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A COMPLETE GAUGE THEORY FOR THE WHOLE POINCARÉ GROUP+

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Abstract

Gauge theories for non-semisimple groups are examined. A theory for the Poincaré group with all the essential characteristics of a Yang-Mills theory possesses necessarily extra equations. Inonü -Wigner contractions of gauge theories are introduced which provide a Lagrangian formalism, equivalent to a Lagrangian de Sitter theory supplemented by weak constraints.

1. Introduction

The recent advances of gauge theories for electroweak interactions and the promising approach of chromodynamics to strong processes have put forward expectations that also gravitation would, in not too remote a future, leave its splendid isolation and find a formulation in the language of gauge fields. The analogies between Yang-Mills theory at the classical level and General Relativity, reflecting their common geometrical basic setting, have been noticed since long, but the essential fact remains that the Hilbert-Einstein Lagrangian is not of the Yang-Mills type and the dynamical aspects of the two theories are qualitatively different.

Despite its charm and success, General Relativity is not beyond criticism from a theoretical point of view. We shall not go into this matter here. Reviews on the subject have been made, among others, by Hehl (1976,1979) and Zhenlong (1979) and, from a different standpoint, by Logunov and collaborators (Logunov and Folomeshkin 1978; Denisov and Logunov 1980 and references therein). A very general point frequently made is that General Relativity does not do justice to the entire Poincaré local symmetry of space-time. This is a common thread linking (sometimes loosely) the old Cartan (1922) theory, through the classical papers by Kibble (1961) and Sciama (1962), to the more recent developments (Trautman 1970,1979; Hehl 1979; Camenzind 1975, 1978; Wallner 1980; Cho 1975, 1976a). We shall in the following simply accept the general lines of this criticism as justifying further research and take as granted the interest of building a gauge theory for the Poincaré group, sticking however to a very orthodox gauge-field point of view. Although allowing for the specificity of gravitation, we try to preserve as far as possible the essential characteristics of Yang-Mills theories, not the least being the duality symmetry and the consequent conformal invariance. It will be not question of "gauging" an abstract Poincaré group: this is to be taken as acting on the frames defined on space-time, wherefrom the above mentioned specificity arises. This peculiarity is usually referred to by saying that gauge theories involve groups acting on internal spaces while gravitation is concerned with space-tame itself. Such a phrasing is to be taken cum grano salis: the Poincaré group will act on the tangent spaces of space-time or, maybe better, on the spaces formed by frames defined on these tangent spaces. The isomorphism between Minkowski space and the space tangent to each one of its points is not canonical and the presence of a gravitational field is precisely what makes its frame-dependence ineluctable (Kaempffer 1968). That gravitation is more intimately connected to space-time comes from soldering, a property of the bundle of frames which is absent in the bundles lying behind the usual gauge theories (Trautman 1979). It is related to the affine character of the tangent spaces and to torsion, and shows itself in any differentiable manifold. Its main consequence is the existence of an extra Bianchi identity and, if duality symmetry is to remain valid, an extra Yang-Mills equation.

In section 2 we describe the main features of what we take as a complete (classical, sourceless) gauge theory, stressing the role of duality symmetry. The point is made that the absence of a non-degenerate Killing-Cartan metric on the group is not by itself an impediment: theories for the non-semisimple linear groups $GL(n,R)$ are quite feasible through the use of the general invariants of the adjoint representation. It is, however, a hindrance for groups including a translation subgroup, like the affine linear group $AL(n,R) = GL(n,R) \oslash T_n$ and groups of the Poincaré type $P_n = SO(n-1,1) \oslash T_{n-1,1}$, which act on affine frames. This case is analysed in section 3, where the Yang-Mills equations are obtained by using the duality symmetry. Concerning the Lagrangian, however, the difficulty remains: if the invariants introduced in section 2 are used, the translational sector does not contribute to the dynamics. In order to face this problem, we proceed along the following line of thought:

i) the Bianchi identities are purely mathematical statements, independent of any dynamical assumption; nevertheless, they can be seen as consequences, via a variational approach, of the second-order invariant of the adjoint representation, for linear, unitary and (pseudo-)orthogonal groups;

ii) for the same groups, Yang-Mills equations follow from a similar treatment, the corresponding Lagrangian being obtained from the second-order invariant if account is taken of duality symmetry;

iii) in the case of affine frames there exists an extra Bianchi identity, which does not follow from the second-order invariant; this invariant misses it in just the same way the corresponding Lagrangian misses the translational contribution;

iv) because we know the missing Bianchi identity to be true anyhow, we look for an enlarged formalism in which it does come from a second-order invariant and use the corresponding Lagrangean to obtain the Yang-Mills equations; these result to be just those obtained by direct use of the duality symmetry.

The formalism is presented in section 4. It requires viewing the Poincaré group as the Wigner-Inonü (1954) contraction of the de Sitter group. Inhomogeneous groups are precisely the usual outputs of such contractions (Inonü 1964, Gilmore 1974). In a way, going to the de Sitter group puts translations and(pseudo-) rotations on an equal footing and it is finally the de Sitter second-order invariant which gives the Lagrangian wished for. The formalism corresponds to a de Sitter gauge theory supplemented by weak (in the sense of Dirac) constraints ensuring the commutation between translations.

2. General Structure of Gauge Theories

Our objetive is to obtain a theory for the Poincaré group with all the essential characteristics of a gauge theory. In this section we shall describe the general structure (Popov 1976; Cho 1975) we would like to preserve. Because it makes life so much simpler, the compact notation of differential forms will be used.

A gauge potential is a 1-form A with values in the Lie algebra G' of the gauge group G: given for G' a basis $\{J_a\}$ of generators,

$$A = J_a A^a \tag{2.1}$$

where the A^a are usual real-valued 1-forms, which in a given coordinate system $\{x^\mu\}$ are

$$A^a = A^a_\mu \; dx^\mu \tag{2.2}$$

The componentes A^a_μ are the usual gauge potentials. Our potential A is consequently a matrix of 1-forms. Mathematically, it is a connexion on a fibre-bundle with space-time as the base manifold and the gauge group as structure group. To simplify matters, we shall consider the forms as already projected to the base manifold, which pressuposes a local choice of gauge (or section). The equations are formally the same in any gauge.

A connexion (Bishop and Crittenden 1965) defines covariant derivatives of tensors belonging to any representation of

G. The potential A is G'-valued and belongs to the adjoint representation. For a form $X = J_a X^a$ in this representation, the covariant derivative is

$$\mathcal{D} X = d X + [\![A, X]\!] \quad . \tag{2.3}$$

Here, d is the exterior derivative and the bracket (rather peculiar because forms of odd degrees anticommute) is defined by

$$[\![X, Y]\!] = [J_a, J_b] X^a \wedge Y^b = J_c \, f_{ab}{}^c \, X^a \wedge Y^b \, , \tag{2.4}$$

where $f_{ab}{}^c$ are the structure constants of G. In words, the bracket is a commutator if at least one of the matrices has as elements forms of even order, and an anticommutator otherwise.

The gauge field strength is the curvature of the connexion A, that is, its own covariant derivative. Because in this particular case $[\![A, A]\!] = A \wedge A$, it takes the simple form

$$F = d A + A \wedge A \tag{2.5}$$

It is a 2-form in the adjoint representation which, in the particular system of coordinates $\{x^\mu\}$, has the components $F^a{}_{\mu\nu}$ given by

$$F = \frac{1}{2} J_a F^a{}_{\mu\nu} \, dx^\mu \wedge dx^\nu =$$

$$= \frac{1}{2} J_a \left[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{bc}{}^a A^b_\mu A^c_\nu \right] dx^\mu \wedge dx^\nu . \tag{2.6}$$

An important operation on forms is the dual transformation : given a metric $g_{\mu\nu}$ on an n-dimensional manifold the dual $*P$ of a p-form

$$P = \frac{1}{p!} P_{\mu_1\mu_2\cdots\mu_p}\, dx^{\mu_1}\wedge dx^{\mu_2}\wedge\ldots dx^{\mu_p}$$

is the (n-p)-form

$$*P = \frac{\sqrt{g}}{p!(m-p)!}\, g^{\mu_1\lambda_1} g^{\mu_2\lambda_2}\cdots g^{\mu_p\lambda_p} P_{\lambda_1\lambda_2\cdots\lambda_p}\, \varepsilon_{\mu_1\mu_2\cdots\mu_m}\, dx^{p+1}\wedge\ldots\wedge dx^{m}, \qquad (2.7)$$

where $g = det(g_{\mu\nu})$ and $\varepsilon_{\mu_1\mu_2\cdots\mu_m}$ is the Levi-Civita anti-symmetric symbol. In particular, for a 2-form on a 4-dimensional space,

$$*F^{a} = \frac{1}{2!}\left[\frac{1}{2!}\sqrt{g}\, g^{\mu\lambda} g^{\nu\rho} F^{a}{}_{\lambda\rho}\, \varepsilon_{\mu\nu\sigma\omega}\right] dx^{\sigma}\wedge dx^{\omega}. \qquad (2.8)$$

The Bianchi identity comes by differentiation of (2.5):

$$dF + [A, F] = 0 \quad . \qquad (2.9)$$

The covariant derivative of F is so automatically zero. All gauge theories exhibit duality symmetry, which says that (for the sourceless case) the dynamical (Yang-Mills) field equations are just (2.9) written for the dual of F:

$$d{*F} + [A, {*F}] = 0. \qquad (2.10)$$

In the presence of sources, the covariant derivative of $*F$ is equal to the Noether current densities whose corresponding charges generate the gauge group. This procedure amounts to a practical rule to obtain the field equations from the Bianchi identity. Notice that, unlike (2.9), the Yang-Mills equation depends on the space-time metric, necessary to define the dual. However, as a simple inspection of (2.8) shows, the operator*, when applied on a 2-form in a 4-dimensional space, depends only on the conformal class of the metric: it gives the same result for any metric $h_{\mu\nu} = f^2 g_{\mu\nu}$ conformally equivalent to $g_{\mu\nu}$. This is the origin of the conformal invariance of classical sourceless gauge theories (Atiyah 1979). A complete Yang-Mills theory will be, for us, one whose fundamental equations are (2.9) and (2.10) in the absence of sources. When a source current is present in (2.10) one might be tempted to add convenient sources also to (2.9) in order to preserve duality. This would mean that (2.5) fails to be true everywhere. We prefer to adopt the point of view that duality is a symmetry of the sourceless theory, broken by the source currents.

Now, equations (2.9) and (2.10) have very different origins. The former is an identity of purely geometrical content, coming from the very definition (2.5) of curvature. The latter is a physical equation,* resulting from the choice of the invariant action

$$S = -\frac{1}{4}\int \mathrm{Tr}\,(F \wedge *F) \qquad (2.11)$$

However, also the Bianchi identity can be obtained as the Euler Lagrange equation in a variational approach. In order to see it, a digression on the invariants of the adjoint representation will be necessary here (Kobayashi and Nomizu 1969). Given a matrix $X = J_a X^a$, the invariants are certain polynomials in the traces of powers of X. More precisely, the k-order invariant I_k is the coefficient of z^k in the expansion of $\det[I + zX]$. Take for instance the Lie algebra GL'(n,R) of the linear group GL(n,R) of real matrices n x n. If $X \in$ GL'(n,R),

$$\det[I + zX] = \sum_{k=0}^{m} z^k I_k = 1 + \frac{z}{1!} \operatorname{Tr} X + \frac{z^2}{2!}\left[(\operatorname{Tr} X)^2 - \operatorname{Tr} X^2\right] +$$

$$+ \frac{z^3}{3!}\left[(\operatorname{Tr} X)^3 - 3(\operatorname{Tr} X)(\operatorname{Tr} X^2) + 2 \operatorname{Tr} X^3\right] + \dots \qquad (2.12)$$

So, the first-order invariant is $\operatorname{Tr} X$. It is a simple matter to see that the n-order invariant is $\det X$. For unitary and (pseudo-) orthogonal Lie algebras analogous procedures apply, although in these cases $\operatorname{Tr} X = 0$ for $m \geqslant 2$. The second-order invariant

$$I_2 = \frac{1}{2}\left[(\operatorname{Tr} X)^2 - \operatorname{Tr} X^2\right] \qquad (2.13)$$

reduces then to

$$I_2 = -\frac{1}{2} \operatorname{Tr} X^2 = -\frac{1}{2} \operatorname{Tr}(J_a J_b) X^a X^b \quad . \qquad (2.14)$$

Usually, gauge theories deal with semisimple groups, for which $\gamma_{ab} = Tr(J_a J_b)$ is the well-defined metric of Killing - Cartan. Some criticism to Poincaré gauge theories(and non-semisimple groups in general) has been based on the non-existence of a bi-invariant metric on the group (Basombrio 1980), which would make it impossible to write down a Lagrangian.We shall see that, by using the invariants above, such a difficulty can be circumvented for the linear group but that for the Poincaré case an enlargement of the group is required, at least as an intermediate step.

As 2-forms commute with each other, F as given by (2.6) behaves just as a numerical matrix belonging to the vector space of the Lie algebra. For $X = F$,(2.12) gives a series of invariant forms involving the curvature. A first fundamental mathematical result is the Weil Lemma: roughly speaking, it says that each such invariant form has vanishing divergence. So for instance the case of electrodynamics with G=U(1) : there $Tr F$ is F itself and the Lemma says that $dF = 0$, which incorporates the first pair of Maxwell's equations. The second-order invariant will be a 4-form, forcibly divergenceless on a 4-dimensional space, so that the Lemma gives nothing new in this case. A second important mathematical result is that these invariant forms define cohomology classes (Chern, Pontrjagin or Euler classes, depending on the bundle considered) and their integrals, besides being invariant under transformations of the gauge group, are numbers invariant under continuous deformations (and so, variations) of the connexion. Such is the case for unitary and orthogonal groups, for which these invariant numbers are

$$\int I_2 (F, F) = - \frac{1}{2} \int Tr (F \wedge F) \quad . \qquad (2.15)$$

We come now to the point we wish to make : if we apply the usual variational procedure to (2.15), taking the potential components A^a_μ as independent fields, we obtain just the Bianchi identity. In the case of the linear group, for which the whole expression (2.13) is to be used, we obtain (2.9) and, due to the $(Tr\, F) \wedge (Tr\, F)$ term, the additional equation

$$d\, (Tr\, F) = 0 \qquad (2.16)$$

This would come anyway from the Weil Lemma for the first-order invariant.

The action (2.11) is a particular case of the invariant

$$I_2 (F, {*F}) = \frac{1}{2} \left[Tr\, F \wedge Tr\, {*F} - Tr \left(F \wedge {*F} \right) \right] , \qquad (2.17)$$

although in this case no theorem exists ensuring the invariance of the integral under continuous deformations of the connexion: this invariance is now a physical assumption. Again, this is where the difference between the Bianchi and Yang-Mills formulae lies: the first is an identity because it comes from the "variation" of an invariant number while the latter is a consequence of a physical assumption.

A full gauge theory can be obtained for the (non-semisimple) group GL(n,R). If we take the variation of $\int I_2(F, *F)$, we find (2.10) plus an extra equation

$$d\,(*\,Tr\,F) = 0 \tag{2.18}$$

As $Tr\,F = d(Tr\,A)$ in this case, the field traces give a one-dimensional subtheory, a consequence of the nonvanishing first-invariant: the last equation is just the dual counterpart of (2.16). The linear group GL(n,R) can be seen as acting on functions defined on the n-dimensional euclidean space R^n. On this space a global system of coordinates $\{x^i\}$ can be used and the generators of GL(n,R) can be realized by the differential operators $\Delta^i{}_j = -x^i\,\partial_j$. The trace is then the well-known dilatation operator $-x^i\partial_i$. [The sign is really irrelevant here. It has been chosen so as to agree with the matricial representation we shall be using later on (see equation (3.2))]. The subtheory is therefore related to dilatation invariance.

So, a complete gauge theory can be obtained for this particular kind of non-semisimple group. It is not quite alike the usual theories, as it has additional equations coming from the first order invariant. Nevertheless, extra difficulties arise in the case of inhomogeneous groups, semidirect products including translation subgroups, of which the most distinguished examples are the affine linear group AL(n,R) and the Poincaré group $\mathcal{P}_m$. The trouble comes from the fact, to be examined later on, that the translational part does not contribute to the invariants given above. The invariants for the $\mathcal{P}_4$ group, for instance, are just those of the homogeneous Lorentz group SO(3,1). An extra Bianchi identity exists in these cases which is not obtainable from the invariants, and one is led to suspect that a gauge theory obtained along the lines sketched above will be incomplete for such groups. Our objective will be to find a way of arriving at all the Bianchi identities also in this case and then, by the duality requirement, establish a complete Yang-Mills theory.

3. Groups of frame transformations

The groups AL(n,R) and $\mathcal{P}_m$ act on the affine frames (Kobayashi and Nomizu 1963) defined on space-time or, more conveniently, on the affine basis of its tangent spaces. The affine character, or the translational invariance, accounts for the arbitraryness in the choice of the origins in tangent spaces. Such groups are primary, always present and at work in any differentiable manifold. They are consequently more closely related to space-time than the "internal" groups of the usual gauge theories. This deeper intimacy is characterized by that very peculiar trait of the bundle of frames which is soldering. In order to examine this property and expose its relation to translations, some use of the bundle language (Lichnerowicz 1962 ;Bishop and Crittenden 1965) seems unavoidable. Let us proceed to a (very crude) description of the bundle of linear frames, in the meantime seeking which fundamental equations a P_4 theory should have in order to comply with the general pattern of the previous section.

Given a differentiable manifold M of dimension m the tangent space T_pM at a fixed $p \in M$ is a vector space of the same dimension. Vector (or linear) frames on T_pM (sets of m linearly independent vectors) can in principle be chosen at will. We can choose one of them for initial reference and specify every other frame by the $m \times m$ matrix whose elements are the components of its members. This corresponds of course to a frame transformation. The set of all such transformations on T_pM constitutes the linear group GL(m,R), which can in this way be identified with the set F_pM of linear frames on T_pM . We want that frames (and components of vector fields with respect to them) be differentiable. Mathematically, this presupposes that the union of the F_pM for all $p \in M$ has itself been made into a differentiable manifold. This larger, $(m+m^2)$-dimensional manifold is the bundle of linear frames $BLF(M)$. A point on this manifold can be specified by $(\{x^\alpha\}, \{h^a_\alpha\})$ where $\{x^\alpha\}$ are the coordinates of $p \in M$ in some local patch and $\{h_i^\alpha\}$ is the matrix corresponding to the frame. Notice however that BLF(M) is not a direct product of manifolds: in the process of making BLF(M) into a smooth manifold the ("base") manifold M is blended into BLF(M) in such a way that its identity is somehow lost. It can only locally (that is, on a local coordinate patch) be unblended out again. This is done by a local section, a mapping of a coordinate patch into BLF(M), which corresponds to a local choice of linear frame.

The spaces tangent to M can, however, be associated to subspaces (called "horizontal") of the spaces tangent to BLF(M), although in infinitely many ways. Extricating spaces tangent to M from all this entanglement is precisely the task of a linear connexion: each connexion defines a horizontal space for every $p \in M$, and associates it to $T_p M$. A linear connexion is a 1-form Γ on BLF(M) with values on the Lie algebra GL'(m,R) of the group GL(m,R). The horizontal spaces are characterized by the vanishing of Γ when applied to their vectors. Once a local choice of frames is made on a particular coordinate patch, Γ can be made into a GL'(m,R)-valued 1-form on the patch, that is, locally on M. It is convenient to use for GL'(m,R) the canonical basis $\{\Delta^j{}_i\}$, where the matrix $\Delta^j{}_i$ has elements given by

$$\left(\Delta^j{}_i\right)^p{}_q = \delta^j_q \quad \delta^p_i \qquad (i,j,p,q = 1,\dots m) \quad . \tag{3.1}$$

These matrices obey the Lie algebra commutation rules

$$[\Delta^{j}{}_{i}, \Delta^{l}{}_{k}] = (\delta^{l}_{i}\,\delta^{s}_{k}\,\delta^{j}_{r} - \delta^{j}_{k}\,\delta^{s}_{i}\,\delta^{l}_{r})\,\Delta^{r}{}_{s} \qquad (3.2)$$

The advantage of this basis is twofold: the matrix elements coincide with the components and the equations to be written later on will have the usual expressions in the particular case of Riemannian geometry. On a patch with coordinates $\{x^{\mu}\}$, the connexion can then be written as

$$\Gamma = \Delta^{j}{}_{i}\,\Gamma^{i}{}_{j\mu}\,dx^{\mu} \quad , \qquad (3.3)$$

which is a matrix of 1-forms (compare with (2.1) and (2.2)), a "gauge potential" for the linear group. It is a 1-form in the adjoint representation of GL(m,R). Just as for the gauge potentials, it defines a covariant derivative and its curvature

$$F = d\Gamma + \Gamma \wedge \Gamma \qquad (3.4)$$

is a GL'(m,R)-valued 2-form, whose components in a coordinate system are those in (2.6) with the structure constants given in (3.2). This is perhaps the place to insist on some trivial points: curvature (as torsion, to be defined later) is not a property of space, but a characteristic of a connexion. Connexions are in principle highly arbitrary, each corresponding to one of the (infinitely many) ways of retrieving the spaces tangent to M from those tangent to BFL(M). Only if submitted (as they will here) to extra equations and boundary conditions will they become fixed.

Up to this point, the analogy with gauge theories is complete. Gauge transformations correspond here to linear transformations of the frames on the tangent space at the point $p \in M$. The peculiar character of the present case can be seen pictorially as follows: intuitively, we think of the tangent space as "touching" the manifold M at the point p, which is "shared" by M and TpM. Else, we tend to look at p as the origin of TpM. However, any point of TpM can be chosen as the one "touching" M at p. This means that the choice of origin in TpM is arbitrary or, if we prefer, that TpM is to be taken as an affine space, or, still, that on TpM an extra translational invariance is at work. Accounting for it, a 1-form on BLF(M) exists, with values in the Euclidean space R^m. This form, named "canonical" or "solder" form, is independent of any connexion and is "horizontal" in the following sense: given any connexion, it will vanish when applied to any vector which is not horizontal. Given a connexion and this always present solder form, an isomorphism is established between (i) horizontal spaces and spaces tangent to M; (ii) vertical spaces (the linear complements to the horizontal spaces in the spaces tangent to BLF(M)) and GL'(m,R). This makes the group GL(m,R) much more tightly tied to M than would the gauge group of an internal symmetry.

Let us examine in some detail the above mentioned isomorphism of vector spaces. To begin with, it is not canonical: it depends on the choice of local frames. If we choose for R^m the vector basis $\{I_j\}$,where I_j is the vector columm with 1 in the j-th row and 0 everywhere else, each local frame $\{h_i\}$ defines an isomorphism

$h: R^m \rightarrow T_pM$ by $h(I_j)=h_j$. In a local coordinate system $\{x^\mu\}$, $h_i = h_i^\mu \partial_\mu$. This makes it possible to transform indices $(i,j,k,\ldots)$ in R^m into indices $(\mu, \nu, \rho, \ldots)$ in T_pM by contracting with h_i^μ (or h^j_λ, the elements of the matrix inverse to (h_i^μ)). For instance, the matrices $\Delta^j{}_i$ may be seen as operating on the columm vectors of R^m, $\Delta^j{}_i I_k = \delta^j_k I_i$, and can be translated into matrices operating on vectors of T_pM, $\Delta^\mu{}_\nu = h^\mu_j h^i_\nu \Delta^j{}_i$. These matrices provide a realization of GL'(m,R) on the tangent space. In the same way, a metric η on R^m, $\eta(I_i, I_j) = \eta_{ij}$, is taken into a metric $g_{\mu\nu} = h^i_\mu h^j_\nu \eta_{ij}$ on the tangent space. The isomorphism between horizontal and tangent spaces is the composition of the solder form with the mapping h. Given the local frame $\{h_i\}$, the solder form can be made into an R^m-valued form om M, with an extra property: it will have the expression

$$S = I_j h^j_\mu dx^\mu , \tag{3.5}$$

so that $S(h_j)=I_j$.

The extra translational invariance on T_pM forces us to enlarge the group GL(m,R) to the affine group. The most convenient way to do it to recall the additive group structure of R^m. This corresponds to identifying it to the translation group T_m, of which the $\{I_j\}$ above are taken as generators. The complete Lie algebra AL'(m,R) generators will obey, in addition to (3.2), the rules

$$[\Delta^j{}_i , I_k] = - \delta^j_k I_i , \tag{3.6}$$

$$[I_i , I_j] = 0 . \tag{3.7}$$

In order to represent this algebra it is necessary to resort to (m+1)x(m+1) real matrices of the form

$$\bar{\Delta} = \left(\begin{array}{ccc|c} & \Delta^{j}{}_{i} & & I_{k} \\ \hline 0 & \dots & 0 & 0 \end{array}\right) \tag{3.8}$$

Once this is done, the bracket (2.4) can be used. The torsion of Γ is the covariant derivative of S:

$$T = dS + [\Gamma, S] = dS + \Gamma\wedge S + S\wedge\Gamma \quad . \tag{3.9}$$

By differentiation of (3.4) and (3.9), the two Bianchi identities of differential geometry result:

$$dF + [\Gamma, F] = 0 \tag{3.10}$$

$$dT + [\Gamma, T] + [S, F] = 0 \quad . \tag{3.11}$$

Due to the absence of soldering in the bundles with general groups, only (3.10) (which is (2.9)) appears in the usual gauge theories.

The Poincaré group P_4 is a subgroup of AL(4,R) and the above considerations can be applied to it by reducing GL(4,R) to the Lorentz group. The P_4 theory is a subtheory. This is not a trivial statement: not every subgroup yields a sub-bundle with the connexion a particular subconnexion of the above Γ . This is true only under some strict conditions which the Lorentz group happens to satisfy (Kobayashi and Nomizu 1963). The translation group $T_{3,1}$ does not, so that a purely translational gauge theory (Cho 1976b) is not a reduction of the AL(4,R) case and does not give a sub-theory. Suppose the Lorentz metric η is given on R^4. The subgroup of GL(4,R) preserving η is generated by $J^k{}_j = (\delta^k_i \delta^\ell_j - \eta_{ji}\eta^{k\ell})\Delta^i{}_\ell$, satisfying $\eta^{j\ell} J^i{}_j J^k{}_\ell = \eta^{ik}$. The isomorphism h defined by the frame $\{h_i\}$ will take these generators into $J^\mu{}_\nu = h^\mu_k h^j_\nu J^k{}_j$, satisfying $g^{\nu\sigma} J^\mu{}_\nu J^\lambda{}_\sigma = g^{\mu\lambda}$ and providing a realization of the Lorentz algebra on T_pM. The Lorentz connexion will be the reduced $\Gamma_0 = J^k{}_j \Gamma_0{}^j{}_k$, a particular linear connexion of the form $\Gamma = \Delta^i{}_\ell(\delta^\ell_k \delta^j_i - \eta_{ik}\eta^{\ell j})\Gamma^k{}_j$. The curvature $F_0 = J^j{}_i F_0{}^i{}_j$ is obtained accordingly. In component form, the usual expressions for the curvature and the torsion are easily obtained from (3.4) and (3.9). Equation (3.10) gives the usual Bianchi identities for $F_0{}^i{}_j = (1/2) R^i{}_{j\mu\nu}\, dx^\mu \wedge dx^\nu$ and, after conversion of indices by putting $R^\rho{}_{\sigma\mu\nu} = h^\rho_i h^j_\sigma R^i{}_{j\mu\nu}$, the extra Bianchi identity (3.11) is ,for vanishing torsion, the origin of the well known cyclic symmetry $R^\rho{}_{[\sigma\mu\nu]} = 0$. If Γ_0 has vanishing torsion and the metric $g_{\mu\nu}$ has zero covariant derivative according to Γ_0 (so that Γ_0 is the Levi-Civita connexion related to $g_{\mu\nu}$), the usual expressions for Riemannian geometry result. Notice however that there is no compelling reason for doing that here: from the point of view of gauge theories, the dynamical variables are the connexions.

The solder form appears much as a gauge potential for the translation sector in (3.5). Furthermore, the isomorphism h, related to a local frame $\{h_i\}$, although providing the realization $\{\Delta^\mu{}_\nu\}$ of the linear algebra on the tangent spaces, fails in general to do the same for the translations: the choice of an anholonomic frame "breaks" the translational invariance. In Yang-Mills language, this choice of frame corresponds to a choice of gauge. The translational field strength would be ds, but the non-commutativity (3.6) of linear transformations and translations creates a coupling between the two sectors which is automatically accounted for in the torsion. As a vector space, the Lie algebra AL'(m,R) is a direct sum of GL'(m,R) and R^m. We can define a gauge potential for the affine group by

$$\bar{\Gamma} = \Delta^j{}_i \, \Gamma^i{}_{j\mu} \, dx^\mu + I_\ell \, h^\ell_\mu \, dx^\mu \tag{3.12}$$

for which the field strength will be

$$\bar{F} = d\bar{\Gamma} + \bar{\Gamma}\wedge\bar{\Gamma} = F + T = \Delta^j{}_i \, F^i{}_j + I_\ell \, T^\ell . \tag{3.13}$$

The Bianchi identity for $\bar{F}$,

$$d\bar{F} + [\bar{\Gamma}, \bar{F}] = 0 \quad , \tag{3.14}$$

decomposes just into a linear part which is (3.10) and a translational part which is (3.11). All this remains true for the Poincaré case, which we shall consider from now on, omitting the indices in Γ_0, F_0.

Let us now apply the considerations of section 2 to determine the Yang-Mills equations for a Poincaré gauge theory. They will come from the Bianchi identities by duality symmetry:

$$d * F + [\Gamma, * F] = 0 \tag{3.15}$$

$$d * T + [\Gamma, * T] + [S, * F] = 0 \tag{3.16}$$

These expressions show once again that the Lorentz sector does constitute a subtheory, but not the translational sector. These equations have been proposed by Popov and Daikhin (1976) on the basis of a heuristic argument. They have pointed out that, for a Levi-Civita connexion Γ, they reduce to

$$R_{\mu\nu;\lambda} - R_{\mu\lambda;\nu} = 0$$

and

$$R_{\mu\nu} = 0$$

respectively. So, this theory includes Yang (1974) and Einstein theories. Of course, in this sourceless case, they are redundant.

Not so in the presence of sources. The sources of the Yang-Mills equations are the Noether current densities whose charges are the generators of the gauge group. Therefore, here they will be the density of relativistic angular momentum $\mathcal{M} = J^{j}{}_{i}\, \mathcal{M}^{i}{}_{j\lambda}\, dx^{\lambda}$ and the energy-stress tensor $\theta = I_{j}\, \theta^{j}{}_{\sigma}\, dx^{\sigma}$:

$$d * F + [\Gamma, *F] = *\mathcal{M} \tag{3.17}$$

$$d * T + [\Gamma, *T] + [S, *F] = *\theta \ . \tag{3.18}$$

In gauge theories, the conservation of the source currents follows directly from the field equations. In electrodynamics, the Maxwell's equation $d*F = *j$ implies $d*j = d^2 * F = 0$. In more general gauge theories the covariant derivative is to be used. From the above equations it can be directly verified that

$$d * \mathcal{M} + [\![\Gamma, *\mathcal{M}]\!] = 0 \tag{3.19}$$

$$d * \theta + [\![\Gamma, *\theta]\!] + [\![S, *\mathcal{M}]\!] = 0 \tag{3.20}$$

This last expression is a somehow "mixed" covariant derivative which takes into account the coupling between (pseudo)-rotations and translations. Defining $\bar{\bar{J}} = \mathcal{M} + \theta$, these equations can be combined into the expression

$$d * \bar{\bar{J}} + [\![\bar{\Gamma}, *\bar{\bar{J}}]\!] = 0 \ . \tag{3.21}$$

4. Contractions of gauge fields

If now we look for a Lagrangian formalism leading to (3.15) and (3.16) along the lines discussed in section 2, we come face to face with a serious difficulty. From the roles played by $\bar{\Gamma}$ and $\bar{F}$, we could expect to be able to proceed in the usual way, taking $Tr(\bar{F}\wedge *\bar{F})$ for the Lagrangian density. This is not so, as only the Lorentz sector (3.15) comes out as the resulting Euler-Lagrange equations. The same happens to the Bianchi identities: (3.10) comes alone from the variations of $Tr(\bar{F}\wedge\bar{F})$. The reason is trivial: the matrix $\bar{F}$ has the form

$$\left(\begin{array}{ccc|c} & F^{i}{}_{j} & & T^{k} \\ \hline 0 & \cdots & 0 & 0 \end{array}\right)$$

and $Tr(\bar{F}\wedge\bar{F}) = Tr(F\wedge F)$, $Tr(\bar{F}\wedge *\bar{F}) = Tr(F\wedge *F)$

These traces ignore the translational sector and equations (3.11) and (3.16), precisely those peculiar to the theory, are missed. We shall follow the general ideas exposed in the introduction to solve this problem.

Notice to begin with that groups including translation subgroups are the normal result of Inonü-Wigner contractions, which were originally introduced to explain how the Poincaré group is changed into the Galilei group when the velocity of light is allowed to tend to infinity. On the other hand, to give Lorentz transformations and translations the same status it is very convenient (Gürsey 1964) to consider the Poincaré group as a contraction of a de Sitter group. We are so led to examine the behaviour of gauge fields under group contractions.

In the process of contraction, some convenient coordinates are chosen in the Lie algebra which, via the exponential mapping, give local coordinates for the group ("group parameters"). A limit is then taken which contracts some of the coordinates to zero while eventually letting other parameters go to infinity. Infinite values of parameters of the original group are absorbed in the parameters of the resulting group.

To fix the ideas, consider the 2-dimensional Poincaré group $P_2 = ISO(2) = SO(2) \oslash T_2$. Its Lie algebra P_2' has generators $\{J_3, T_1, T_2\}$ satisfying $[J_3, T_1] = T_2$; $[J_3, T_2] = T_1$; $[T_1, T_2] = 0$. We can use for them the matrix representation

$$J_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ; \; T_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ; \; T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} . \qquad (4.1)$$

A matrix B belonging to P_2' is

$$B = J_3 B^3 + T_1 B^1 + T_2 B^2 = \begin{pmatrix} 0 & B^3 & B^1 \\ B^3 & 0 & B^2 \\ 0 & 0 & 0 \end{pmatrix} . \qquad (4.2)$$

It corresponds to a group element

$$g = \exp B \doteq \begin{pmatrix} \mathrm{ch}\, B^3 & \mathrm{sh}\, B^3 & a_x \\ \mathrm{sh}\, B^3 & \mathrm{ch}\, B^3 & a_t \\ 0 & 0 & 1 \end{pmatrix} , \qquad (4.3)$$

where $a_x = \frac{B^1}{B^3} \mathrm{sh}\, B^3 + \frac{B^2}{B^3}(\mathrm{ch}\, B^3 - 1)$; $a_t = \frac{B^2}{B^3} \mathrm{sh}\, B^3 + \frac{B^1}{B^3}(\mathrm{ch}\, B^3 - 1)$.

The usual parametrization is obtained for $\mathrm{sh}\, B^3 = \frac{v}{c}\gamma$; $\mathrm{ch}\, B^3 = \gamma = (1 - v^2/c^2)^{-1/2}$. If we try to obtain the Galilei group by taking the limit $c \to \infty$, only its translational part comes out. In order to obtain the whole group, a similarity transformation $S\, g\, S^{-1}$, with $S = \begin{pmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, has to be done beforehand, which changes (4.3) to

$$\begin{pmatrix} \gamma & v\gamma & c a_x \\ v\gamma/c^2 & \gamma & a_t \\ 0 & 0 & 1 \end{pmatrix} \qquad (4.4)$$

The limit now gives

$$\begin{pmatrix} 1 & v & b_x \\ 0 & 1 & b_t \\ 0 & 0 & 1 \end{pmatrix} \qquad (4.5)$$

which represents the 2-dimensional Galilei group when applied to columm vectors $(x, t, 1)$. Notice that the Galilean translation parameter $b_x = c\,a_x$ absorbs the infinity of c (Inonü 1964). The elements of any matrix in the Lie algebra will be accordingly contracted or stretched, their eventual infinities being absorbed in the elements of the corresponding matrices of the final Lie algebra.

We shall here be mainly concerned with contractions of Lie algebras. As any finite Lie algebra is a subalgebra of some linear algebra GL'(n,R) (by Ado's theorem), we shall use the device of embedding both the initial and final algebras in a convenient matrix algebra. This is perhaps an unusual way of looking at contractions, but it helps to see them at work and shows in what sense they are singular transformations. We have already done it above, as (4.1) is a particular realization of P_2' in GL'(3,R). The similarity transformation leading from (4.3) to (4.4) corresponds to a change of the basis (4.1) to $J_3 = c\,\Delta^1{}_2 + c^{-1}\Delta^2{}_1$; $T_1 = c\,\Delta^1{}_3$; $T_2 = \Delta^2{}_3$, with the $\Delta^i{}_j$ given in (3.1). This possibility of different realizations comes from the fact that the commutations rules do not fix them completely.

Our interest will be the contraction of the de Sitter groups SO(4,1) or SO(3,2) to the Poincaré group P_4, but, to show the procedure in some detail, we shall examine the simpler case of the contraction of a gauge theory for the de Sitter group in 2 dimensions SO(2,1) to a P_2 gauge theory, while retaining a 4-dimensional space-time. We shall keep the basis (4.1) for P_2'. The gauge potential and the field strength for a P_2 gauge theory will then be written

$$\bar{A} = \begin{pmatrix} 0 & \bar{A}^3 & \bar{A}^1 \\ \bar{A}^3 & 0 & \bar{A}^2 \\ 0 & 0 & 0 \end{pmatrix} \; ; \quad \bar{F} = \begin{pmatrix} 0 & \bar{F}^3 & \bar{F}^1 \\ \bar{F}^3 & 0 & \bar{F}^2 \\ 0 & 0 & 0 \end{pmatrix} \quad . \tag{4.6}$$

The Lie algebra SO'(2,1) of the de Sitter group has generators obeying $[J_1, J_2] = -J_3$; $[J_2, J_3] = -J_1$; $[J_3, J_1] = J_2$. These commutation relations by themselves do not fix the basis of generators. If we look for the most general linear combinations of the $\Delta^i{}_j \in$. GL'(3,R), we find that they will be satisfied by any set $\{J_i\}$ of the form

$$J_3 = \frac{\beta}{\alpha}\Delta^1{}_2 + \frac{\alpha}{\beta}\Delta^2{}_1 \;\; ; J_2 = \beta\Delta^1{}_3 - \frac{1}{\beta}\Delta^3{}_1 \; ; \; J_1 = \alpha\Delta^2{}_3 + \frac{1}{\alpha}\Delta^3{}_2 \, ,$$

for arbitrary real values of α , β . As we shall contract SO'(2,1) to P_2' preserving the SO'(2) generated by J_3, we choose $\alpha = \beta$. The gauge potential and the field strength will be

$$A_0 = \begin{pmatrix} 0 & A_0^3 & \alpha A_0^2 \\ A_0^3 & 0 & \alpha A_0^1 \\ -\frac{A_0^2}{\alpha} & \frac{A_0^1}{\alpha} & 0 \end{pmatrix} \; ; \quad F_0 = \begin{pmatrix} 0 & F_0^3 & \alpha F_0^2 \\ F_0^3 & 0 & \alpha F_0^1 \\ -\frac{F_0^2}{\alpha} & \frac{F_0^1}{\alpha} & 0 \end{pmatrix} \tag{4.7}$$

Notice that the basis chosen for SO'(2,1) is a choice of vectors in a 6-dimensional subspace of the vector space GL'(3,R). The P_2' basis (4.1) is a choice of vectors in a 4-dimensional subspace included in the above one. The contraction $SO'(2,1) \rightarrow P_2'$ is the limit $\alpha \rightarrow \infty$. Let us look, for instance, at J_1: it can be any point on a hyperbola branch on the plane $(\Delta^2{}_3, \Delta^3{}_2)$. The contraction corresponds to a transition to the asymptote $\Delta^2{}_3$. In this sense, it is a singular transformation strongly reminding the passage to infinite-momentum frames. The components of any matrix will stretch or shrink as shown in (4.7), which is a kind of interpolation between the initial (say, for $\alpha = 1$) and the final algebras. Comparison with (4.6) shows how the P_2 fields absorb the infinities: in the limit, $\alpha F_0^2 = \bar{F}^1$; $\alpha F_0^1 = \bar{F}^2$; $F_0^3 = \bar{F}^3$; $\bar{A}^1 = \alpha A_0^2$; $\bar{A}^2 = \alpha A_0^1$ and $\bar{A}^3 = A_0^3$. Always in the limit, the fields and potentials are related by

$$F_0^1 = dA_0^1 + A_0^3 \wedge A_0^2 = \alpha^{-1}\bar{F}^2 = \alpha^{-1}(d\bar{A}^2 + \bar{A}^3 \wedge \bar{A}^1) \; ;$$

$$F_0^2 = dA_0^2 + A_0^3 \wedge A_0^1 = \alpha^{-1}\bar{F}^1 = \alpha^{-1}(d\bar{A}^1 + \bar{A}^3 \wedge \bar{A}^2) \; ;$$

$$F_0^3 = dA_0^3 + A_0^2 \wedge A_0^1 = \bar{F}^3 = d\bar{A}^3 + \alpha^{-2}\bar{A}^1 \wedge \bar{A}^2 \; .$$

Notice that $\bar{A}^1$, $\bar{A}^2$ are parts (because we are working with only a subgroup P_2 of P_4) of the solder form; $\bar{F}^1, \bar{F}^2$, of the torsion form; and $\bar{F}^3$ and $\bar{A}^3$ the field and potential for the untouched SO(2) sector. Changing notation accordingly,

$$\left.\begin{aligned} F_0^1 &= \alpha^{-1} T^2 = \alpha^{-1} (dS^2 + A \wedge S^1) \ ; \\ F_0^2 &= \alpha^{-1} T^1 = \alpha^{-1} (dS^1 + A \wedge S^2) \ ; \\ F_0^3 &= F = dA + \alpha^{-2} S^1 \wedge S^2 \quad . \end{aligned}\right\} \qquad (4.8)$$

The SO(2,1) second-order invariant is

$$I_2 = -\frac{1}{2} \, Tr \left(F_0 \wedge F_0 \right) = -\frac{1}{2} \, Tr \left(F \wedge F - \alpha^{-2} \, T \wedge T \right) , \qquad (4.9)$$

for any value of α and consequently as near as we may wish of the asymptotic limit meant by the contraction. The translational contribution is lost only when $\alpha^{-1} = 0$, the invariant reducing to the sole SO(2) invariant. Suppose however that we integrate (4.9) as it is, and take variations in the potentials A, S^1 and S^2: the resulting equations are

$$d F + \alpha^{-2} [S , T] = 0$$

$$d T + [A , T] + [S , F] = 0 \quad .$$

When $\alpha^{-1} = 0$, these are just (3.10) and (3.11) for this particular case. Notice also from (4.8) that F reduces to the SO(2) field strength. So,we learn here the following: the correct Bianchi identities are obtained from the de Sitter invariant if the variations are proceeded to as for the SO(2,1) theory and the contraction is accomplished afterwards. The same is true for the Lagrangian

$$L = -\frac{1}{2}\, Tr\left(F\wedge{*F} - \alpha^{-2}\; T\wedge{*T}\right) , \qquad (4.10)$$

which leads to the SO(2,1) Yang-Mills equations and, with contraction as the last step, to equations (3.15) and (3.16) for this particular case.

Notice that, looking in GL'(3,R), the components affected by the contraction are those along $\Delta^3{}_1$, $\Delta^1{}_3$, $\Delta^3{}_2$ and $\Delta^2{}_3$, involving the index "3". We could call them the "third" components. The components which relate only to the fixed subgroup SO(2) remain intact. All the above procedure holds for the 4-dimensional Poincaré group P_4, whose equations are obtained from those for SO(4,1) or SO(3,2) in just the same may as above those for P_2 have been derived from those for SO(2,1). Of course, the fixed subgroup will be SO(3,1) and the distorted components are those along $\Delta^i{}_5$ and $\Delta^5{}_i$ in the embedding GL'(5,R) ("fifth" components), but nothing essential is changed. Before contraction, the equations are (3.11), (3.16) and

$$dF + [\Gamma, F] + \alpha^{-2}\,[S, T] = 0 \qquad (4.11)$$

$$d{*F} + [\Gamma, {*F}] + \alpha^{-2}\,[S, {*T}] = 0 \quad . \qquad (4.12)$$

The torsion (the "fifth" components of the de Sitter field strengths) appears directly as in (3.9), because both sides of the equations are distorted in the same may (as in (4.8)). The field strength is now

$$F = d\Gamma + \Gamma \wedge \Gamma + \alpha^{-2}\, S \wedge S\ , \tag{4.13}$$

S being the "fifth" components of the de Sitter potential.

From these interpolating expressions an alternative interpretation comes forth: the P_4 theory is a de Sitter gauge theory supplemented by the constraints

$$[\,S\,,\,T\,] \approx 0 \quad ; \quad S \wedge S \approx 0\ , \tag{4.14}$$

whose role is to enforce the commutation between translations. One could forget about contraction, use the action

$$S = \frac{1}{8}\int Tr\left[F \wedge * F - T \wedge * T\right] \tag{4.16}$$

taking T and S as the "fifth" components in a de Sitter theory(which means that F depends on S for variations) and use (4.14) as weak constraints in the sense of Dirac (1964), to become effective only after the variations are performed. This is similar to the relativistic kinematics for a free particle, where the explicitly covariant equations of motion result from the action $S = m\int d\tau\, u_\lambda u^\lambda$ and only after that the weak constraint $u_\lambda u^\lambda = -c^2$ is reinstated.

5. Final comments and speculations

If we accept the arguments favoring a P_4 theory,it is difficult to evade the conclusion that the scheme above describes the general lines of what a classical gauge theory for gravitation should be. Being an asymptotic limit of an usual gauge theory, it will probably avoid many of the problems faced by General Relativity, such as those related to nonconservation of overall energy-momentum (Denisov and Logunov 1980) and the Newtonian limit (Denisov and Logunov 1981). It is also probable that the contraction procedure turn out to be helpful in analysing the question of quantization. One could even conjecture that contractions would help clarifying the issue of symmetry breaking in general. Higgs fields appear in a very natural way in the interplay of internal and space-time symmetries (Forgács and Manton 1980). In the approach above, a group is contracted and (as discussed in section 3) the resulting translational sector is "broken" by choice of gauge (local frames). One could take the arguments given by Trautman (1979), by which the four-legs are Higgs fields breaking the natural GL(4,R) invariance down to Lorentz invariance and argue tail-end backward: the better known vierbein fields could be helpful in clarifying the meaning of the far less understood Higgs fields.

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References

Atiyah M F 1979 "Geometry of Yang-Mills Fields" Accademia Nazionale dei Lincei, Pisa, Italy.
Basombrio F G 1980 Gen.Rel. Grav. 12 109
Bishop R L and Crittenden R J 1965 "Geometry of Manifolds", Academic Press, New York
Camenzind M 1975 J.Math.Phys, 16 1023
Camenzind M 1978 Phys.Rev. D18 1068
Cartan E 1922 C.R.Acad.Sci. 174 593
Cho Y M 1975 J.Math.Phys. 16 2029
Cho Y M 1976a Phys.Rev. D14 3335
Cho Y M 1976b Phys.Rev. D14 2521
Denisov V I and Logunov A A 1980 Theor.Math.Physics 43 401
Denisov V I and Logunov A A 1981 Theor.Math.Physics 45 1035
Dirac P A M 1964 "Lectures on Quantum Mechanics" Yeshiva University Press, New York.
Forgács P and Manton N S 1980 Commun.Math.Phys. 72 15
Gilmore R 1974 "Lie Groups, Lie Algebras and Some of Their Applications" J.Wiley, New York
Gürsey F 1964, in Gürsey F (ed) "Group Theoretical Concepts and Methods in Elementary Particle Physics", Gordon and Breach, New York, p.365
Hehl F H, von der Heyde P, Kerlick G D and Nester J M 1976 Rev. Mod. Phys. 48 393
Hehl F H 1979, "Four Lectures on Poincaré Field Theory", in the 6th Course of the International School of Cosmology and Gravitation, Erice, Italy, (preprint)
Inönü E 1964, in Gürsey F (ed.) "Group Theoretical Concepts and Methods in Elementary Particle Physics", Gordon and Breach, New York p.391.
Käempffer F A 1968 Phys.Rev. 165 1420
Kibble T W B 1961 J.Math.Physics 2 212
Kobayashi S and Nomizu K1963 "Foundations of Differential Geometry", Interscience Pub. New York, vol.I

Kobayashi S and NomizuK1969 same as above, vol.II
Lichnerowicz A 1962 "Théorie Globale des Connexions et des Groupes d'Holonomie ", Edizioni Cremoneze, Roma
Logunov A A and Folomeshkin 1978 Theor.Math.Physics 32 653
Popov D A and Daikhin L I 1976 Doklady 20 818
Sciama D W 1962 in "Recent Developments in General Relativity", Pergamon, Oxford.
Trautman A 1970 Rep.Math.Physics 1 29
Trautman A 1979 Czech.J.Phys. B29 107
Wallner R P 1980 Gen.Rel.Grav. 12 719
Wigner E P and Inönü E 1954 Proc.Mat.Acad.Sci.40 119
Yang C N 1974 Phys.Rev. Letters 33 445
Zhenlong Z et al 1979 Scientia Sinica 22 628