the context of $\mathcal{L}_+^f$: Proposition 6 does not hold and $Maxdeg(\Sigma, A)$ may be not in $\{0\} \cup N_\Sigma$.

First, we generalise fuzzy propositions to the form $[A \geq \lambda]$, where λ is a fuzzy value defined as follows. Let $\mathcal{X}$ be a new alphabet of *fuzzy variables* (with metavariable x). A *multiset*, (with metavariable L) is a finite set of fuzzy variables in which a variable x can occur more than once. A *fuzzy value* (with metavariable λ) is a pair (n, L) where $n \in [0,1]$ and L is a multiset. An interpretation $\mathcal{I}$ is such that $x^{\mathcal{I}} \in [0,1]$, $\{x_1, \ldots, x_n\}^{\mathcal{I}} = x_1{}^{\mathcal{I}} \cdot \ldots \cdot x_n{}^{\mathcal{I}}$, $\emptyset^{\mathcal{I}} = 1$ and $(n, L)^{\mathcal{I}} = n \cdot L^{\mathcal{I}}$. We extend the multiplication function $\cdot$ to fuzzy values by defining $(n, L_1) \cdot (m, L_2)$ as $(n \cdot m, L_1 \cup L_2)$. For ease of notation we will write 0, n and x in place of $(0, L)$, $(n, \emptyset)$ and $(1, \{x\})$, respectively. Moreover, we will allow fuzzy values $\frac{\lambda_1}{\lambda_2}$ and $\lambda^{\frac{m}{n}}$ (n, m are positive integers) with obvious semantics. The greater equal relation $\geq$ is extended to fuzzy values as follows: $\lambda_1 \geq \lambda_2$ iff for all interpretations $\mathcal{I}$, $\lambda_1{}^{\mathcal{I}} \geq \lambda_2{}^{\mathcal{I}}$. Similarly for the relation $>$. Checking whether $\lambda_1 \geq \lambda_2$ can be done by observing that $(n, L) \geq (m, L')$ iff $n \geq m$ and if $m \neq 0$ then $L \subseteq L'$. The reader can verify that it is determinable whether $\frac{\lambda_1}{\lambda_2} \geq \frac{\lambda_3}{\lambda_4}$ and $\lambda_1^{\frac{m_1}{n_1}} \geq \lambda_2^{\frac{m_2}{n_2}}$. For instance, $0.7 \cdot x_1 \geq 0.6 \cdot x_1 \cdot x_2$, whereas $(0.9 \cdot x_1)^{\frac{1}{3}} \geq (0.5 \cdot x_1 \cdot x_2)^{\frac{1}{2}}$, since $(0.9 \cdot x_1)^2 \geq (0.5 \cdot x_1 \cdot x_2)^3$.

In what follows, we will use the obvious extension of the definition of satisfiability with the following clauses on signed propositions involving $\Rightarrow$: (i) $\mathtt{NT}[A \Rightarrow B \geq \lambda]$ is of type α and $\mathtt{T}[A \geq x]$ and $\mathtt{NT}[B \geq \lambda \cdot x]$ are its α_1 and α_2 components (for a "new" fuzzy variable x), and (ii) $\mathtt{T}[A \Rightarrow B \geq \lambda_1]$ is of type β and $\mathtt{NT}[A \geq \lambda_2]$ and $\mathtt{T}[B \geq \lambda_1 \cdot \lambda_2]$ are its β_1 and β_2 components (for a all λ_2). Moreover, $\mathtt{T}[A \geq \lambda_1]$ and $\mathtt{NT}[A \geq \lambda_2]$ are called *conjugated signed propositions* if $\lambda_1 \geq \lambda_2$.

Algorithm MaxVal(S, A) below computes the set of maximal fuzzy values $\lambda_1, \ldots, \lambda_n$ for which $S \cup \{\mathtt{NT}[A \geq \lambda_i]\}$ is not satisfiable. If MaxVal$(\mathtt{T}\Sigma, A) = \{n\}$, where $n \in [0,1]$, then $Maxdeg(\Sigma, A) = n$, otherwise $Maxdeg(\Sigma, A) = 0$.

Let ϕ_i be a not closed and completed branch of a deduction tree and x a fuzzy variable. Let N_i be the set of all fuzzy values λ such that (i) both $\mathtt{T}[A \geq x^{n_1} \cdot \lambda_1]$ and $\mathtt{NT}[A \geq x^{n_2} \cdot \lambda_2]$ are in S^{ϕ_i}, where $n_1 + n_2 \geq 1$ and $n_1 \neq n_2$; (ii) if $n_1 < n_2$ then $\lambda = (\frac{\lambda_1}{\lambda_2})^{\frac{1}{n_2 - n_1}}$; (iii) if $n_1 > n_2$ then $\lambda = (\frac{\lambda_2}{\lambda_1})^{\frac{1}{n_1 - n_2}}$. N_i is just the set of fuzzy values λ such that ϕ_i is closed whenever x is substituted by λ, *i.e.* $\mathtt{T}[A \geq x^{n_1} \cdot \lambda_1]$ and $\mathtt{NT}[A \geq x^{n_2} \cdot \lambda_2]$ will be a conjugated pair. It can be verified that $0 \leq \lambda \leq 1$.

Algorithm 3 (MaxVal(S, A))
Let the root node be labelled with $S \cup \{\mathtt{NT}[A \geq x]\}$, where x is a new fuzzy variable. At each step of the construction of a deduction tree the following steps are performed[6]*:*

1. *select a branch ϕ which is not yet completed;*
2. *expand ϕ by means of the rules (A), (B1) and (B2) until it becomes AB-completed. Let ϕ' be the resulting branch;*
3. *if ϕ' is neither closed nor completed then*
   (a) *select a proposition of type β which is not yet fulfilled in the branch;*

[5] Note that $(p, \mathtt{N}m)$, $(p \Rightarrow q, \mathtt{N}n) \models_2^{pos} (q, \mathtt{N}\min\{m, n\})$ holds and $n \cdot m \leq \min\{m, n\}$.

[6] The branches ϕ will be maintained maximal, *i.e.* not both $\mathtt{T}[A \geq \lambda_1]$ and $\mathtt{T}[A \geq \lambda_2]$ are in S^ϕ with $\lambda_2 > \lambda_1$. Moreover, $\mathtt{T}[A \geq 0] \notin S^\phi$. Similarly for case $\mathtt{NT}$.

(b) *if* β_1 *is of type* $\mathtt{T}[A \Rightarrow B \geq \lambda]$ *then let* $\{\lambda_1, \ldots, \lambda_l\}$ *be* $MaxVal(S^{\phi'} \backslash \{\mathtt{T}[A \Rightarrow B \geq \lambda]\}, A)$, *expand* ϕ' *by means of one children node labelled* $\mathtt{T}[B \geq \lambda \cdot \lambda_1], \ldots, \mathtt{T}[B \geq \lambda \cdot \lambda_l]$ *and go to step 1;*

(c) *otherwise apply rule (PB) with* β_1 *and* β_1^c *as PB-formulae and go to step 1;*

otherwise, go to step 1.

4. *for all not closed and completed branches* ϕ_i $(1 \leq i \leq h)$ *let* $n_i := \max N_i$ $(\max \emptyset = 0)$; *for all closed and completed branches* ϕ_i $(h+1 \leq i \leq k)$, *let* $n_i := 1$. $MaxVal(S, A) := \min\{n_1, \ldots n_k\}$. ■

Just notice that Step 3b is not needed in $\mathcal{L}^f$. Moreover, the procedure can be improved by performing Step 4. during Step 2. - 3. It can be shown that

Proposition 11 *Let* Σ *be a* $\mathcal{L}_+^f$ *KB and* $A \in \mathcal{L}_+$. $Maxdeg(\Sigma, A) = n > 0$ *iff* $MaxVal(\mathtt{T}\Sigma, A) = \{n\}$, *and checking* $Maxdeg(\Sigma, A) \geq n$ *is a coNP-complete problem.* ⊣

Example 6 Consider Example 2. MaxVal($\mathtt{T}\Sigma, A$) generates two branches ϕ_1 and ϕ_2, where $S^{\phi_1} := S \cup \{\mathtt{T}[q \geq 0.6]\}$, $S^{\phi_2} := S \cup \{\mathtt{NT}[q \geq 0.6], \mathtt{T}[r \geq 0.6]\}$ and $S = \{\mathtt{T}[p \geq 0.5], \mathtt{T}[q \geq 0.5], \mathtt{NT}[p \geq x], \mathtt{NT}[r \geq x]\}$. Now, $N_1 = \{0.5\}$, n_1=0.5, N_2=$\{0.5, 0.6\}$ and $n_2 = 0.6$. Hence, $Maxdeg(\Sigma, A)$=$\min\{n_1, n_2\} = 0.5$. ■

Example 7 Let A be $(p \Rightarrow q) \vee ((p \Rightarrow q) \Rightarrow q)$. We have seen that $\models_4 A$, whereas $\not\approx_4 [A \geq n]$, for all $n > 0$. Compute MaxVal($\emptyset, A$). The computation generates an unique branch ϕ such that S^ϕ contains $\mathtt{T}[p \geq x_1]$, $\mathtt{NT}[q \geq x \cdot x_1]$, $\mathtt{T}[p \Rightarrow q \geq x_2]$ and $\mathtt{NT}[q \geq x \cdot x_2]$. By Step 3b, a recursive call MaxVal($S^\phi \setminus \{\mathtt{T}[p \Rightarrow q \geq x_2]\}, p$) will be performed answering with $\{x_1\}$, *i.e.* the set of maximal degrees of p with respect to $S^\phi \setminus \{\mathtt{T}[p \Rightarrow q \geq x_2]\}$. Hence, the computation proceeds with branch ϕ', where $S^{\phi'}$ is $S^\phi \cup \{\mathtt{T}[q \geq x_2 \cdot x_1]\}$. Finally we will have MaxVal($\emptyset, A$)=$\{x_1, x_2\}$. Therefore, $Maxdeg(\emptyset, A) = 0$. ■

Finally, let $\overline{\mathcal{L}}_+^f$ be the extension of $\overline{\mathcal{L}}_+$ to the fuzzy case. In this case, it can be shown that MaxVal(S, A), without Step 3. and such that N_i is computed during Step 2., can be modified in such a way that it runs in *polynomial time*. Roughly, given *e.g.* $\{\mathtt{T}[A \wedge B \Rightarrow C \geq 0.6], \mathtt{T}[A \geq 0.7], \mathtt{T}[B \geq 0.8], \mathtt{NT}[C \geq x]\}$, rule $(B1)$ can be applied and we add $\mathtt{T}[C \geq \min\{0.7, 0.8\} \cdot 0.6]$ to the branch and finally we get $x = 0.42$.

Proposition 12 *Let* Σ *be a* $\overline{\mathcal{L}}_+^f$ *KB and* $A \in \overline{\mathcal{L}}_+$. *Computing* $Maxdeg(\Sigma, A)$ *can be done in time* $O(|\Sigma||A|)$. ⊣

Note that the KB in Example 5 is a $\overline{\mathcal{L}}_+^f$ KB.

5 Conclusions

There are two main contributions in this paper. The first one is an alternative procedure to Levesque's algorithm for deciding entailment in $\mathcal{L}$ (with same complexity on propositions in CNF), but which works too for $\mathcal{L}_+$, *i.e.* $\mathcal{L}$ with modus ponens. The second one is the definition of the logic $\mathcal{L}_+^f$ for reasoning in presence of vague concepts and inconsistencies with an expressively powerful and computationally tractable case. These two parts can be furthermore combined, without affecting the computational complexity, by combining fuzzy propositions $[A \geq n]$ with the operators $\wedge, \vee, \neg$ and $\Rightarrow$ and, thus, allowing fuzzy propositions of type, *e.g.* $[A \geq n] \vee [B \geq m]$ and $[A \geq n] \Rightarrow [B \geq m]$. A decision procedure can simply be obtained by combining the algorithms for deciding entailment in $\mathcal{L}_+$ and the one for $\mathcal{L}_+^f$.

Uncertain fuzzy propositions can be obtained by allowing expressions of type $(\gamma, \mathtt{P}n)$ and $(\gamma, \mathtt{N}n)$, where γ is a fuzzy proposition. The development of both a precise semantics within our four-valued framework and a calculus for automated reasoning in it, can be seen as interesting topics of further research.

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References

[Belnap, 1977] N. D. Belnap. A useful four-valued logic. In G. Epstein and J. M. Dunn, ed., *Modern uses of multiple-valued logic,* pages 5-37. Reidel, Dordrecht. NL, 1977.

[Chen and Kundu, 1996] J. Chen and S. Kundu. A sound and complete fuzzy logic system using Zadeh's implication operator. In *Proc. of the 9th Int. Syrup, on Methodologies for Intelligent Systems,* LNAI 1079. pages 233-242. Springer, 1996.

[D'Agostino and Mondadori, 1994] M. D'Agostino and M. Mondadori. The taming of the cut. Classical refutations with analytical cut. *Journal of Logic and Computation,* 4(3):285-319, 1994.

[Dubois and Prade, 1986] D. Dubois and H. Prade. Possibilistic logic. In D. M. Gabbay and C. J. Hogger, ed.. *Handbook of Logic in Artificial Intelligence,* 3:439-513. Clarendon Press, Oxford, Dordrecht* NL, 1986.

[Hollunder, 1994] B. Hollunder. An alternative proof method for possibilistic logic and its application to terminological logics. In *10th Annual Conference on Uncertainty in Artificial Intelligence,* Seattle, WA, 1994.

[Levesque, 1984] H. J. Levesque. A logic of implicit and explicit belief. In *Proc. of AAAI-84,* pages 198-202. Austin, TX, 1984.

[Meghini *et al,* 1997] C. Meghini, F. Sebastiani, and U. Straccia. Reasoning about the form and content for multimedia objects (extended abstract). In *Proc. of AAAI 1997 Spring Symp. on Intelligent Integration and Use of Text, Image, Video and Audio,* pages 89-94, Stanford University, CA, 1997.

[Smullyan, 1968] R. M. Smullyan. *First Order Logic.* Springer, 1968.

[Wagner, 1991] Gerd Wagner. Ex contradictione nihil sequitur. In *Proc. of IJCAI-91,* pages 538-543, Sydney, Australia, 1991.