CNRS-CPT-80/P-1178
FR8001718

THE SPONTANEOUS BREAKDOWN OF CHIRAL SYMMETRY IN QCD

K. YOSHIDA[*]

ABSTRACT

It is suggested that the usual path integral representation of Euclidean vacuum amplitude (tunneling amplitude) in QCD must be supplemented by the explicit boundary condition corresponding to the spontaneous breaking of chiral SU(N) x SU(N). Adopting the trial wave function introduced by Nambu and Jona-Lasinio, one sees that such a path integral automatically breaks also the additional chiral U(1) symmetry of massless quarks. The catastrophe of semi-classical approach to QCD and "U(1) problem" would be avoided in this way and one has, in principle, a better starting point for the self-consistent calculation.

FEBRUARY 1980
80/P.1178

[*] Centre de Physique Théorique, CNRS, Luminy, Marseille
On leave of absence from Istituto di Fisica, Università di Salerno, Salerno, Italy

Postal Address : Centre de Physique Théorique
CNRS - Luminy - Case 907
13288 MARSEILLE CEDEX 2 (France)

I. INTRODUCTION

One of the popular methods in the theoretical study of quantum chromodynamics (QCD) is (or used to be) the semi-classical method and its generalizations [1] .

Such an approach gives very good qualitative description when applied to the quantum mechanics problems [2] , even if used in a rather crude form. But this technics seems to suffer various defects when applied to the problems of quantum field theory in general, and Q.C.D. in particular.

In latter case, while there have been the series of works on the possibility of the spontaneous breaking of chiral symmetry and the generation of quark masses [3] , [4] many of which follow the classical observation of 't Hooft [5] , one is also worried by the fact that, as soon as one tries to analyze the situation by semi-classical method even in its most general form [6] , one gets the results completely contrary to the expectation [7] .

It was Crewther who examined this and related problems ("U(1) problem") in the greatest detail [8] and his conclusion was that, even if one is to reject the most general assumption of semi-classical method such as the importance of classical solutions with finite Euclidean action, and thus the whole idea of integer topological numbers, one is still left with quite severe chiral selection rules which may minimize the significance of "gauge non invariance" of U(1) axial charge. Thus, in spite of observation by 't Hooft [5] , one would be in difficulty so long as one does not admit the unwanted U(1) Goldstone boson [8] [9].

On the other hand, recently there appeared the series of works based on 1/N expansion [10] of QCD which have shown that the appearance of U(1) Goldstone boson, after all, may not be so disastrous and one can get on quite happily with normal current algebra type phenomenology as long as one does not really insist on the quantitative explanations of, for instance, η' mass or, indeed, pion decay constant [11] .

At the same time, Witten has shown the possible unreliableness of semi-classical method in the problems of quantum field theory [12] . If one defines the semi-classical method as the Gaussian expansion around the arbitrary (well separated) real minima of the Euclidean action, Witten's idea was confirmed by the exact calculation by Lüscher and Berg on the special model [13] [14]. It is quite possible that one must interpret η' as U(1) Goldstone boson [11, 12, 15] and moreover that one cannot ask for the quantitative explanation beyond the consistency argument offered by $1/N$ approximation [11, 12] .

However, even if the most familiar method of the dilute gas approximation is shown to be definitely misleading in some cases [13] [14] , there seems to be still quite a few unsolved problems as well as the possibilities of computational improvement in the semi-classical technics in field theory [16] .

In the following note, I would like to present the arguments to show that the conclusion of Crewther and others is not the most general one which one can expect within the framework of conventional QCD. Even the seemingly clear-cut conclusion from dilute gas approximation [7] of QCD may partly originate from the way in which basic "path integral" representation is written down without due regard for the boundary conditions.

I shall begin with the summary of Crewther's argument [19] and the simple minded derivation of the chiral selection rule in the functional language.

The path integral representation for the Euclidean expectation value with respect to so-called θ -vacua [1] of operator (or the product of operators) $X(\psi, \bar{\psi}, A_\mu)$ is

$$\langle X(\psi, \bar{\psi}, A_\mu) \rangle_\theta = \frac{1}{N} \int \mathcal{D}A_\mu \; e^{-S_E^{YM}} \cdot \int \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \; e^{-\frac{i\theta}{16\pi^2}\int \mathrm{Tr} \hat{F} \hat{\tilde{F}}} \; e^{-S_E^F} \cdot X(\psi, \bar{\psi}, A_\mu) \tag{1}$$

(N ; normalization factor)

where S_E^{YM} = pure Yang-Mills action

$$= \frac{1}{2g^2} \int d^4x \; \mathrm{Tr} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \tag{2}$$

$$S_E^F = \int d^4x \; \bar{\psi}_{su} \delta_{st} (\delta_{uv} \partial_\mu + (\hat{A}_\mu)_{uv}) \gamma^{\mu\prime} \psi_{tv} + \sum_{s \geq L+1} \int d^4x \; m_s \bar{\psi}_{su} \psi_{su}$$

with

$$\hat{A}_\mu = \frac{g}{i} A_\mu^a \cdot T^a \tag{3}$$

$$[T^a, T^b] = i f^{abc} T^c \tag{4}$$

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + [\hat{A}_\mu, \hat{A}_\nu]$$

and

$$^*\hat{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} \hat{F}_{\lambda\sigma} .$$

The Euclidean γ matrices $\{\gamma_\mu'\}_{\mu=1}^4$ satisfy

$$\{\gamma_\mu', \gamma_\nu'\} = 2\delta_{\mu\nu}$$

The indices s,t... and u,v... refer to the flavours and colours of quarks and run for $1,\ldots,N_F$ and $1,\ldots,N_C$ respectively. The letter L in the second term of S_E^F (3) represents the number of light quarks. Physically $L \sim 2$ [17] . One can also define the expectation value with respect to the generalized "topological" sector [18] $\langle \cdots \rangle_\nu$ by

$$\langle X \rangle_\nu = \frac{1}{N'} \int \mathcal{D}A_\mu \, \delta\left(-\frac{1}{16\pi^2}\int \mathrm{Tr}\, \overset{\star}{\hat{F}} \hat{F} \, d^4x - \nu\right) e^{-S_E^{YM}} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \; X \; e^{-S_E^F} \quad (5)$$

One can derive the chiral Ward-Takahashi identity (CWT.I) for $\langle \cdots \rangle_\theta$ or $\langle \cdots \rangle_\nu$ by applying the transformation of the integral variables

$$\begin{aligned} A_\mu^a(x) &\to A_\mu^a(x) \\ \psi_s(x) &\to e^{i\alpha_s(x)\gamma_5}\psi_s(x) \\ \bar{\psi}_s(x) &\to \bar{\psi}_s(x)\, e^{i\alpha_s(x)\gamma_5} \end{aligned} \quad (6)$$

where
$$\alpha_s(x) = \begin{cases} \alpha(x) & s = 1, 2, \cdots L \\ 0 & s = L+1, \cdots N_F \end{cases}$$

Only non trivial point is the regularization of the formally divergent integral

$$\mathrm{Det}\left[\frac{\partial(\psi'(x), \bar{\psi}'(x))}{\partial(\psi(x), \bar{\psi}(x))}\right]$$

which is the Jacobian of the transformation (6).

As has been shown by Fujikawa [19] , this can be done by ζ -function regularization [20] of Hawking and one has the regularized expression

$$\int \mathcal{D}\psi' \mathcal{D}\bar{\psi}' = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \; \exp i \left\{ 2L \cdot \frac{1}{16\pi^2} \int \alpha(x) \, \mathrm{Tr}(\overset{*}{F} F) d^4x \right\} \tag{7}$$

Thus one arrives at the equality

$$\begin{aligned} \langle X \rangle_\theta = & \frac{1}{N} \int \mathcal{D}A_\mu \; e^{-S_E^{YM}} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \\ & \times X \left(e^{i\alpha(x)\gamma_5} \psi(x), \; \bar{\psi}(x) e^{i\alpha(x)\gamma_5}, \; A_\mu \right) \\ & \times \exp \left\{ \frac{-i}{16\pi^2} \int (\theta - 2L\,\alpha(x)) \, \mathrm{Tr} \, \hat{\overset{*}{F}} \hat{F} \, d^4x \right\} \\ & \times \exp - \left\{ S_E^F + \int \partial_\mu \alpha(x) \, i \bar{\psi} \gamma_\mu \gamma_5 \psi(x) \right\} \end{aligned} \tag{8}$$

where one has used the fact that the quark-gluon action (3) is invariant under (6), except the term coming from the space-time dependence of $\alpha(x)$.

Since LHS is independent of $\alpha(x)$, one can write

$$\frac{\delta}{\delta \alpha(x)} \left[R.H.S \right] \Big|_{\alpha(x) \to 0} = 0$$

and this gives the well-known local WTI ,

$$\frac{1}{N}\int \mathcal{D}A_\mu \cdot e^{-S_E^{YM}} \int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}$$
$$\times \Big[\frac{\delta X}{\delta \alpha(x)}\Big|_{\alpha=0} + \Big\{ 2L \cdot \frac{1}{16\pi^2} \mathrm{Tr}\, {}^{*}\hat{F}\hat{F}(x)$$
$$+ \partial^\mu \sum_{s=1}^{L} (\bar{\psi}_s \gamma_\mu \gamma_5 \psi_s)(x) \Big\} X \Big]$$
$$\times \exp\Big\{ -\frac{i\theta}{16\pi^2} \int \mathrm{Tr}\, {}^{*}\hat{F}\hat{F}\, d^4x \Big\} \exp -S_E^F \qquad (9)$$

Now assume the operator X is of the form

$$X = \prod_k X_k(x_k)$$

with

$$X_k(x_k) \longrightarrow e^{i\alpha(x)\chi_k} X_k(x_k) \qquad (10)$$

under (6).

Then (9) becomes

$$\frac{1}{N}\int \mathcal{D}A_\mu\, e^{-S_E^{YM}} \int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}$$
$$\times \Big[\delta(x-x_k)\chi_k + 2L \cdot \frac{1}{16\pi^2} \mathrm{Tr}\, {}^{*}FF(x)$$
$$+ \partial^\mu \sum_{s=1}^{L} (\bar{\psi}\gamma_\mu \gamma_5 \psi)(x) \Big] X$$
$$\times \exp\Big\{ -\frac{i\theta}{16\pi^2} \int \mathrm{Tr}\, {}^{*}\hat{F}\hat{F} \Big\} \exp -S_E^F \qquad (9')$$

At this point, one puts the all-important assumption that the gauge invariant axial U(1) current $\bar{\psi}\gamma_\mu\gamma_5\psi$ does not couple to zero mass particle for any value of θ [21] .

Then one can drop the current divergence term in (9') after integrating over $\mathcal{X}$, and one ends up with

$$[\, i\,\chi_{tot} - 2L \frac{d}{d\theta}\,] \langle X \rangle_\theta = 0 \qquad (11)$$

where $\chi_{tot} \doteq \sum_k \chi_k$

Written in term of ν vacuum (4), one has

$$[\, \chi_{tot} - 2L\nu\,] \langle X \rangle_\nu = 0 \qquad (12)$$

i.e. $\langle X \rangle_\nu \equiv 0$ unless the chirality of X satisfies

$$\chi_{tot} = 2L\nu$$

If one further assumes that the topological number ν can take only the integer value

$$\int \mathcal{D}A_\mu = \sum_{\nu=-\infty}^{\infty} \int [\mathcal{D}A_\mu]_{-\frac{1}{16\pi^2}\int Tr\, \tilde{F}F\, d^4x = \nu} \qquad (13)$$

then one needs to think only of $\{X_\nu\}_{\nu=-\infty}^{\infty}$ and $\langle X \rangle_\theta$ becomes Fourier series [22]

$$\langle X \rangle_\theta = \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta} \langle X \rangle_\nu \qquad (14)$$

One of the first "difficulties" which worried some authors [7] were

$$\langle \bar{\Psi}(1 \pm \gamma_5)\Psi \rangle_\theta = 0 \qquad \text{for any } \theta$$

$$\text{if} \qquad L \geq 2 \qquad (15)$$

This is the immediate consequence of (12) since $\langle X \rangle_{\nu=0} = 0$ trivially unless $\chi(X) = 0$. But here one can obtain (15) only assuming the boundary condition : $A_\mu(x) \to g\,\partial_\mu g^{-1}(x)$ (pure gauge) as $|x| \to \infty$ which implies (13). (15) was taken as implying the impossibility of spontaneous mass generation but it can be interpreted as the impossibility of whole current algebra, since $\langle \bar{\psi} \psi \rangle_{\theta=0} \sim F_\pi$. Apart from this, it has been pointed out that (12) implies that the vacuum expectation value $\langle \cdots \rangle_0$ is not the periodic function of θ with period 2π but with $2\pi L$ [23].

It is not easy to see what has really gone wrong in the derivation discussed above (which is conventional and presented here only for explanatory purpose) of chiral selection rule in the limit of $U(L)\times U(L)$symmetry.

If one is to accept the above derivation (or the more careful current algebra derivation [8] of the same selection rule), then it seems that one cannot despense with physical U(1) Goldstone boson, in spite of 't Hooft [5] observation that the colour gluons can break chiral U(1) symmetry via Adler-Bell-Jackiw anomaly.

In what follows, I would like to suggest that the conventional path integral (1) is, to say the least, a rather inconvenient starting point for any approximation scheme one may like to apply. Unless somehow one can solve the problem exactly, one would never reach the desired "symmetry breaking phase" from expression such as (1).

II. THE PATH INTEGRAL IN THE QUANTUM MECHANICS

To illustrate the possible modification of the path integral representation (1), which leads to the selection rules (11) et (12), I would like to discuss first a relatively trivial problem from quantum mechanics.

Let us consider the system of two fermionic oscillators of the same frequency or "bare mass" (particle and antiparticle), coupled to each other through a constant potential.

The Hamiltonian is

$$H = \omega (a^* a + b^* b) + A (a^* + b)(a + b^*) + \text{const.} \qquad (16)$$

where the creation and the annihilation operators satisfy the anti-commutation relations

$$\{a^*, a\} = 1 \qquad \{b^*, b\} = 1$$

$$a^2 = a^{*2} = b^2 = b^{*2} = 0 \qquad (17)$$

The constant in (16) is added to make the ground state energy to take some convenient value. Hamiltonian like (16) (with many degrees of freedom) makes frequent appearance in the problems of statistical mechanics, such as Ising model [24] .

Taking the arbitrary states expressed as the linear combinations of bases in Fock space

$$|I\rangle = (\alpha^I_{00} + \alpha^I_{10} a^* + \alpha^I_{01} b^* + \alpha^I_{11} a^* b^*)|0\rangle$$
$$|F\rangle = (\alpha^F_{00} + \alpha^F_{10} a^* + \alpha^F_{01} b^* + \alpha^F_{11} a^* b^*)|0\rangle \qquad (18)$$

(where $|0\rangle$ is the Fock-vacuum defined by $a|0\rangle = 0$, $b|0\rangle = 0$), one can express the Euclidean transition amplitude as the functional integral over the path defined in the Grassmann algebra [25] , viz.

$$
\begin{aligned}
&\langle F | e^{-H(t''-t')} | I \rangle \\
&= \lim_{N\to\infty} \prod_{j=0}^{N} \int d\eta_j^a \, d\xi_j^a \, d\eta_j^b \, d\xi_j^b \\
&\times \Psi_F(\xi_N) \exp - \sum_{j=1}^{N} \Big[\eta_j^a(\xi_j^a - \xi_{j-1}^a) + \eta_j^b(\xi_j^b - \xi_{j-1}^b) \\
&\qquad + \Delta t \left\{ \omega(\eta_j^a \xi_{j-1}^a + \eta_j^b \xi_{j-1}^b) + A(\eta_j^a + \xi_{j-1}^b)(\xi_{j-1}^a + \eta_j^b) \right\} \Big] \\
&\qquad \times \exp - (\eta_0^a \xi_0^a + \eta_0^b \xi_0^b) \\
&\qquad \times \Psi_I(\eta_0) \\
&\qquad\qquad \left(\Delta t = \frac{t''-t'}{N} \right)
\end{aligned}
\tag{19}
$$

where Ψ_I and Ψ_F are the polynomials of variables at end points $t = t' = t_0$ and $t = t'' = t_N$ representing the final and initial "wave functions" (Surface term).

$$
\begin{aligned}
\Psi_I(\eta_0) &= \alpha_{00}^I + \alpha_{10}^I \eta_0^a + \alpha_{01}^I \eta_0^b + \alpha_{11} \eta_0^a \eta_0^b \\
\Psi_F(\xi_N) &= \alpha_{00}^F + \alpha_{10}^{F*} \xi_N^a + \alpha_{01}^{F*} \xi_N^b + \alpha_{11}^{F*} \xi_N^b \xi_N^a
\end{aligned}
\tag{20}
$$

In the expression (19), the integration variables $(\xi_i^{a,b}, \eta_i^{a,b})_{i=0}^{N}$ are $4(N+1)$ generators of $2^{4(N+1)}$ dimensional Grassmann algebra and the integral over them is defined according to the measure on Grassmann variables introduced by Berezin [24] .

The "Euclidean Action" in the first exponential in (19) takes more familiar form in the formal continuum limit [36] $(\Delta t \to 0)$

$$S_E \sim \int_{t'}^{t''} dt \left[\eta_t^a \frac{d\xi_t^a}{dt} + \eta_t^b \frac{d\xi_t^b}{dt} + \omega(\eta_t^a \xi_t^a + \eta_t^b \xi_t^b) + A(\eta_t^a + \xi_t^b)(\xi_t^a + \eta_t^b) \right] \tag{21}$$

The extra surface factor at $t = t'$, $e^{-\sum_{\alpha=a,b} \eta_0^\alpha \xi_0^\alpha}$ is important when one makes the correspondence with usual canonical formalism, such as the transformation of variables [25]. However, it does not play important role in the consideration of symmetry breaking and I will omit this factor in the most of next section.

In particular, the Euclidean expectation value with respect to ordinary Fock vacuum is given by (19) with $\Psi_I = \Psi_F = 1$, and then (19) is reduced to a familiar (except the last factor) form.

$$\langle 0 | e^{-H(t''-t')} | 0 \rangle = \prod_{t=t'}^{t''} \int d\eta_t^a d\xi_t^a d\eta_t^b d\xi_t^b \times \exp - S_E \cdot \exp - (\eta_{t'}^a \xi_{t'}^a + \eta_{t'}^b \xi_{t'}^b) \tag{22}$$

On the other hand, it is clear that the Fock vacuum $|0\rangle$ is not the lowest energy eigenstate of total Hamiltonian H. Thus the simple formula (22) is not expected to give the "vacuum" amplitude $\exp - E_0(t''-t')$, except in the limit $t'' \to \infty$ and $t' \to -\infty$.

To obtain the correct energy levels, one diagonalizes the Hamiltonian by introducing the new creation and annihilation operators by the transformation [26]

$$a'(\alpha) = \cos\alpha \; a - \sin\alpha \; b^{*}$$
$$b'(\alpha) = \sin\alpha \; a^{*} + \cos\alpha \; b \qquad (23)$$

All the anti-commutation relations (17) are unchanged for a' and b'. They annihilate new vacuum defined by [27]

$$|\underline{0}\rangle_{\alpha} = (\cos\alpha + \sin\alpha \; a^{*} b^{*}) \, |0\rangle \qquad (24)$$

$$a'(\alpha)\,|\underline{0}\rangle_{\alpha} = 0$$
$$b'(\alpha)\,|\underline{0}\rangle_{\alpha} = 0 \qquad (25)$$

(Note also ${}_{\alpha}\langle\underline{0}|\underline{0}\rangle_{\alpha} = \langle 0|0\rangle = 1$).

The original Hamiltonian (16) is diagonalized if α satisfies

$$\tan 2\alpha(A) = -\frac{A}{\omega} \qquad (26)$$

Then, in term of $a'(\alpha)$ and $b'(\alpha)$, H becomes

$$H = (\sqrt{\omega^2 + A^2} + A)\, a'^{*} a' + (\sqrt{\omega^2 + A^2} - A)\, b'^{*} b'$$
$$+ \text{const.} \qquad (27)$$

This shows that the new "vacuum" $|\underline{0}\rangle_{\alpha(A)}$ is the true ground state of H . Measuring from the energy of $|\underline{0}\rangle_{\alpha(A)}$, there are also three excited states with energies

$$\begin{cases} \sqrt{\omega^2 + A^2} \pm A \\ 2\sqrt{\omega^2 + A^2} \end{cases}$$

Note that one can express the new vacuum (24) as the unitary transformation [28]

$$|\underline{0}\rangle_{\alpha} = e^{-iG}\,|0\rangle \qquad (28)$$

with $$G = i\alpha\,(a^{*}b^{*} - b\,a)$$

The true ground state $|\Omega\rangle_{\alpha(A)}$ has the overlap with Fock vacuum.

$$\langle 0|\Omega\rangle_{\alpha(A)} = \cos\alpha(A) \tag{29}$$

Combining (19), (26), (24), one can express the true Euclidean vacuum amplitude as

$$\begin{aligned}
&\langle\Omega|\,e^{-H(t''-t')}\,|\Omega\rangle \\
&= \lim_{N\to\infty}\prod_{j=0}^{N}\int d\eta_j^{a}\,d\xi_j^{a}\,d\eta_j^{b}\,d\xi_j^{b} \\
&\times(\cos\alpha + \sin\alpha\,\xi_N^{b}\xi_N^{a}) \\
&\times(\cos\alpha + \sin\alpha\,\eta_0^{a}\eta_0^{b}) \\
&\times\ \exp - S_E \cdot \exp-(\eta_0^{a}\xi_0^{a} + \eta_0^{b}\xi_0^{b})
\end{aligned} \tag{30}$$

The transformation (23) is the simplest example of Bogoliubov transformation [26] [27] familiar in the BCS [29] theory of superconductivity as well as in the study of 2 dim Ising model [36].

The example of eq.(30) does not necessarily imply that one must start worrying about the possible presence of non trivial wave functions in the expression such as (1) for QCD. Path integral (1) is the field theoretical generalization of simpler example like (30) but in the limit $t''\to\infty$ and $t'\to-\infty$. In this limit

$$\lim_{\substack{t''\to\infty\\ t'\to-\infty}} e^{-H(t''-t')} = |\Omega\rangle\langle\Omega|\,e^{-E_0(t''-t')} \tag{31}$$

where $|\Omega\rangle$ is the time ground state of H.
Thus, for the Euclidean vacuum expectation value of the operator X, such as the ones discussed in Chapter I, one can write

$$\langle \underline{0} | X | \underline{0} \rangle = \frac{\langle F | \underline{0} \rangle \langle \underline{0} | X | \underline{0} \rangle \langle \underline{0} | I \rangle}{\langle F | \underline{0} \rangle \langle \underline{0} | I \rangle}$$

$$= \frac{\lim_{\substack{t'' \to \infty \\ t' \to -\infty}} \langle F | e^{-t'' H} X e^{t' H} | I \rangle}{\lim_{\substack{t'' \to \infty \\ t' \to -\infty}} \langle F | e^{-(t''-t')H} | I \rangle} \tag{32}$$

with any pair of states $|I\rangle$ and $|F\rangle$ such that

$$\langle \underline{0} | F \rangle \neq 0 \qquad \langle \underline{0} | I \rangle \neq 0 \tag{33}$$

From this argument, one can see that normally the conventional path integral formula for field theory with Euclidean field $\varphi(x)$

$$\langle \underline{0} | X(\varphi) | \underline{0} \rangle = \frac{\int \mathcal{D}\varphi \, X(\varphi) \, e^{-S_E(\varphi)}}{\int \mathcal{D}\varphi \, e^{-S_E(\varphi)}}$$

$$S_E = \int_{-\infty}^{\infty} d^4x \, \mathcal{L}_E(\varphi) \tag{34}$$

$$(\Psi_F = \Psi_I = 1)$$

can be used to represent the true vacuum expectation value.

But there are exceptions when

(1) $\langle \underline{0} | 0 \rangle = 0$ between true vacuum $|\underline{0}\rangle$ and Fock vacuum $|0\rangle$ (35)

or

(2) vacuum is degenerate, so

$$\lim_{t \to \infty} e^{-Ht} \sim \sum_i C_i |0\rangle_i \langle 0| \tag{36}$$

when there are many degrees of freedom, the arguments given above become much more delicate due to these problems.

As the trivial example, one may generalize our simple model (16) to the case with many components of oscillators

$$(a, b) \rightarrow (a_i, b_i)_{i=1}^{\infty}$$

with the hamiltonian

$$H = \sum_{i=1}^{\infty} [\omega_i (a_i^\dagger a_i + b_i^\dagger b_i) + A_i (a_i^\dagger + b_i)(a_i + b_i^\dagger)] \quad (37)$$

where one can imagine that ω_i is concentrated just below some "ionization energy".

e.g., $\omega_i = -\lambda/i^2$ while the couplings and A_i are such that

$$|\cos \alpha(A_i, \omega_i)| < \gamma < 1 \quad (38)$$

where $\alpha(A_i, \omega_i)$ is the Bogoliubov angle given by (26).

The model can have finite 0-point energy. But the overlap between true vacuum $|\underline{0}\rangle$ and the Fock vacuum $|0\rangle$, given by $a_i|0\rangle = 0$, $b_i|0\rangle = 0$, is

$$|\langle 0|\underline{0}\rangle_{\tan 2\alpha_i = -\frac{A_i}{\omega_i}}| < \lim_{M \to \infty} |\gamma|^M = 0 \quad (39)$$

One will then never arrive at the correct vacuum expectation value with the trivial wave function $\Psi_I = \Psi_F = 1$. Indeed, in such a case, one will observe the collapse of the vacuum amplitude

$$\lim_{t \to \infty} \langle 0| e^{-Ht} |0\rangle = 0 \quad (40)$$

($|0\rangle$; Fock vacuum)

simply because

$$\lim_{t \to \infty} \| e^{-Ht} |0\rangle \| = 0 \quad (41)$$

(taking the ground state energy as $E_0 = 0$).

As for the presence of vacuum degeneracy (which we hope to find in (1)), this normally necessitates the non trivial boundary condition in the path integral so that one may arrive at one of the degenerate vacua instead of a statistical mixture. (Remember, for instance, linear σ-model with scalar fields). For fermionic path integral, such a boundary condition can be simply put only through the explicit wave functions.
In any case, the question is then what sort of wave function or the boundary condition can survive the limit $\tau \to \infty$ in Euclidean path integral.

Since it is not possible to analyze the problem with full mathematical rigour, I am going to present in the next section the phenomenological guess for the trial vacuum wave function which is consistent with the idea of spontaneous breaking of chiral $SU(N_F)$ symmetry and current algebra. In this way, one will be led to the original formalism of chiral symmetry breaking model as it has been conceived by Nambu and Jona-Lasinio [31] .

III. THE PATH INTEGRAL IN QCD.

The discussions at the end of last section suggest that the way to modify the path integral (1), so that the resultant chiral selection rules may be less severe than (12), is to add the non trivial wave function which can contribute to the Euclidean path integral in the limit $t'' \to \infty$ and $t' \to -\infty$. Such a wave function must be able to induce the system to fall into one of degenerate vacua and thus must contain the germ of chiral SU(N) symmetry breaking in itself.

The discussion of Section I shows that the non trivial boundary condition on the gauge field integration $\mathcal{D}A_\mu$, although it breaks chiral U(1) symmetry [30], does not have enough symmetry breaking in it. Thus the simplest possibility would be to look for the wave functions which depend on the "fermionic" variables ψ and $\bar{\psi}$ at the boundary surface $t = \pm\infty$. From the way in which our path integral is defined (i.e. as the generalization of (19) to infinitely many degrees of freedom), this wave function should express the relationship between the Fock vacuum of massless quarks and antiquarks and the true physical vacuum where the chiral symmetry is spontaneously broken and quarks are massive.

Now, just such a relationship has been considered in the classical paper by Nambu and Jona-Lasinio [31] introducing for the first time the "Goldstone pions" in the theory of strong interaction.

According to these authors, the chiral symmetry is spontaneously broken through the "super conducting" states where the massless quark and anti-quark pairs (nucleon-anti-nucleon of Ref. 31) of same helicity and opposite momenta form the "Cooper pairs" [35].

In analogy with the coherent trial states of Refs. [26] and [27], Nambu and Jona-Lasinio give the explicit expression in the simplest case of $N_F = 1$. [32].

$$|\Omega^m\rangle = \prod_{p,\lambda} \left\{ \sqrt{\tfrac{1}{2}(1+\beta_p)} + \sqrt{\tfrac{1}{2}(1-\beta_p)}\; a^*(p,\lambda)\, b^*(-p,\lambda) \right\} |\Omega^0\rangle \tag{42}$$

p, quark momentum

λ , helicity

and $\beta_p = |p| / \sqrt{p^2 + m^2}$ (m : parameter).

$|\Omega^0\rangle \equiv |0\rangle$ is the Fock vacuum of the massless "nucleons" or quarks

$$a(p, \lambda)|\Omega^0\rangle = 0$$

$$b(p, \lambda)|\Omega^0\rangle = 0 \quad \text{for all } p \text{ and } \lambda \tag{43}$$

Writing

$$\sin\theta(p) = \sqrt{\tfrac{1}{2}(1-\beta_p)}$$

$$\cos\theta(p) = \sqrt{\tfrac{1}{2}(1+\beta_p)}$$

one sees that the formula (42) corresponds to the Bogoliubov transformation

$$\left.\begin{aligned} a'(p,\lambda) &= \cos\theta(p)\, a(p,\lambda) + \sin\theta(p)\, b^*(-p,\lambda) \\ b'(p,\lambda) &= -\sin\theta(p)\, a^*(-p,\lambda) + \cos\theta(p)\, b(p,\lambda) \end{aligned}\right\} \tag{44}$$

The new annihilation operators satisfy

$$a'(p,\lambda)|\Omega^m\rangle = 0$$

$$b'(p,\lambda)|\Omega^m\rangle = 0 \tag{45}$$

for all p and λ .

The parameter m , which is related to the Bogoliubov angle as

$$\tan\theta(p) = \left(\sqrt{p^2+m^2} - |p|\right)/m \tag{46}$$

corresponds to the spontaneously generated mass of quarks.
This can be in principle calculated with self-consistent method [33].

The chiral symmetry breaking trial state (42) of Nambu Jona-Lasinio is of the form discussed in Section II. Moreover, if one calculates the overlap with Fock vacuum [31],

$$\langle 0 | \underline{\Omega} \rangle \equiv \langle \Omega^0 | \Omega^m \rangle$$

$$= \exp 4\pi \int_0^\infty dp\, p^2 \, l - \sqrt{1+\beta_p} = 0 \tag{47}$$

because the exponent is negative at large momentum and diverges linearly with the ultra-violet limit of integral.

The Fock vacuum has the zero overlap with the "super conducting" state of Nambu and Jona-Lasinio just because of the contribution from large number of states with high momentum. According to the discussion of Section II, this may be taken as the indications that any approximation scheme starting from the conventional path integral (1) would run into difficulties [26] . We are in the situation where one cannot hope to proceed trivial wave function and reach the true vacuum "perturbatively". (Even with non trivial configuration of A_μ).

Before writing down the modified path integral which should replace (1), I generalize the Nambu-Jona-Lasinio representation (42) to chiral SU(N_F) with $N_F > 1$

$$|\Omega^{(m)}\rangle_{\underline{\Omega}} = \prod_{p,\lambda=\pm 1} \{ \cos\theta(p)$$

$$+ \sin\theta(p) \sum_{s,t} a^{*}_{su}(p,\lambda) \left(e^{2i\lambda\underline{\Omega}\cdot I}\right)_{st} e^{2i\lambda\alpha} b^{*}_{tu}(-p,\lambda) \} |\Omega^0\rangle \tag{48}$$

where the angle $\theta(p)$ is chosen as before, and

$\{T_a\}_{a=1}^{N_F^2-1}$; the generators of SU(N_F) in the quark representation.

$\{\Omega_a\}_{a=1}^{N_F^2-1}$; parametrize vacuum degeneracy with respect to the chiral part of SU(N_F) x SU(N_F).

α : parametrize va uum degeneracy with respect to chiral U(1). Corresponding to the global chiral transformat'on of the field operators

$$e^{iQ_f\cdot\Omega'}\psi_{su}\,e^{-iQ_f\cdot\Omega'} = \left[e^{iI\cdot\Omega'\gamma_5}\right]\psi_{tu} \qquad (49)$$

One has

$$e^{iQ_f\cdot\Omega'}|\Omega^{(m)}\rangle_{\Omega\alpha} = |\Omega^{(m)}\rangle_{\Omega'';\alpha}$$

where Ω'' is given by

$$e^{i\Omega' I}\,e^{2i\Omega\cdot I}\,e^{i\Omega' I} = e^{2i\Omega'' I} \qquad (50)$$

$|\Omega^m\rangle$ also breaks chiral U(1) which amounts to the change of parameter

$$\alpha \to \alpha + \alpha' \qquad (51)$$

Now I can put the wave function corresponding to the trial state (48) into (1) and obtain the following modified path integral representation of the vacuum expectation value of operator $X(\psi,\psi^\dagger,A_f)$

$$\begin{aligned}
&\langle 0|X(\psi,\psi^\dagger,A_f)|0\rangle_{\theta,\Omega}\\
&= \prod_{x,\tau}\int \mathcal{D}A_f(x,\tau)\,e^{-S_E^{YM}}\int\mathcal{D}\psi(x,\tau)\,\mathcal{D}\bar\psi(x,\tau)\\
&\quad\prod_{p,\lambda=1}\left\{\cos\theta(p)+\sin\theta(p)\left[\eta_t^{\dagger}(-p,\lambda)\right]_{\tau=\infty}\left(e^{-2i\lambda\Omega\cdot I}\right)_{ts}\left[\xi_s^{1}(p,\lambda)\right]_{\tau=\infty}\right\}\\
&\quad\times\prod_{p,\lambda=-1}\left\{\cos\theta(p)+\sin\theta(p)\left[\eta_s'(p,\lambda)\right]_{\tau=-\infty}\left(e^{2i\lambda\Omega\cdot I}\right)_{st}\left[\xi_t^{\dagger}(p,\lambda)\right]_{\tau=-\infty}\right\}\\
&\quad\times X(\psi,\psi^\dagger,A_f)\,e^{-S_E^F}\,e^{-\frac{i\theta}{16\pi^2}\int \mathrm{Tr}\,\hat F\hat F\,d^4x}
\end{aligned}$$

$$\times\ 1/(\text{normalization factor}) \qquad (52)$$

The letters (θ, Ω) in LHS indicate the degeneracy of vacuum with respect to θ as well as the direction in the space of chiral transformation Ω, the Grassmann variables $[\xi^{1,2}_{su}(p,\lambda)]_\tau$ and $[\eta^{1,2}_{su}(p,\lambda)]_\tau$ correspond to the Fourier components of local 4-component Dirac variables $\psi(x,\tau)$ and $\psi^\dagger(x,\tau)$ at the given Euclidean time τ. One can write

$$\psi_{su}(x,\tau) = \int d^3p\, e^{ipx}\Big\{ [\xi^1_{su}(p,+1)]_\tau \begin{pmatrix} 0 \\ w_+(p) \end{pmatrix} + [\xi^1_{su}(p,-1)]_\tau \begin{pmatrix} w_-(p) \\ 0 \end{pmatrix}$$

$$+ [\xi^2_{su}(-p,+1)]_\tau \begin{pmatrix} w^*_-(p) \\ 0 \end{pmatrix} + [\xi^2_{su}(-p,-1)]_\tau \begin{pmatrix} 0 \\ -w^*_+(p) \end{pmatrix} \Big\}$$

$$\psi^\dagger_{su}(x,\tau) = \int d^3p\, e^{-ipx}\Big\{ [\eta^1_{su}(p,+1)]_\tau (0, w^\dagger_+(p)) + [\eta^1_{su}(p,-1)]_\tau (w^\dagger_-(p), 0)$$

$$+ [\eta^2_{su}(-p,+1)](w^T_-(p), 0) + [\eta^2_{su}(-p,-1)](0, -w^T_+(p)) \Big\}$$

$$(\psi^\dagger = \bar\psi\gamma_0) \tag{53}$$

The vectors $\begin{pmatrix} 0 \\ w(p) \end{pmatrix}$ etc are the massless spinors in the representation where γ_5 matrix is diagonal. One can choose, for instance,

$$w_+(p) = \sqrt{\frac{|p|+p_3}{2|p|}} \begin{pmatrix} 1 \\ \frac{p_1+ip_2}{|p|+p_3} \end{pmatrix}, \quad w_-(p) = \sqrt{\frac{|p|+p_3}{2|p|}} \begin{pmatrix} -\frac{p_1-ip_2}{|p|+p_3} \\ 1 \end{pmatrix} \tag{54}$$

and

$$\left.\begin{aligned} w_+ \otimes w_+^* &= \tfrac{1}{2}\left(1 + \frac{p\cdot\sigma}{|p|}\right) \\ w_- \otimes w_-^* &= \tfrac{1}{2}\left(1 - \frac{p\cdot\sigma}{|p|}\right) \end{aligned}\right\} \tag{55}$$

(52) is the main result of this section.
Note that one can write the coherent state (48) as unitary transformation (as in (28) of Section II)

$$|\Omega^{m}\rangle_{\underline{\Omega}\,\alpha} = \exp i\, G(a, b, a^{*}, b^{*}; \underline{\Omega}, \alpha)\, |0\rangle$$

where

$$G(a, b, a^{*}, b^{*}; \underline{\Omega}, \alpha) = i \int d^{3}p\, \theta(p) \sum_{\lambda=\pm 1} \Big\{ a^{\dagger}_{st}(p,\lambda)\left(e^{2i\lambda\underline{\Omega}\cdot\underline{T}}\right)_{st} e^{2i\lambda\alpha}\, b^{*}(p,\lambda) - b_{ts}(-p,\lambda)\left[e^{-2i\lambda\underline{\Omega}\cdot\underline{T}}\right]_{ts} e^{-2i\lambda\alpha}\, a_{st}(p,\lambda) \Big\}$$

(56)

although it is not so simple to introduce the object like (56), which is not normal ordered, into the path integral. (Normal ordered form of (56) is, of course, just the original (48)).

The "current algebra" vacuum of Nambu-Jona-Lasinio breaks chiral SU(N) x SU(N) according to (50). It also breaks the chiral U(1), i.e. under the global transformation

$$\psi(x) \rightarrow e^{i\alpha'\gamma_5}\psi(x)$$
$$\bar{\psi}(x) \rightarrow \bar{\psi}(x)\, e^{i\alpha'\gamma_5}$$

(57)

one has

$$|\Omega^{m}\rangle_{\underline{\Omega}\,\alpha} \rightarrow |\Omega^{m}\rangle_{\underline{\Omega},\,\alpha\alpha'} \qquad (58)$$

Just as in QCD Lagrangian of quarks and gluons, the invariance under the chiral SU(N) x SU(N) implies automatically the chiral U(1) invariance (unlike the Gell-Mann-Levy linear σ-model with σ, $\vec{\pi}$ and N), the spontaneous breaking of former (by coupling to the wave functions (48)) entails the breaking of the latter.

The factors $a^{\dagger}(p,\lambda)b^{\dagger}(-p,\lambda)$ (λ, helicity) in (48) have the chirality 2λ . So to study the chiral transformation property of (48), it is convenient to write this as

$$|\Omega^{m}\rangle = \prod_{p} [\cos\theta(p) + \sin\theta(p)\hat{U}^{+}(p)] \times \prod_{p} [\cos\theta(p) + \sin\theta(p)\hat{U}^{-}(p)] \ |0\rangle \tag{59}$$

where the operators $\hat{U}^{\pm}(p)$ transform as

$$U^{\pm}(p) \rightarrow e^{\pm 2i\alpha'} U^{\pm}(p)$$

under (57).
(59) means that the Nambu-Jona-Lasinio's trial states are the coherent superposition of chiral U(1) eigenstates

$$|\Omega^{m}\rangle = \sum_{\nu=-\infty}^{\infty} \hat{W}_{2\nu} \ |0\rangle \tag{60}$$

where $\hat{W}_{\chi} \rightarrow e^{i\chi\alpha'} \hat{W}_{\chi}$

under (57).
The vacuum expectation value according to the modified expression (52), $\langle\Omega^{m}| \cdot \cdots |\Omega^{m}\rangle$ can be written as

$$\sum_{\nu_f, \nu_i \to \infty} \langle 0| \hat{W}^{*}_{2\nu_f} \cdots \hat{W}_{2\nu_i} |0\rangle$$

where $|0\rangle$ is still "Fock vacuum" or the factor 1 in the path integral.

The assumption of spontaneous breaking of chiral $\underline{SU(N_F) \times SU(N_F)}$ (or the current algebra) is that the chiral sectors $\langle 0| \hat{W}^{*}_{2\nu_f} \cdots \hat{W}_{2\nu_i} |0\rangle$ with more or less arbitrary values of ν_F and ν_I should be able to contribute to the vacuum amplitude.

Now it is easy to see what has gone wrong in the analysis of Section I. Applying the same technics of WTI and taking care of chiral <u>non</u> invariance of the vacuum wave functions in (52), one immediately arrives at the modified selection rule

$$[\chi_{td} - 2L\nu - 2(\nu_F - \nu_t)] \langle X \rangle_\nu^{(\nu_F)(\nu_t)} = 0 \qquad (62)$$

which should replace (12). Here, ν_F and ν_t (positive or negative integers) refer to the component

$\langle 0 | \hat{W}^{\psi}_{2\nu_F} \cdots \hat{W}_{2\nu_t} | 0 \rangle$ of the total vacuum expectation value $\langle X \rangle_\nu$.

ν, as before, refers to the chiral symmetry breaking from the boundary condition on $\mathcal{D} A_\Gamma$.

The validity of the idea of spontaneous breaking of chiral $SU(N_F) \times SU(N_F)$ implies that the total vacuum expectation value of the operator X (ν-vacuum) must be non trivially contributed by many components (ν_F, ν_t). For each of such component terms, there will be different chiral $U(1)$ selection rules

$$\chi_{td}(X) = 2L\nu + 2(\nu_F - \nu_t) \qquad (63)$$

Conversely, the rigid chiral selection rule of (12) implies not so much as the existence or the non existence of U(1) Goldstone boson but the negation of the idea of spontaneous breaking of chiral $SU(N_F) \times SU(N_F)$ and thus the impossibility of current algebra.

The modification (52) and resultant change in the selection rule (63) of course do not affect the <u>local</u> chiral WTI with current divergences.

The only difference between path integral (1) and (52) is the addition of surface terms at $t = \pm\infty$.

Thus the standard manipulation of Section 1 with the local group parameter $\alpha(x)$, concentrated around some finite space-time point x_0 , will lead to the exactly same form of (anomalous) WTI

$$\langle \partial^\mu J_\mu^5(x_0), X \rangle = -\left\langle \frac{\delta X}{\delta\alpha(x_0)}\bigg|_{\alpha\to 0}\right\rangle - \frac{L}{16\pi^2}\langle \mathrm{Tr}\,{}^*\hat{F}\hat{F}(x_0), X\rangle$$

where $$J_\mu^5(x) = \sum_{s=1}^{L} \bar{\psi}_s \gamma_\mu \gamma_5 \psi_s(x) \qquad (64)$$

IV. SUMMARY AND CONCLUSION.

It is suggested that the controversy over U(1) chiral symmetry in the usual formalism of QCD is in fact the controversy over the possibility of chiral SU(N) x SU(N) current algebra itself in this framework. Thus the present author is in complete agreement with the spirit of Ref. [8] on this point.

The reason why one is allowed to think separately of the spontaneous breaking of chiral SU(N) x SU(N) and of that of the chiral U(1) x U(1) in some of the published works on "U(1) problem" seems to be partly due to the rather careless way in which the path integral representation of the Euclidean amplitude is understood. It must be remembered that even in the linear scalor σ -model and its equivalents in Statistical Mechanics, one must specify the boundary condition in the corresponding path integral to obtain the specific ground state which manifests the spontaneous symmetry breaking

$$\langle \sigma \rangle = F \neq 0$$

In fact, the path integral which does not specify such a boundary condition would lead to the statistical mixture of degenerate vacua where one would not observe any order parameter.

The simplest example would be the $\lambda\phi^4$ theory in two-dimension where the Euclidean action is given by

$$S_E = \int_{-\infty}^{\infty} d^2x \left[\frac{1}{2} (\partial_a \phi)^2 - \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4 \right]$$

$$a = 1, 2. \qquad \mu^2, \lambda > 0$$

One expects (and can demonstrate rigorously) the spontaneous breaking of the discrete symmetry

$$\phi \longleftrightarrow -\phi$$

But if one carelessly identifies the vacuum as

$$\lim_{t\to\infty} \langle 0| e^{-Ht} |0\rangle \sim \int_{-\infty}^{\infty} \mathcal{D}\phi \; e^{-S_E(\phi)}$$

then one has trivially

$$\langle 0| \phi |0\rangle = \int_{-\infty}^{\infty} \mathcal{D}\phi \; \phi \, e^{-S_E} = 0$$

In QCD, probably it is possible to demonstrate rigourously (independently of semi-classical approximation) that $\langle \bar{\psi}\psi \rangle_0 = 0$ if the path integral is not supplemented by the suitable boundary condition other than the topological sectors on $\mathcal{D}A_\mu$.

To try to obtain U(1) symmetry breaking only through the non-trivial (topological) boundary condition on $\mathcal{D}A_\mu$ (gauge fields) is equivalent to try to obtain the whole of SU(N) x SU(N) spontaneous breaking in the same way. Although 't Hooft mechanism singles out the chiral U(1), if one says that there is nothing else needed to define the path integral for QCD, then one would first of all lose chiral SU(N) x SU(N) breaking and current algebra before the question of U(1) Goldstone boson. Of course, the present analysis cannot demonstrate this statement mathematically by the rigorous analysis of field theory defined by the path integral (1) or (52). It would be certainly very beautiful if one could reduce all of the chiral SU(N) x SU(N) breaking as well as the chiral U(1) x U(1) breaking to some topological structure of gauge fields.

On the other hand, the closer analysis (when it is possible) of the path integral such as (52) may result in the existence of U(1) Goldstone boson which somehow becomes much heavier than its SU(N) counterparts, like π , K , η . This would be in accordance with the speculation from 1/N analysis mentioned before [10].

In principle, one can obtain the indication of the answer to these questions by assuming the integer topological number on gauge fields and then by applying the usual analysis with special gauge field dependent quark variables (η_n, ξ_n), corresponding to the Dirac eigenvalue problem

$$(\not{\partial} + \not{A})\phi_n = \lambda_n \phi_n$$

The behaviour of integrand with respect to zero mode, i.e. those quark variables (η_0^ν, ξ_0^ν) which correspond to

$$(\not{\partial} + \not{A})\phi_0^\nu = 0 \qquad \nu = 1, 2 \cdots \quad n^+ + n^-$$

depends entirely on the wave functions Ψ_F and Ψ_I, since S_E^F now takes the form

$$\sum_{\lambda_n \neq 0} \lambda_n \eta_n \xi_n$$

But the complicated non local structure of Ψ_I and Ψ_F prevents so far to see if the vacuum expectation value such as $\langle \bar{\psi}\psi \rangle$ has the term which is proportional to

$$\prod_{\nu=1}^{n^+ + n^-} \int d\eta_0^\nu d\xi_0^\nu \prod_{\nu=1}^{n^+ + n^-} \eta_0^\nu \xi_0^\nu \neq 0$$

The coefficients of some of just such terms seem to cancel.

Finally, it is usual to consider the pure Yang-Mills fields as all important for the qualitative understanding of QCD, such as the possible phases of the system with or without confinement. Nevertheless, the correlation between U(1) problem and the spontaneous breakdown of chiral SU(N_F) of Nambu-Jona-Lasinio seems to suggest that the existence of fermions is essential to understand any part of physics of QCD. One example of the speculation in this direction is the phase transition depending on the number of fermion proposed by Callan-Dashen and Gross [34].

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ACKNOWLEDGEMENTS

The idea of using the Bogoliubov transformations to deal with the zero mode problem in semi-classical approach to QCD was suggested to the author in a conversation with F. Strocchi.

The author also acknowledges his indebtness to R. Crewther for instructive discussions on the U(1) problem.

Special thanks are due to D. de Falco, F. Guerra and G. Immirzi for many discussions on the mathematical problems related to Feynman path integral.

Lastly, the hospitality of the Centre de Physique Théorique, CNRS, Marseille, where this work was reconsidered and written up is gratefully acknowledged.

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