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Cut-off parameters in the one-dimensional two-fermion model

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Abstract . It is shown that the usual cut-off procedure (α cut-off parameter) employed in the boson representation of the fermion field operators of the one-dimensional two-fermion model (TFM) is an incorrect one as the commutator of the hermitean-conjugate field operators at the same space-point fails to fulfil a certain relationship which was pointed out long ago by Jordan . The complete form of the boson representation (including the zero-mode) of a single fermion field and the correct use of the cut-off parameter α is reviewed following Jordan and generalized to the TFM. The cut-off parameter α corresponds to a bandwidth cut-off and Jordan's boson representation is exact only in the limit $\alpha \to 0$. The additional zero-mode terms make the exact solution of the backscattering and umklapp scattering problem to be valid only if a supplementary condition is imposed on the coupling constants.Using the present bosonization technique all the inconsistencies of the TFM are removed. The one-particle Green's function and response functions of the Tomonaga-Luttinger model (TLM) are calculated and found to be identical with those obtained by direct diagram summation. The energy gap appearing in the spectrum of the TFM with backscattering and umklapp scattering for certain values of the coupling constants is shown to be proportional to the momentum transfer cut-off γ^{-1} which has to be kept finite while α goes to zero. Under such conditions the anticommutation relations and Jordan's commutator are invariant under the canonical transformation on the boson operators that diagonalizes the Hamiltonian of the TLM The charge-density response function of the TFM with backscattering is perturbationally calculated up to the first order. The cut-off α^{-1} applies in the same way to terms which differ only by their spin indices in the expression of this response function. The charge-density response function is also evaluated at low frequencies for the exactly soluble TFM with backscattering by using Jordan's cut-off procedure.

1. INTRODUCTION.

Although the investigation of the one-dimensional problem of interacting fermions started long time ago it was only recently that the contact was made between theory and experiment with the attempts for understanding the unusual properties of the quasi-one-dimensional materials[1]. This aroused a great deal of interest in the many-fermion system in one dimension. The present paper deals with the one-dimensional two-fermion model (TFM) proposed many years ago by Luttinger[2] and generalized by Luther and Emery[3] to include the backscattering interaction and by Emery, Luther and Peschel[4] to include the umklapp scattering. There is a close analogy between this model and the one-dimensional Fermi gas model (FGM) whose characteristic features are briefly recalled further below.

The one-dimensional FGM consists of weakly interacting spin-half fermions with wavevector p ranging (in the ground-state) from $-k_F$ to $+k_F$, k_F being the Fermi momentum. As the low excited states can be built up by superposing the particle-hole pairs in the neighborhood of the $\pm k_F$ points a bandwidth cut-off k_0 is introduced much smaller than k_F, which restricts the single-particle states participating in the dynamics of the system within the range $2k_0$ around $\pm k_F$, $\pm k_F - k_0 < p < \pm k_F + k_0$. A linear expression is used for the energy of these states, $\varepsilon_p = \mu + v_F(|p| - k_F)$, where μ is the Fermi level and v_F is the Fermi velocity, thus obtaining two linear branches of the fermion spectrum as p lies near $+k_F$ or $-k_F$. The dynamics of the low excited states is governed by two interaction processes. The first one is the forward scattering process that involves a small momentum transfer. This process excites a particle-hole pair in the neighborhood of $\pm k_F$. The second one is the backward scattering process with momentum transfer near $\pm 2k_F$ that excites a particle-hole pair across the Fermi sea. The excitation energies associated with these processes are very small and consequently both processes play an essential role in the physics of the system. If there is an underlying lattice periodicity and the band is half filled there is one more process whose importance can not be neglected. This is the umklapp scattering that excites two particle-hole pairs across the Fermi sea. The momentum transfer in this process is near $\pm 2k_F$ and the momentum conservation is ensured by the reciprocal lattice vector $G = 4k_F$. The FGM is further specified by allowing for a momentum transfer cut-off k_D which differs from k_0. This cut-off is imposed on the processes with momentum transfer near $\pm 2k_F$ which may be interpreted as coming from phonon-mediated effective interaction. Thus the momentum transfer cut-off is reminiscent of the Debye cut-off.

The FGM as formulated before is not an exactly soluble model. Various attempts have been made to get approximate solutions. The model with backscattering and bandwidth cut-off has firstly been treated[5] by summing up the most divergent diagrams (parquet approximation) thus leading to a typical problem with logarithmic singularities. This approach predicts a phase transition which can not be accepted in one dimension. The lower order logarithmic corrections have been taken into account by using the skeleton graph technique[6] and the renormalization group approach[7]. Beyond the parquet approximation it was found that all the singularities of the vertex and response functions are shifted to zero frequency and temperature

The momentum transfer cut-off was introduced by Chui, Rice and Varma[8] and the renormalization group technique was applied to this model[9] as well as to the model with umklapp scattering[10] All this work was recently reviewed by Sólyom[11] The spectrum of the particle-density excitations was also investigated[12] in the model with backscattering in the limit of weak coupling strengths when the Fermi sea is not too strongly distorted by interaction

Unlike the FGM with backscattering and umklapp scattering the model with forward scattering only is an exactly soluble model. Many years ago Tomonaga[13] showed that those parts of the Fourier components of the particle-density operator which correspond to each of the two branches of the fermion spectrum satisfy boson-like commutation relations in the weak coupling limit . A model hamiltonian can be derived to describe the collective excitations of the particle density . This hamiltonian express itself as a bilinear form of two types of boson operators and can straightforwardly be diagonalized (Tomonaga model). The FGM with forward scattering was further developed by Dzyaloshinsky and Larkin[14] in a very interesting way . They assumed that the two linear branches of the fermion spectrum may be interpreted as being approximately described by two independent fields of fermions with linear spectrum of the form $\mu \pm v_F (p \mp k_F)$. Here p is confined to the whole energy band which is of the order of k_F . In order to get physical results for the correlation functions and momentum distribution of the fermions near $\pm k_F$ a momentum transfer cut-off is needed. Both these quantities and the structure of the excitation spectrum were derived by means of the Ward identity[14,15] and a version of the functional integral method[16]. It is known that these methods are equivalent to a direct diagram summation The first precise

statement of the one-dimensional TFM was made by Luttinger[2]
The Luttinger model consists of two types of fermions whose energy levels are $\pm v_F p$. The non-interacting ground-state is filled from $-\infty$ to $+k_F$ with fermions of the first type and from $-k_F$ to $+\infty$ with fermions of the second type. It is argued that this extension of the allowable fermion states does not modify the physical results- at least in the weak coupling case - as the newly introduced states are far away from the Fermi points. Mattis and Lieb[17] showed that this infinite filling of the Fermi sea causes the Fourier components of the particle-density operator to satisfy rigorously the boson-like commutation relations . The kinetic part of the hamiltonian was shown to be equivalent to a model hamiltonian which contains only boson operators . The model with forward scattering interaction (expressed as a bilinear form in boson operators) can be easily treated by means of the canonical transformation method and the results turn out to be those of the Tomonaga model . This is why both these models will be hereafter referred to as the Tomonaga-Luttinger model (TLM). However it is worth remarking that there is a difference between these models : whereas in the Tomonaga model the forward scattering process excites a particle-hole pair near $\pm k_F$ in the Luttinger model this excited pair may be placed everywhere . By using the boson algebra the momentum distribution[17, 18] of fermions and the one-particle Green's function[19] was calculated in the TLM . A momentum transfer cut-off was required in such calculations to get finite results The TLM was recently reviewed by Bohr[20] . An interesting development of this model was attempted by Haldane[21] who added non-linear terms to the fermion dispersion relation. The concept of "Luttinger liquid" was introduced and argued to apply to a wide class of one-dimensional systems.

The boson algebre of the Fourier components of the particle-density operator was fully exploited when Luther and Peschel[22] and Mattis[23] introduced a boson representation for the fermion fields operators . This representation was used to treat the model with backscattering[3, 24] and umklapp scattering[4] . It was shown that for particular values of the coupling constants both these models are exactly soluble. A gap is opened in the spin- and charge-density wave spectrum , respectively , which has an important effect on the infrared behavior of the correlation functions . It is worth mentioning here that , despite the formal resemblance of the backscattering and umklapp scattering terms in the hamiltonian of the TFM to the corresponding terms in the FGM , there are some important differences between these models[25-27] . First , an ambiguity reveals itself when one attempts to assign a momentum transfer to these processes in the TFM . Secondly, whereas the momentum transfer involved by these processes in the FGM is near $\pm 2k_F$ there is no such a restriction for the momentum transfer whatever it would be , in the TFM .

Although the boson representation of the fermion fields operators proved to be of great use in treating the one-dimensional TFM there are nevertheless some difficulties in dealing with it . All these difficulties are related to the cut-off parameter α introduced by Luther and Peschel[22] The boson representation given by Luther and Peschel[22] is not normal-ordered in boson operators When normal-ordering is attempted factors appear which contain divergent summations over an infinite range of wavevectors. Luther and Peschel[22] introduced a cut-off parameter α in their boson representation in such a way as to simply ensure the convergence of these sums . It was shown that the boson representation is correct

only in the limit $\alpha \rightarrow 0$. However this cut-off procedure leads to some inconsistencies which will be successively sketched here[28]
The one-particle Green's function and response functions of the TLM can be calculated by using the boson representation of the fermion fields operators and the bosonized hamiltonian. When compared with the same quantities calculated by the usual direct diagram summation[14, 15] one can see that the two cut-offs (bandwidth and momentum transfer) appearing in these latter expressions are both replaced by the cut-off α^{-1} . Thus α^{-1} can be interpreted neither as a bandwidth cut-off nor as a momentum transfer cut-off , but appears in place of both of them. This suggests that the cut-off parameter α is a too strong one as it leaves no room for the dissociation of the bandwidth cut-off from the momentum transfer cut-off. Another type of difficulty arises when the backscattering and umklapp scattering are introduced . As is well known these models are exactly soluble for particular values of the coupling constants and have a gap in the excitation spectrum of the spin - and charge-density degrees of freedom , respectively . This gap is proportional to α^{-1} and letting α go to zero the gap becomes infinite , a physically meaningless result . Instead of making α equal to zero Luther and Emery kept it finite and interpreted α^{-1} as a bandwidth cut-off . But still Theumann[29] showed that in order to preserve the anticommutation relations of the fermion fields under the canonical transformation on the boson operators that diagonalizes the hamiltonian of the TLM a momentum transfer cut-off γ^{-1} is needed which must be kept finite while α goes to zero The momentum transfer cut-off γ^{-1} proves to be essential to the preservation of sum rules for the spectral density[19(b)] and in fact, the cut-off parameter γ was earlier used by Luther and Peschel[22]

for deriving the correlation functions of the TLM by means of the bosonization technique . However it was pointed out by Theumann[29] that the backscattering hamiltonian (as well as the umklapp scattering one) can be diagonalized only if the limiting process is inverted, that is by letting $r \to 0$ while keeping α finite. Grest[25] calculated perturbationally the first order contributions to the charge-density response function of the TFM with backscattering by using the Luther and Peschel boson representation . He found that the expression of this function does not coincide with that corresponding to the FGM (calculated both with bandwidth cut-off and with bandwidth and momentum transfer cut-offs). The discrepancy relates to the cut-off parameter α which does not apply in the same way to the contributions that differ only by their spin indices ([illegible] and [illegible]). As Grest[25] correctly pointed out this discrepancy arises from the nature of the parameter α . as it is used by Luther and Peschel[22] which is not a true bandwidth cut-off parameter but merely a parameter introduced ad-hoc in order to remove divergencies.

Recently Haldane[21(a), 26] showed that a major lack of the previos[22, 23] boson representation is the zero-modes terms associated with the particle-number operators. He consistently taken into account these terms and obtain the complete form of the boson representation. This boson representation looks very much the same as that encountered in the field-theoretical literature[30] and , in fact , it was derived long time ago by Jordan[31] for a single field of fermions with energy levels $+p$ in his attempt of constructing a neutrinic theory of light[32] The boson representation given by Haldane[21(a), 26] is normal-ordered so that there is no need of the cut-off parameter α in this expression . However , products of two or more field operators are to be calculated and the normal-ordering problem arises again. In order to make finite the summations over wavevectors appearing in the problems of this type Haldane[21(a), 26] pointed out an essentially the same cut-off procedure as that given by Luther and Peschel[22] although the parameter α has a different interpretation. The boson representation and the cut-off procedure given by Haldane[21(a), 26] remove all the aforementioned inconsistencies of the TFM. However, there is a quantity pointed out by Jordan[31] (and hereafter referred to as Jordan's commutator) which has been overlooked so far by all these boson representations (Haldane's included). Owing to the fact that the Fermi sea of the TFM has an infinite number of particles some operators may have infinite values when acting upon the states of the system. Jordan[31] redefined these operators in such a way as they should be finite and the resulting infinite c-numbers be controlled by the cut-off parameter α. As a result commutator of the hermitean conjugate fields at the same space-point must satisfy a certain relationship. This Jordan commutator plays the role of a supplementary condition which has to be satisfied by the boson representation. The importance of Jordan's commutator is directly connected to the renormalization of the infinitely large density of particles. The cut-off procedure given by Luther and Peschel[22] and by Haldane[21(a), 26] do not make the bosonized fermion fields to satisfy Jordan's commutator. The proper cut-off procedure was suggested by Jordan[31]

The aim of this paper is to generalize the Jordan theory to the TFM (which is described by four fermion operators, spin included) and to introduce the proper cut-off procedure. Using Jordan's cut-off procedure it is shown that the aforementioned inconsistencies of the TFM are also removed. The one-particle Green's function and response functions of the TLM are calculated by using Jordan's cut-off procedure and found to be identical with their expressions as derived by direct diagram summation. Jordan's cut-off parameter α turns out to correspond to a bandwidth cut-off . It is shown that the exact solutions given by Luther and Emery[3] and Emery , Luther and Peschel[4] are valid only if the zero-mode terms are absent . This requires an additional condition imposed on the coupling constants ($g_{4\parallel} \mp g_{4\perp} = 3\pi v_F$, respectively). Under such conditions the diagonalization of the hamiltonian can be done without keeping α finite . The energy gap appearing in these models is shown to be proportional to r^{-1} (not α^{-1}) having thus a finite value. Thus we may safely let α go to zero while keeping r finite. It follows that the anticommutation relations of the fermion operators and the Jordan's commutator are invarient under the canonical transformation on the boson operators that diagonalizes the hamiltonian of the TLM as it should be[29] . It is worth remarking here that Sólyom[28] interpreted an argument advanced by Lee[24(a)] as pointing to the necessity of keeping finite the cut-off parameter α appearing in the expression of the energy gap . But a closer examination of the Lee's argument, as derived from the BCS gap equation , leads to the conclusion that if a momentum transfer cut-off r^{-1} is introduced such as $r^{-1} \lesseqgtr \alpha^{-1}$ the gap becomes proportional to this latter cut-off r^{-1} , as results also from the present theory ; and therefore r^{-1} is the cut-off which has to be kept finite , as it was emphasized before . The charge-density response function of the TFM with backscattering is perturbationally calculated up to the first order by using the Jordan cut-off procedure . It is found that the bandwidth cut-off parameter

α applies in the same way to both $g_{1\parallel,\perp}$ contributions , the inconsistency pointed out by Great[25] being thereby removed.

Having introduced the correct form of the Jordan's boson representation and the cut-off procedure one can attempt to compare the results of the TFM with backscattering and umklapp scattering to the results corresponding to the FGM . As it is suggested by our results there is no major difference between these two models, at least in the overall behavior and the leading contributions to the response functions. This conclusion seems to be supported by a recent work[27]. where the general features of the TFM are shown to belong also to the FGM , although this latter model is used with an ultraviolet cut-off procedure which differs from the conventional one. However Haldane[26] showed that the bosonisation technique applied to the FGM with the conventional bandwidth cut-off leads to a residual coupling between spin-and charge-degrees of freedom in contrast to the TFM . This residual coupling is expected to be effective for large values of the coupling constants. There is one more point worth mentioning when one compares the results of the TFM with those of the FGM . This is related to the scaling equations of the renormalization group approach[25, 33] . The correct use of the cut-off parameter α presented in this paper will surely throw light upon this unsettled problem . This point is left to a forthcoming investigation.

The paper is organized as follows . The Jordan's boson representation is reviewed and generalized to the TFM in Sec. II. Section III. is devoted to the calculation of the one-particle Green's function and response functions of the TLM . The TFM with backscattering and umklapp scattering is diagonalized in Sec.IV. The charge-density response function of the TFM with backscattering is perturbationally calculated in Sec.V. The same response function is evaluated at low frequencies for the exactly soluble TTM with backscattering also in Sec.V. A summary of the results is included in Sec.VI. The paper ends with an Appendix in which four objects are introduced in such a way as to ensure the anticommutation relations of the four different field operators.

II. JORDAN'S BOSON REPRESENTATION.

Let α_{jq} , $j = 1, 2$, $q = 2\pi L^{-1}(n + 1/2)$, n integer , be the destruction operators of two types of fermions with the properties

$$\alpha_{jq} = \alpha^{+}_{j-q} \qquad \{\alpha_{jq}, \alpha_{j'q'}\} = \delta_{jj'}\delta_{q,-q'} \tag{2.1}$$

L being the length of the box the system is confined to . Under such circumstance Jordan[31] proved that the operator

$$b_k = i\sum_q \alpha_{1q}\alpha_{2k-q} \; , \quad b_{-k} = b^{+}_{k} \; , \tag{2.2}$$

where $k = 2\pi L^{-1} n$, n integer , satisfies boson-like commutation relations :

$$\left[b_k , b^{+}_{k'} \right] = (2\pi)^{-1} L k \,\delta_{kk'} \tag{2.3}$$

The proof is as follows. Let us firstly suppose $k, k' \geq 0$. The operators b_k and $b^{+}_{k'}$ may be written as

$$b_k = i\sum_{q>0}\alpha^{+}_{1q}\alpha_{2k+q} + i\sum_{0<q<k}\alpha_{1q}\alpha_{2k-q} + i\sum_{q>k}\alpha_{1q}\alpha^{+}_{2q-k} \, ,$$

$$b^{+}_{k'} = -i\sum_{q>0}\alpha^{+}_{2k'+q}\alpha_{1q} - i\sum_{0<q<k'}\alpha^{+}_{2k'-q}\alpha^{+}_{1q} - i\sum_{q>k'}\alpha_{2q-k'}\alpha_{1q} \, .$$

. For $k > k' \geqslant 0$ we have

$$[b_k, b^+_{k'}] = \sum_{j,q>0} \alpha^+_{jq} \alpha_{jq+k-k'} - \sum_{j,0<q<k} \alpha^+_{jq} [illegible] - \sum_{j,q>k'} \alpha^+_{jq} \alpha_{jq+k-k'} +$$

$$+ \sum_{j,0<q<k-k'} \alpha_{jq} \alpha_{jk-k'-q} + \sum_{0<q<k} \delta [illegible] \, k \, \delta_{kk'}$$

since we noticed that

$$\sum_{0<q<k-k'} \alpha_{jq} \alpha_{jk-k'-q} \; [illegible] \sum_{0<q<k-k'} \alpha_{jq} \alpha_{jk-k'-q} = 0 \, .$$

Similarly we have for [illegible]

$$[b_k, b^+_{k'}] = [illegible] \alpha_{jq+k-k'} - \sum_{j,q>k'} \alpha^+_{jq} \alpha_{jq+k-k'} +$$

$$[illegible] \, k \, \delta_{kk'} \, .$$

For [illegible] follows immediately

$$[b_k, b^+_{k'}] = [illegible] \, k \, \delta_{kk'} \, .$$

For completing the proof we have still to consider $k > 0$, $k' \leq 0$, in that case we have $[b_k, b^+_{k'}] = [b_k, b_{-k'}]$ and for $k, k' \geqslant 0$ we get

$$[b_k, b_{k'}] = \sum_q \alpha_{1q} \alpha_{1k+k'-q} - \sum_q \alpha_{2q} \alpha_{2k+k'-q} = \sum_q \alpha_{1k+k'-q} \alpha_{1q} -$$

$$- \sum_q \alpha_{2k+k'-q} \alpha_{2q} = - \sum_q \alpha_{1q} \alpha_{1k+k'-q} + \sum_q \alpha_{2q} \alpha_{2k+k'-q} = 0 \, .$$

Let $\psi(x) = L^{-1/2} \sum_p a_p e^{ipx}$ be the fermion field operator whose Fourier components a_p obey the anticommutation relations

$$\{a_p, a_{p'}\} = 0 \quad , \quad \{a^+_p, a_{p'}\} = \delta_{pp'} \, , \tag{2.4}$$

the wavevector p being given by $p = 2\pi L^{-1} n$, n integer. We define the operators α_{jq} by the following relations :

$$\alpha_{1q} = \frac{1}{\sqrt{2}}\left(a_{q-\pi L^{-1}} + a^{+}_{-q-\pi L^{-1}}\right), \qquad a_p = \frac{1}{\sqrt{2}}\left(\alpha_{1\,p+\pi L^{-1}} + i\alpha_{2\,p+\pi L^{-1}}\right),$$
$$\alpha_{2q} = \frac{i}{\sqrt{2}}\left(a^{+}_{-q-\pi L^{-1}} - a_{q-\pi L^{-1}}\right), \qquad a^{+}_p = \frac{1}{\sqrt{2}}\left(\alpha_{1\,-p-\pi L^{-1}} - i\alpha_{2\,-p-\pi L^{-1}}\right), \tag{2.5}$$

where $q = \pm(p+\pi L^{-1}) = 2\pi L^{-1}(n+1/2)$, n integer. One can easily see by using Eqs. (2.4) and (2.5) that the operators α_{jq} fulfil the conditions (2.1) . Let us introduce the Fourier components $\varrho(-k)$ of the particle-density operator

$$\varrho(-k) = \sum_p a^{+}_p a_{p+k} \;,\; \varrho^{+}(-k) = \sum_p a^{+}_p a_{p-k} = \varrho(k), \quad k>0 \;. \tag{2.6}$$

With the aid of Eqs. (2.5) we get

$$\varrho(-k) = \sum_p a^{+}_p a_{p+k} = i\sum_q \alpha_{1q}\alpha_{2\,k-q} = b_k \;, \tag{2.7}$$

where we used again the property $\sum_q \alpha_{jq}\alpha_{j\,k-q} = -\sum_q \alpha_{jq}\alpha_{j\,k-q} = 0$ for $k>0$. It follows from Eqs. (2.3) and (2.7)

$$\left[\varrho(-k), \varrho^{+}(-k')\right] = (2\pi)^{-1} L k \delta_{kk'} \;,\; \left[\varrho(-k), \varrho(-k')\right] = 0 \;,\; k, k' > 0 \;, \tag{2.8}$$

that is the well-known[13, 17] boson-like commutation relations of the Fourier components of the fermion-density operator in one dimension. Tomonaga[13] derived these relations within the approximation of weak coupling strengths (when the Fermi sea is not too strongly distorted by interaction) and Mattis and Lieb[17] used a "unitarily inequivalent" particle-hole representation to get them.

We pass now to the Jordan boson representation. Let us assume that the field operator $\psi(x)$ corresponds to a one-dimensional many-fermion system with cyclic boundary conditions on the box of length L , $-L/2 < x \leq L/2$. Throughout this paper the calculations are performed under the assumption $L \to \infty$ so that the sum

$\sum_p$ may be replaced by $(2\pi)^{-1}L\int dp$. The single-particle energy levels are $v_F p$, v_F being the Fermi velocity and $p=2\pi L^{-1}n$, n integer , the wavevector . This system is governed by the kinetic hamiltonian

$$H_o = v_F\sum_{p>0} p\, a_p^+ a_p - v_F\sum_{p\leq 0} p\, a_p a_p^+ = v_F\sum_{p>0} p\, a_p^+ a_p + v_F\sum_{p\leq 0} p\,(a_p^+ a_p - 1)\,, \qquad (2.9)$$

where $a_p(a_p^+)$ is the destruction (creation) operator of the single-particle state labeled by the wavevector p . These operators obey the anticommutation relations given by Eqs. (2.4). The ground - state $|0\rangle$ is filled with particles from $-\infty$ to k_F , k_F being the Fermi momentum , so that the ground-state energy is $E_o = \langle 0|H_o|0\rangle =$ $= (4\pi)^{-1}L v_F k_F^2$. Instead of working with the particle-number operator $\sum_p a_p^+ a_p$ which has an infinite value when acting upon $|0\rangle$ Jordan[31] used the "charge" operator

$$B = \sum_{p>0} a_p^+ a_p - \sum_{p\leq 0} a_p a_p^+ = \sum_{p>0} a_p^+ a_p + \sum_{p\leq 0}(a_p^+ a_p - 1) \qquad (2.10)$$

which counts the particles with $p>0$ minus the holes with $p\leq 0$. When applied to the ground-state this operator yields $B|0\rangle = (2\pi)^{-1}Lk_F|0\rangle$. Let us introduce also the quantities

$$V(x) = -i\,2\pi L^{-1}\sum_{k>0} k^{-1} e^{ikx}\varrho(-k)\,, \quad F(x) = \frac{\partial V(x)}{\partial x} = 2\pi L^{-1}\sum_{k>0} e^{ikx}\varrho(-k)\,, \qquad (2.11)$$

where $\varrho(-k)$ is defined by Eqs. (2.6) . The particle-density operator can easily be expressed as

$$\psi^+(x)\psi(x) = L^{-1}\sum_{p,k} e^{ikx} a_p^+ a_{p+k} = L^{-1}\sum_{p\leq 0} 1 + L^{-1}B + (2\pi)^{-1}\left[F(x) + F^+(x)\right]. \qquad (2.12)$$

In order to control the divergent sum in Eq. (2.12) Jordan introduced the cut-off parameter $\alpha>0$ by

$$\psi^+(x)\psi(y) = \lim_{\alpha\to 0}\left[\psi(x - i\alpha/2)\right]^+ \psi(y - i\alpha/2) \qquad (2.13)$$

and found

$$\left[\psi(x-i\alpha/2)\right]^{+}\psi(x-i\alpha/2) = L^{-1}\sum_{p>0} e^{p\alpha} a_p^{+} a_p - L^{-1}\sum_{p\le 0} e^{p\alpha}\left(a_p a_p^{+} - 1\right) + \tag{2.14}$$
$$+ L^{-1}\sum_{p,k>0} e^{(p+k/2)\alpha} e^{ikx} a_p^{+} a_{p+k} + L^{-1}\sum_{p,k>0} e^{(p+k/2)\alpha} e^{-ikx} a_{p+k}^{+} a_p$$

which for small α can be written as

$$\left[\psi(x-i\alpha/2)\right]^{+}\psi(x-i\alpha/2) = \frac{1}{2\pi\alpha} + L^{-1}B + (2\pi)^{-1}\left[F(x) + F^{+}(x)\right] + O(\alpha). \tag{2.15}$$

Similarly we define

$$\psi(x)\psi^{+}(y) = \lim_{\alpha\to 0} \psi(x+i\alpha/2)\left[\psi(y+i\alpha/2)\right]^{+} \tag{2.16}$$

and have

$$\psi(x+i\alpha/2)\left[\psi(x+i\alpha/2)\right]^{+} = \frac{1}{2\pi\alpha} - L^{-1}B - (2\pi)^{-1}\left[F(x) + F^{+}(x)\right] + O(\alpha), \tag{2.17}$$

so that

$$\left[\psi^{+}(x),\psi(x)\right] = \lim_{\alpha\to 0}\left\{\left[\psi(x-i\alpha/2)\right]^{+}\psi(x-i\alpha/2) - \psi(x+i\alpha/2)\left[\psi(x+i\alpha/2)\right]^{+}\right\} = \tag{2.18}$$
$$2L^{-1}B + \pi^{-1}\left[F(x) + F^{+}(x)\right]$$

This commutator was pointed out by Jorden[31] and so far overlooked by the theory of the TFM . It represents an additional condition which has to be satisfied by the boson representation of the fermion field. Let us note a useful relation which can be derived from Eqs. (2.14) and (2.15) :

$$L^{-1}\int dx\left[\psi(x-i\alpha/2)\right]^{+}\psi(x-i\alpha/2) = L^{-1}\sum_{p>0} e^{p\alpha} a_p^{+} a_p - L^{-1}\sum_{p\le 0} e^{p\alpha}\left(a_p a_p^{+} - 1\right) = \frac{1}{2\pi\alpha} + L^{-1}B + O(\alpha). \tag{2.19}$$

Using the anticommutator $\{\psi^{+}(x),\psi(y)\} = \delta(x-y)$ and Eqs. (2.15) and (2.18) we remark that $(\pi\alpha)^{-1}$ stands for $\delta(0)$.

One can easily verify that the conditions

$$[\psi(x), \rho(-k)] = e^{-ikx}\psi(x)\ , \quad [\psi(x), \rho^+(-k)] = e^{ikx}\psi(x)\ , \quad [\psi(x), B] = \psi(x) \qquad (2.20)$$

are satisfied if $\psi(x)$ is of the form

$$\psi(x) = \chi(x)\, e^{iV^+(x)}\, e^{iV(x)} \qquad (2.21)$$

where $\chi(x)$ should be chosen in such a way as

$$[\chi(x), \rho(-k)] = [\chi(x), \rho^+(-k)] = 0\ , \quad [\chi(x), B] = \chi(x) \qquad (2.22)$$

We used here the fact that B commutes with $\rho(-k)$ and $\rho^+(-k)$. Let us introduce the unitary operator S which is defined by

$$S a_p S^{-1} = a_{p+2\pi L^{-1}}\ , \quad S a_p^+ S^{-1} = a^+_{p+2\pi L^{-1}}\ , \quad S\psi(x)S^{-1} = e^{-i2\pi L^{-1}x}\psi(x)\ , \quad S\psi^+(x)S^{-1} = e^{i2\pi L^{-1}x}\psi^+(x). \qquad (2.23)$$

One can easily see that

$$[S, \rho(-k)] = [S, \rho^+(-k)] = 0 \qquad (2.24)$$

and

$$S B S^{-1} = \sum_{p > 2\pi L^{-1}} a_p^+ a_p - \sum_{p \leq 2\pi L^{-1}} a_p a_p^+ = B - 1 \qquad (2.25)$$

that is

$$[S, B] = -S \qquad [S^{-1}, B] = S^{-1}$$

Similarly we have[34]

$$S H_0 S^{-1} = v_F \sum_{p > 2\pi L^{-1}} p\, a_p^+ a_p + v_F \sum_{p \leq 2\pi L^{-1}} p\,(a_p^+ a_p - 1) - 2\pi L^{-1} v_F \Big[\sum_{p > 2\pi L^{-1}} a_p^+ a_p + \qquad (2.26)$$

$$+ \sum_{p \leq [illegible]} (a_p^+ a_p - 1)\Big] = H_0 - v_F \sum_{0 < p \leq 2\pi L^{-1}} p - 2\pi L^{-1} v_F (B \ [illegible]) =$$

$$= H_0 - 2\pi L^{-1} v_F (B - [illegible])$$

or

$$[S, H_0] = -2\pi L^{-1} v_F (B - 1/2) S = -2\pi L^{-1} v_F S (B + 1/2) . \qquad (2.26)$$

Looking at Eqs. (2.22) and (2.25) we find that $\chi(x)$ must be of the form

$$\chi(x) = S^{-1} \chi_0 (B, x) , \qquad (2.27)$$

where $\chi_0(B,x)$ has to be further specified. Moreover

$$S \chi(x) S^{-1} = S^{-1} \chi_0 (B-1, x) = e^{-i 2\pi L^{-1} x} S^{-1} \chi_0 (B, x)$$

whence

$$\chi_0 (B, x) = e^{i 2\pi L^{-1} x} \chi_0 (B-1, x) ,$$

that is

$$\chi_0(B,x) = K(x) \, e^{i 2\pi L^{-1} B x} , \qquad (2.28)$$

$K(x)$ being a undetermined function of x. In order to find $K(x)$ we investigate the equation of motion for the fermion field

$$[\psi(x), H_0] = -i v_F \frac{\partial}{\partial x} \psi(x) = -i v_F \frac{\partial \chi(x)}{\partial x} e^{i V^+(x)} e^{i V(x)} - \qquad (2.29)$$

$$- i v_F \chi(x) \frac{\partial}{\partial x} \left[e^{i V^+(x)} e^{i V(x)} \right] = [\chi(x), H_0] \, e^{i V^+(x)} e^{i V(x)} +$$

$$+ \chi(x) \left[e^{i V^+(x)} e^{i V(x)}, H_0 \right] .$$

Using Eq. (2.26) we get straightforwardly

$$[\chi(x), H_0] = 2\pi L^{-1} v_F S^{-1} (B - 1/2) \chi_0 (B, x) ,$$

where we used the commutator $[B, H_0] = 0$. Taking into account the relation

$$[\rho(-k), H_0] = v_F k \rho(-k) \qquad (2.30)$$

we get similarly

$$\left[e^{iV^{+}(x)}e^{iV(x)}, H_0\right] = -iv_F \frac{\partial}{\partial x}\left[e^{iV^{+}(x)}\, e^{iV(x)}\right] .$$

Introducing these results into Eq.(2.29) we obtain the equation

$$-i\frac{\partial}{\partial x}\chi_0(B,x) = 2\pi L^{-1}(B-1/2)\,\chi_0(B,x)$$

whose solution is

$$\chi_0(B,x) = c\, e^{i2\pi L^{-1}(B-1/2)\,x} , \tag{2.31}$$

c being a constant. Therefore $K(x) = c\, e^{-i\pi L^{-1}x}$ as one can see by comparing Eqs. (2.28) and (2.31) . Bringing together the results given by Eqs. (2.11).(2.21) , (2.27) and (2.31) we obtain the Jordan's boson representation

$$\psi(x) = c\, S^{-1} \exp\left[i2\pi L^{-1}(B-1/2)x\right] \exp\left[-2\pi L^{-1}\sum_{k>0} k^{-1} e^{-ikx} \rho^{+}(-k)\right] \exp\left[2\pi L^{-1}\sum_{k>0} k^{-1} e^{ikx} \rho(-k)\right] . \tag{2.32}$$

It still remains to check up whether the anticommutation relations

$$\{\psi^{+}(x), \psi(y)\} = \delta(x-y) \quad , \{\psi(x), \psi(y)\} = 0 \tag{2.33}$$

and the Jordan commutator given by Eq. (2.18) are satisfied by this boson representation. In order to do this we follow the Jordan prescription (2.13) and (2.16) of introducing the cut-off parameter α . When using this cut-off procedure and the boson representation (2.32) for calculating products of two fermion fields we encounter sums of the type

$$f(z) = 2\pi L^{-1}\sum_{k>0} k^{-1} e^{-kz} , \quad Re\, z \geq 0 , \; z \neq 0 . \tag{2.34}$$

For $L^{-1}|z| \ll 1$ (condition fulfilled for any fixed z and $L \to \infty$) this sum may be approximated by

$$f(z) \cong -\ln(2\pi L^{-1}z) + \pi L^{-1}z , \tag{2.35}$$

and this approximation will be used throughout this paper. By straightforward calculation we get for $x \neq y$

$$\{\psi(x),\psi(y)\} = c^2 S^{-2} \exp\left[i 2\pi L^{-1}(B-1/2)(x+y)\right] \exp\left[-2\pi L^{-1}\sum_{k>0} k^{-1}\left(e^{-ikx}+e^{-iky}\right)\rho^{+}(-k)\right] \cdot \tag{2.36}$$
$$\cdot \exp\left[2\pi L^{-1}\sum_{k>0} k^{-1}\left(e^{ikx}+e^{iky}\right)\rho(-k)\right]\left\{ e^{-i2\pi L^{-1}x - f[-i(x-y)]} + e^{-i2\pi L^{-1}y - f[i(x-y)]}\right\} = 0$$

and

$$\psi^2(x) = c^2 S^{-2} \exp\left[i 4\pi L^{-1}(B-1/2)x\right] \exp\left[-4\pi L^{-1}\sum_{k>0} k^{-1} e^{-ikx}\rho^{+}(-k)\right] \exp\left[4\pi L^{-1}\sum_{k>0} k^{-1} e^{ikx}\rho(-k)\right] e^{-f(0)} = 0, \tag{2.37}$$

due to the last exponential factor which is equal to zero. Using the cut-off procedure given by Eqs. (2.13) and (2.16) we obtain

$$[\psi(x-i\alpha/2)]^{+}\psi(y-i\alpha/2) = |c|^2 \exp\left[-i 2\pi L^{-1}(B-1/2)(x-y)\right] \exp\left[2\pi L^{-1}(B-1/2)\alpha\right] \cdot \tag{2.38}$$
$$\cdot \exp\left[2\pi L^{-1}\sum_{k>0} k^{-1}\left(e^{-ikx+\alpha k/2} - e^{-iky-\alpha k/2}\right)\rho^{+}(-k)\right] \exp\left[-2\pi L^{-1}\sum_{k>0} k^{-1}\left(e^{ikx-\alpha k/2} - e^{iky+\alpha k/2}\right)\cdot\right.$$
$$\left.\cdot\, \rho(-k)\right]\, e^{f[\alpha - i(x-y)]}$$

and

$$\psi(y+i\alpha/2)[\psi(x+i\alpha/2)]^{+} = |c|^2 \exp\left[-i 2\pi L^{-1}(B+1/2)(x-y)\right] \exp\left[-2\pi L^{-1}(B+1/2)\alpha\right] \cdot \tag{2.39}$$
$$\exp\left[2\pi L^{-1}\sum_{k>0} k^{-1}\left(e^{-ikx-\alpha k/2} - e^{-iky+\alpha k/2}\right)\rho^{+}(-k)\right] \exp\left[-2\pi L^{-1}\sum_{k>0} k^{-1}\left(e^{ikx+\alpha k/2} -\right.\right.$$
$$\left.\left. - e^{iky-\alpha k/2}\right)\rho(-k)\right] e^{f[\alpha + i(x-y)]}$$

so that

$$\{\psi^{+}(x),\psi(y)\} = |c|^2 L \exp\left[-i2\pi L^{-1}B(x-y)\right] \exp\left[2\pi L^{-1}\sum_{k>0} k^{-1}\left(e^{-ikx} - e^{-iky}\right)\rho^{+}(-k)\right] \cdot \tag{2.40}$$
$$\cdot \exp\left[-2\pi L^{-1}\sum_{k>0} k^{-1}\left(e^{ikx} - e^{iky}\right)\rho(-k)\right] \lim_{\alpha\to 0} \frac{\pi^{-1}\alpha}{\alpha^2+(x-y)^2} = |c|^2 L\,\delta(x-y).$$

It follows $c = c_0 L^{-1/2}$, c_0 being a constant with $|c_0| = 1$. Similarly we have from (2.38)

$$
\begin{aligned}
&[\psi(x-i\alpha/2)]^{+}\psi(x-i\alpha/2) = L^{-1}\exp\left[2\pi L^{-1}(B-1/2)\alpha + f(\alpha)\right]\cdot \\
&\cdot \exp\left[4\pi L^{-1}\sum_{k>0} k^{-1} e^{-ikx} \sinh\frac{\alpha k}{2}\, \rho^{+}(-k)\right] \exp\left[4\pi L^{-1}\sum_{k>0} k^{-1} e^{ikx} \sinh\frac{\alpha k}{2}\, \rho(-k)\right] = \\
&= \frac{1}{2\pi\alpha} + L^{-1}B + (2\pi)^{-1}\left[F(x)+F^{+}(x)\right] + \pi\alpha :\left\{L^{-1}B + (2\pi)^{-1}\left[F(x)+F^{+}(x)\right]\right\}^{2}: + \\
&+ O(\alpha^{2}) ,
\end{aligned}
\tag{2.41}
$$

where :...: means the normal ordering of the boson operators; from Eq. (2.39) we get

$$
\begin{aligned}
&\psi(x+i\alpha/2)[\psi(x+i\alpha/2)]^{+} = \frac{1}{2\pi\alpha} - L^{-1}B - (2\pi)^{-1}\left[F(x)+F^{+}(x)\right] + \\
&+ \pi\alpha :\left\{L^{-1}B + (2\pi)^{-1}\left[F(x)+F^{+}(x)\right]\right\}^{2}: + O(\alpha^{2}) .
\end{aligned}
\tag{2.42}
$$

These expressions agree with those given by Eqs. (2.15) and (2.17) and one can easily see that the Jorden commutator (2.18) is obtained by this bosonization technique. We notice that the factor $\exp(\alpha k/2)$ appearing in these calculations may be consider as a shorthand notation for its first-order power expansion $1+\alpha k/2$. In this way the limit $\alpha \to 0$ may be safely transposed with the summation over k. This done, the validity of the Jordan's boson representation (2.32) and the cut-off prescription (2.13) and (2.16) are completely established. We should get now the form of the hamiltonian H_0 given by Eq. (2.9) in the boson representation. By straightforwed calculation we have

$$
\begin{aligned}
&-i\int dx\,[\psi(x-i\alpha/2)]^{+}\frac{\partial}{\partial x}\psi(x-i\alpha/2) = \sum_{p>0} e^{p\alpha}\, p\, a_p^{+}a_p - \sum_{p\le 0} e^{p\alpha}\, p\,(a_p a_p^{+}-1) = \\
&= \frac{\partial}{\partial\alpha}\left[\sum_{p>0} e^{p\alpha} a_p^{+}a_p - \sum_{p\le 0} e^{p\alpha}(a_p a_p^{+}-1)\right] ,
\end{aligned}
\tag{2.43}
$$

and comparing with

$$\int dx\,[\psi(x-i\alpha/2)]^{+}\psi(x-i\alpha/2) = \sum_{p>0} e^{p\alpha} a_p^{+} a_p - \sum_{p\leq 0} e^{p\alpha}(a_p a_p^{+} - 1) \tag{2.44}$$

we get

$$\sum_{p>0} e^{p\alpha} p\, a_p^{+} a_p - \sum_{p\leq 0} e^{p\alpha} p\, a_p a_p^{+} = \frac{L}{2\pi\alpha^2} + \frac{\partial}{\partial\alpha}\int dx\,[\psi(x-i\alpha/2)]^{+}\psi(x-i\alpha/2) . \tag{2.45}$$

From Eq. (2.41) we obtain

$$\int dx\,[\psi(x-i\alpha/2)]^{+}\psi(x-i\alpha/2) = \frac{L}{2\pi\alpha} + B + \pi L^{-1}\alpha\Big[B^2 + 2\sum_{k>0}\rho^{+}(-k)\rho(-k)\Big] + O(\alpha^2), \tag{2.46}$$

and introducing it into Eq. (2.45) we get

$$\sum_{p>0} e^{p\alpha} p\, a_p^{+} a_p - \sum_{p\leq 0} e^{p\alpha} p\, a_p a_p^{+} = \pi L^{-1} B^2 + 2\pi L^{-1}\sum_{k>0}\rho^{+}(-k)\rho(-k) + O(\alpha) \tag{2.47}$$

whence

$$H_0 = v_F\sum_{p>0} p\, a_p^{+} a_p - v_F\sum_{p\leq 0} p\, a_p a_p^{+} = \pi L^{-1} v_F B^2 + 2\pi L^{-1} v_F \sum_{k>0}\rho^{+}(-k)\rho(-k) . \tag{2.48}$$

One can see that Eqs. (2.26) and (2.29) are satisfied by this bosonized form of H_0 . From Eqs. (2.43) , (2.44) , (2.46) and (2.48) one obtains also

$$\int dx\,[\psi(x-i\alpha/2)]^{+}\psi(x-i\alpha/2) = \frac{L}{2\pi\alpha} + B + \alpha v_F^{-1} H_0 + O(\alpha^2) ,$$

which agrees with Eq. (2.19), , and

$$-i\int dx\,[\psi(x-i\alpha/2)]^{+}\frac{\partial}{\partial x}\psi(x-i\alpha/2) = \frac{\partial}{\partial\alpha}\int dx\,[\psi(x-i\alpha/2)]^{+}\psi(x-i\alpha/2) = -\frac{L}{2\pi\alpha^2} + v_F^{-1} H_0 + O(\alpha) .$$

This latter relation can be obtained also by using directly the boson representation of the fermion fields. It is noteworthy that the expectation value of the product $[\psi(x-i\alpha/2)]^{+}\psi(x-i\alpha/2)$ given by Eq. (2.41) on the ground-state is $(2\pi\alpha)^{-1} + (2\pi)^{-1}k_F + O(\alpha)$, whence one may interpret α^{-1} as a bandwidth cut-off.

We pass now to the generalization of the Jordan's boson representation to the set of four fermion operators appearing in the theory of the TFM ,

$$\psi_{js}(x) = L^{-1/2}\sum_p a_{jps} e^{ipx} \ , \ \{a^+_{jps}, a_{j'p's'}\} = \delta_{jj'}\delta_{pp'}\delta_{ss'} \ , \ \{a_{jps}, a_{j'p's'}\} = 0 \ , \tag{2.49}$$

where $j = 1,2$, $p = 2\pi L^{-1} n$, n integer and $s = \pm 1$ is the spin index . The hamiltonian of this system is given by

$$\begin{aligned} H_0 &= v_F \sum_{s,p>0} p\, a^+_{1ps} a_{1ps} - v_F \sum_{s,p\le 0} p\, a_{1ps} a^+_{1ps} - v_F \sum_{s,p<0} p\, a^+_{2ps} a_{2ps} + v_F \sum_{s,p\ge 0} p\, a_{2ps} a^+_{2ps} = \\ &= v_F \sum_{s,p>0} p\, a^+_{1ps} a_{1ps} + v_F \sum_{s,p\le 0} p (a^+_{1ps} a_{1ps} - 1) - v_F \sum_{s,p<0} p\, a^+_{2ps} a_{2ps} - v_F \sum_{s,p\ge 0} p\, (a^+_{2ps} a_{2ps} - 1) \ , \end{aligned} \tag{2.50}$$

and the Fermi sea is filled with particles of the first type ($j=1$) from $p = -\infty$ to $p = +k_F$ and with particles of the second type ($j=2$) from $p = -k_F$ to $p = +\infty$. The "charge" operators are

$$B_{1s} = \sum_{p>0} a^+_{1ps} a_{1ps} + \sum_{p\le 0} (a^+_{1ps} a_{1ps} - 1) \ , \quad B_{2s} = \sum_{p<0} a^+_{2ps} a_{2ps} + \sum_{p\ge 0} (a^+_{2ps} a_{2ps} - 1) \ , \tag{2.51}$$

which commute with H_0 . One can easily see that the operators α_{jqs} and β_{jqs} defined by

$$\begin{aligned} \alpha_{1qs} &= \frac{1}{\sqrt{2}} \left(a_{1q-\pi L^{-1}s} + a^+_{1-q-\pi L^{-1}s}\right), & a_{1ps} &= \frac{1}{\sqrt{2}} \left(\alpha_{1p+\pi L^{-1}s} + i\,\alpha_{2p+\pi L^{-1}s}\right), \\ \alpha_{2qs} &= \frac{i}{\sqrt{2}} \left(a^+_{1-q-\pi L^{-1}s} - a_{1q-\pi L^{-1}s}\right), & a^+_{1ps} &= \frac{1}{\sqrt{2}} \left(\alpha_{1-p-\pi L^{-1}s} - i\,\alpha_{2-p-\pi L^{-1}s}\right), \\ \beta_{1qs} &= \frac{1}{\sqrt{2}} \left(a_{2-q-\pi L^{-1}s} + a^+_{2q-\pi L^{-1}s}\right), & a_{2ps} &= \frac{1}{\sqrt{2}} \left(\beta_{1-p-\pi L^{-1}s} + i\,\beta_{2-p-\pi L^{-1}s}\right), \\ \beta_{2qs} &= \frac{i}{\sqrt{2}} \left(a^+_{2q-\pi L^{-1}s} - a_{2-q-\pi L^{-1}s}\right), & a^+_{2ps} &= \frac{1}{\sqrt{2}} \left(\beta_{1p+\pi L^{-1}s} - i\,\beta_{2p+\pi L^{-1}s}\right), \end{aligned} \tag{2.52}$$

where $q = \pm(p + \pi L^{-1}) = 2\pi L^{-1}(n + 1/2)$, n integer satisfy the conditions (2.1) , so that the Fourier components of the particle-density operator

$$\rho_{1s}(k) = \rho^+_{1s}(-k) = \sum_p a^+_{1ps} a_{1p+ks} - i \sum_q \alpha_{1qs} \alpha_{2k-qs} \tag{2.53}$$
$$\rho_{2s}(k) = \rho^+_{2s}(-k) = \sum_p a^+_{2p+ks} a_{2ps} = i \sum_q \beta_{1qs} \beta_{2k-qs} \; , \quad k>0 \; ,$$

obey the boson-like commutation relations

$$[\rho_{js}(\mp k), \rho_{j's'}(\pm k')] = (2\pi)^{-1} L k \delta_{jj'} \delta_{ss'} \delta_{kk'} \; , \; [\rho_{js}(\mp k), \rho_{j's'}(\mp k')] = 0 \; , \; k, k' > 0 \; , \tag{2.54}$$

where the upper (lower) sign corresponds to $j, j' = 1(2)$ .. In addition any $\rho_{js}(\mp k)$ commutes with any B_{js} and

$$[\rho_{js}(\mp k), H_0] = \mp v_F k \rho_{js}(\mp k) \; , \; [B_{js}, H_0] = 0 \tag{2.55}$$

Likewise as before we introduce the unitary operators $S_{js} = (S^{-1}_{js})^+$

$$S_{js} a_{j'ps'} S^{-1}_{js} = \delta_{jj'} \delta_{ss'} a_{jp \pm 2\pi L^{-1} s} + (1 - \delta_{jj'} \delta_{ss'}) a_{j'ps'} \tag{2.56}$$

with the properties

$$S_{js} B_{j's'} S^{-1}_{js} = \delta_{jj'} \delta_{ss'} (B_{js} \mp 1) + (1 - \delta_{jj'} \delta_{ss'}) B_{j's'} \tag{2.57}$$
$$S_{js} H_0 S^{-1}_{js} = H_0 \mp 2\pi L^{-1} v_F (B_{js} \mp 1/2)$$

and $[S_{js}, \rho_{j's'}(\mp k)] = [S_{js}, \rho^+_{j's'}(\mp k)] = 0$. One can straightforwardly check up that all the properties of the field operators listed below

$$[\psi_{js}(x), \rho_{j's'}(\mp k)] = \delta_{jj'} \delta_{ss'} e^{\mp ikx} \psi_{js}(x) \; , \; [\psi_{js}(x), B_{j's'}] = \delta_{jj'} \delta_{ss'} \psi_{js}(x) \; , \tag{2.58}$$
$$S_{js} \psi_{j's'}(x) S^{-1}_{js} = \delta_{jj'} \delta_{ss'} e^{-i 2\pi L^{-1} x} \psi_{js}(x) + (1 - \delta_{jj'} \delta_{ss'}) \psi_{j's'}(x) \; ,$$
$$[\psi_{js}(x), H_0] = \mp i v_F \frac{\partial}{\partial x} \psi_{js}(x) \; , \; \{\psi^+_{js}(x), \psi_{j's'}(y)\} = \delta_{jj'} \delta_{ss'} \delta(x-y) \; ,$$
$$\{\psi_{js}(x), \psi_{j's'}(y)\} = 0 \; ,$$
$$[\psi^+_{js}(x), \psi_{js}(x)] = 2 L^{-1} B_{js} + \pi^{-1} [F_{js}(x) + F^+_{js}(x)]$$

where $\rho_{js}(x) = 2\pi L^{-1}\sum_{k>0} e^{\pm ikx} \rho_{js}(\mp k)$ are satisfied by the boson representation

$$\psi_{js}(x) = c_{js} L^{-1/2} S_j^{\mp 1} \exp\left[\pm i\, 2\pi L^{-1}(\theta_{js} - 1/2)x\right] \exp\left[-2\pi L^{-1}\sum_{k>0} k^{-1} e^{\mp ikx} \rho^{+}_{js}(\mp k)\right] \exp\left[2\pi L^{-1}\sum_{k>0} k^{-1} e^{\pm ikx} \rho_{js}(\mp k)\right], \tag{2.59}$$

provided that the Jordan's prescription is used for introducing the cut-off parameter α :

$$\psi^{+}_{js}(x)\psi_{js}(y) = \lim_{\alpha \to 0} \left[\psi_{js}(x \mp i\alpha/2)\right]^{+} \psi_{js}(y \mp i\alpha/2) , \tag{2.60}$$

$$\psi_{js}(y)\psi^{+}_{js}(x) = \lim_{\alpha \to 0} \psi_{js}(x \pm i\alpha/2) \left[\psi_{js}(y \pm i\alpha/2)\right]^{+}$$

The coefficients c_{js} are chosen in such a way as to satisfy the relations

$$c_{js} c^{+}_{js} = 1 , \quad \{c_{js}, c_{j's'}\} = \{c_{js}, c^{+}_{j's'}\} = 0, \quad (js) \neq (j's') . \tag{2.61}$$

Their construction is given in Appendix[35] . The Jordan boson representation (2.59) is normal-ordered in the boson operators and it is complete since it consistently includes the modes corresponding to $k = 0$ (through the θ_{js} operators). The boson representation (2.59) has also been derived by Haldane[21(a), 26] by means of an entirely different technique. However Haldane's approach does not include Jordan's commutator and the precise form of the cut-off procedure (2.60) is not specified in Haldane[21(a), 26] The present boson representation differs from the usual one[22] by having not explicitly introduced the cut-off parameter α Instead of this, the representation (2.59) is used together with Jordan's prescription (2.60) and one may easily see that the present cut-off procedure is more specific than the usual one in which only the factor $\exp(-\alpha k/2)$ appears. The hamiltonian H given by (2.50) becomes in the boson representation

$$H_o = \pi L^{-1} v_F \sum_{js} B_{js}^2 + 2\pi L^{-1} v_F \sum_{js,k>0} \rho^+_{js}(\mp k)\, \rho_{js}(\mp k) \tag{2.62}$$

As $[B_{js}, H_o] = 0$ the additional zero-mode contribution appearing in H_o has no notable effect on the energy spectrum of H_o which can be described either in terms of one fermion excitations or in terms of ρ -excitations[17]

Finally, let us investigate the effect of the canonical transformation

$$\rho_{js}(\mp k) \longrightarrow \tilde{\rho}_{js}(\mp k) = v_s(k)\, \rho_{js}(\mp k) + w_s(k)\, \rho^+_{\bar{j}s}(\mp k), \tag{2.63}$$

where $\bar{j} = 1$ for $j = 2$ and $\bar{j} = 2$ for $j = 1$, $v_s^2(k) - w_s^2(k) = 1$ $w_s(k) = w_s\, e^{-rk/2}$, $r^{-1} > 0$ being a momentum transfer cut-off ,on the anticommutation relations of the field operators and on the Jordan's commutator . We shall prove that these relations are preserved by such a transformation provided that $\alpha \to 0$ while r is hold finite . This invariance was proved[29] for the usual cut-off procedure introduced by Luther and Peschel[22] and it is shown here that it holds also for the present Jordan's prescription of introducing the cut-off parameter α . By straightforward calculation we get

$$\begin{aligned}
\left[\tilde{\psi}_{js}(x \mp i\alpha/2)\right]^+ \tilde{\psi}_{js}(y \mp i\alpha/2) = L^{-1} \exp\left[\mp i 2\pi L^{-1} B_{js}(x-y)\right] \exp\left[2\pi L^{-1} (B_{js} - 1/2)\alpha\right]\\
\exp\left[2\pi L^{-1} \sum_{k>0} k^{-1} v_s(k) \left(e^{\mp ikx + \alpha k/2} - e^{\mp iky - \alpha k/2}\right) \rho^+_{js}(\mp k)\right]\\
\exp\left[2\pi L^{-1} \sum_{k>0} k^{-1} w_s(k) \left(e^{\pm iky + \alpha k/2} - e^{\pm ikx - \alpha k/2}\right) \rho^+_{\bar{j}s}(\mp k)\right]\\
\exp\left[2\pi L^{-1} \sum_{k>0} k^{-1} v_s(k) \left(e^{\pm iky + \alpha k/2} - e^{\pm ikx - \alpha k/2}\right) \rho_{js}(\mp k)\right] .
\end{aligned}$$

$$\cdot \exp\left[2\pi L^{-1}\sum_{k>0} k^{-1} w_s(k)\left(e^{\mp ikx + \alpha k/2} - e^{\mp iky - \alpha k/2}\right)\rho_{\bar{j}s}(\mp k)\right]\cdot$$

$$\cdot \exp\left[-2\pi L^{-1}\sum_{k>0} w_s^2 k^{-1} e^{-\alpha k}\left[2 - e^{\mp ik(x-y) - \alpha k} - [illegible] (x-y) - \alpha k\right]\right]\cdot$$

$$\cdot \exp\left\{\pm i\pi L^{-1}(x-y) + f\left[\mp i(x-y) + \alpha\right]\right\}$$

and

$$\psi^{+}_{js}(y \pm i\alpha/2)\left[\psi_{js}(x [illegible])\right]^{+} = L^{-1}\exp\left[\mp i 2\pi L^{-1} B_{js}(x-y)\right]\exp\left[-2\pi L^{-1}(B_{js}+1/2)\alpha\right]\cdot$$

$$\exp\left[-2\pi L^{-1}\sum_{k>0} k^{-1} v_s(k)\left(e^{\mp iky + \alpha k/2} - e^{\mp ikx - \alpha k/2}\right)\rho^{+}_{js}(\mp k)\right]\cdot$$

$$\exp\left[-2\pi L^{-1}\sum_{k>0} k^{-1} w_s(k)\left(e^{\pm ikx + \alpha k/2} - e^{\pm iky - \alpha k/2}\right)\rho^{+}_{\bar{j}s}(\mp k)\right]\cdot$$

$$\exp\left[-2\pi L^{-1}\sum_{k>0} k^{-1} V_s(k)\left(e^{\pm ikx + \alpha k/2} - e^{\pm iky - \alpha k/2}\right)\rho_{js}(\mp k)\right]$$

$$\cdot \exp\left[-2\pi L^{-1}\sum_{k>0} k^{-1} w_s(k)\left(e^{\mp iky + \alpha k/2} - e^{\mp ikx - \alpha k/2}\right)\rho_{\bar{j}s}(\mp k)\right]$$

$$\cdot \exp\left\{2\pi L^{-1} w_s^2 \sum_{k>0} k^{-1} e^{-\alpha k}\left[2 - e^{\pm ik(x-y) + \alpha k} - e^{\mp ik(x-y) - \alpha k}\right]\right\}\cdot$$

$$\cdot \exp\left\{\pm i\pi L^{-1}(x-y) + f\left[\pm i(x-y) + \alpha\right]\right\}$$

whence

$$\{\psi^{+}_{js}(x), \psi_{js}(y)\} = \exp\left[\pm i 2\pi L^{-1} B_{js}(x-y)\right]\exp\left[-2\pi L^{-1}\sum_{k>0} k^{-1} v_s(k)\left(e^{\mp ikx} - e^{\mp iky}\right)\rho^{+}_{js}(\mp k)\right]$$

$$\cdot \exp\left[-2\pi L^{-1}\sum_{k>0} k^{-1} w_s(k)\left(e^{\pm ikx} - e^{\pm iky}\right)\rho^{+}_{\bar{j}s}(\mp k)\right]\exp\left[-2\pi L^{-1}\sum_{k>0} k^{-1} v_s(k)\left(e^{\pm ikx} - e^{\pm iky}\right)\rho_{js}(\mp k)\right]$$

$$\cdot \exp\left[2\pi L^{-1}\sum_{k>0} k^{-1} w_s(k)\left(e^{\mp ikx} - e^{\mp iky}\right)\rho_{\bar{j}s}(\mp k)\right]\left[\frac{r^2}{r^2+(x-y)^2}\right]^{w_s^2}\lim_{\alpha\to 0}\frac{\pi^{-1}\alpha}{\alpha^2+(x-y)^2} = \delta(x-y)$$

and

$$\begin{aligned}
\left[\tilde{\Psi}^{+}_{js}(x), \tilde{\Psi}_{js}(x)\right] = \lim_{\alpha\to 0} L^{-1} \Big\{ & \exp\left[2\pi L^{-1}(B_{js}-1/2)\alpha\right] \exp\Big[\pi L^{-1}\alpha \sum_{k>0} v_s(k)\, e^{\mp ikx} \rho^{+}_{js}(\mp k)\Big] \cdot \\
\cdot & \exp\Big[2\pi L^{-1}\alpha \sum_{k>0} w_s(k)\, e^{\pm ikx} \rho^{+}_{\bar{j}s}(\mp k)\Big] \exp\Big[\pi L^{-1}\alpha \sum_{k>0} v_s(k)\, e^{\pm ikx} \rho_{js}(\mp k)\Big] \cdot \exp\Big[2\pi L^{-1}\alpha \cdot \\
\cdot & \sum_{k>0} w_s(k)\, e^{\mp ikx} \rho_{\bar{j}s}(\mp k)\Big] - \exp\left[-2\pi L^{-1}(B_{js}+1/2)\alpha\right] \exp\Big[-\pi L^{-1}\alpha \sum_{k>0} v_s(k)\, e^{\mp ikx} \rho^{+}_{js}(\mp k)\Big] \cdot \\
\cdot & \exp\Big[-\pi L^{-1}\alpha \sum_{k>0} w_s(k)\, e^{\pm ikx} \rho^{+}_{\bar{j}s}(\mp k)\Big] \exp\Big[-\pi L^{-1}\alpha \sum_{k>0} v_s(k)\, e^{\pm ikx} \rho_{js}(\mp k)\Big] \exp\Big[-\pi L^{-1}\alpha \cdot \\
\cdot & \sum_{k>0} w_s(k)\, e^{\mp ikx} \rho_{\bar{j}s}(\mp k)\Big]\Big\} \left(\frac{r^2}{r^2-\alpha^2}\right)^{w_s^2} \exp\left[f(\alpha)\right] = \\
= \lim_{\alpha\to 0} L^{-1}\, 2\alpha \Big\{ & 2\pi L^{-1} B_{js} + \left[\tilde{F}_{js}(x) + \tilde{F}^{+}_{js}(x)\right]\Big\} \left(\frac{r^2}{r^2-\alpha^2}\right)^{w_s^2} \frac{1}{2\pi L^{-1}\alpha} = \\
= 2L^{-1} B_{js} & + \pi^{-1}\left[\tilde{F}_{js}(x) + \tilde{F}^{+}_{js}(x)\right] .
\end{aligned}$$

Similarly one can see that $\{\tilde{\Psi}_{js}(x), \tilde{\Psi}_{js}(y)\} = 0$ **, so that we may conclude that all the aforementioned commutation relations are invariant under the transformation (2.63) provided that** $\alpha \to 0$ **while** r **is kept finite . It is worth remarking that this conclusion holds also for a more general canonical transformation , of the type we dealing with in Sec. IV, which affects the "charge" operators too .**

III. CORRELATION FUNCTIONS OF THE TLM.

The TLM is described by the hamiltonian $H = H_0 + H_1$,

$$\begin{aligned}
H_1 = {} & g_{2\parallel} \sum_{s,k>0} \left[\rho_{1s}(-k)\rho_{2s}(k) + \rho^{+}_{2s}(k)\rho^{+}_{1s}(-k)\right] + g_{2\perp} \sum_{s,k>0} \left[\rho_{1s}(-k)\rho_{2-s}(k) + \rho^{+}_{2s}(k)\rho^{+}_{1-s}(-k)\right] + \\
& + g_{4\parallel} \sum_{s,k>0} \left[\rho^{+}_{1s}(-k)\rho_{1s}(-k) + \rho^{+}_{2s}(k)\rho_{2s}(k)\right] + g_{4\perp} \sum_{s,k>0} \left[\rho^{+}_{1s}(-k)\rho_{1-s}(-k) + \rho^{+}_{2s}(k)\rho_{2-s}(k)\right] ,
\end{aligned} \tag{3.1}$$

where H_o is given by Eq. (2.50) and L is put equal to unit . Using the canonical transformation

$$\rho_j(\mp k) = \frac{1}{\sqrt{2}}\left[\rho_{j1}(\mp k) + \rho_{j-1}(\mp k)\right], \quad \sigma_j(\mp k) = \frac{1}{\sqrt{2}}\left[\rho_{j1}(\mp k) - \rho_{j-1}(\mp k)\right], \tag{3.2}$$

and the bosonised form (2.62) of H_o the hamiltonian (3.1) becomes

$$H = \pi v_F \sum_{js} B_{js}^2 + H_\rho + H_\sigma , \tag{3.3}$$

$$H_\rho = (g_{4\parallel} + g_{4\perp} + 2\pi v_F)\sum_{k>0}\left[\rho_1^+(-k)\rho_1(-k) + \rho_2^+(k)\rho_2(k)\right] + (g_{2\parallel} + g_{2\perp})\sum_{k>0}\left[\rho_1(-k)\rho_2(k) + \rho_2^+(k)\rho_1^+(-k)\right],$$

$$H_\sigma = (g_{4\parallel} - g_{4\perp} + 2\pi v_F)\sum_{k>0}\left[\sigma_1^+(-k)\sigma_1(-k) + \sigma_2^+(k)\sigma_2(k)\right] + (g_{2\parallel} - g_{2\perp})\sum_{k>0}\left[\sigma_1(-k)\sigma_2(k) + \sigma_2^+(k)\sigma_1^+(-k)\right].$$

One can see that zero-mode term $\pi v_F \sum_{js} B_{js}^2$ does not affect the spectrum of $H_{\rho,\sigma}$. By using the Mattis-Lieb canonical transformations[17] $\exp(S_{\rho,\sigma})$, whose generators are

$$S_\rho = 2\pi \sum_{k>0} k^{-1} \varphi_\rho(k)\left[\rho_1(-k)\rho_2(k) - \rho_2^+(k)\rho_1^+(-k)\right], \tag{3.4a}$$

$$S_\sigma = 2\pi \sum_{k>0} k^{-1} \varphi_\sigma(k)\left[\sigma_1(-k)\sigma_2(k) - \sigma_2^+(k)\sigma_1^+(-k)\right], \tag{3.4b}$$

$\varphi_{\rho,\sigma}(k)$ being real functions of k , the ρ -and σ -operators become

$$\tilde{\rho}_j(\mp k) = e^{S_\rho}\rho_j(\mp k)e^{-S_\rho} = v_\rho(k)\rho_j(\mp k) + w_\rho(k)\rho_{\bar{j}}^+(\mp k), \tag{3.5}$$

$$\tilde{\sigma}_j(\mp k) = e^{S_\sigma}\sigma_j(\mp k)e^{-S_\sigma} = v_\sigma(k)\sigma_j(\mp k) + w_\sigma(k)\sigma_{\bar{j}}^+(\mp k),$$

with $v_{\rho,\sigma}(k) = \cosh\varphi_{\rho,\sigma}(k)$, $w_{\rho,\sigma}(k) = \sinh\varphi_{\rho,\sigma}(k)$, $\bar{j} = 1$ for $j = 2$ and $\bar{j} = 2$ for $j = 1$, and the hamiltonian H given by Eq. (3.3) can be brought into the diagonal form (up to a constant)

$$\tilde{H} = \exp(S_\sigma)\exp(S_\varsigma) H \exp(-S_\varsigma)\exp(-S_\sigma) = \pi v_F \sum_{qs} B_{qs}^2 + \tag{3.6}$$
$$+ 2\pi u_\varsigma \sum_{k>0}\left[\varsigma_1^+(-k)\varsigma_1(-k) + \varsigma_2^+(k)\varsigma_2(k)\right] + 2\pi u_\sigma \sum_{k>0}\left[\sigma_1^+(-k)\sigma_1(-k) + \sigma_2^+(k)\sigma_2(k)\right],$$

$$u^2_{\varsigma\sigma} = \left[v_F + (2\pi)^{-1}(g_{4\parallel} \pm g_{4\perp})\right]^2 - \left[(2\pi)^{-1}(g_{2\parallel} \pm g_{2\perp})\right]^2 ,$$

provided that

$$\tanh 2\varphi_{\varsigma\sigma}(k) = -\frac{g_{2\parallel} \pm g_{2\perp}}{g_{4\parallel} \pm g_{4\perp} + 2\pi v_F} \; , \; |g_{2\parallel} \pm g_{2\perp}| < |2\pi v_F + g_{4\parallel} \pm g_{4\perp}| \; , \tag{3.7}$$

the upper (lower) sign corresponding to $\varsigma(\sigma)$ index. A weak k - dependence is assumed for the coupling constants $g_{2\parallel,\perp}$, of the form $g_{2\parallel,\perp} \sim e^{-rk/2}$ where $r>0$ is a small parameter of the momentum cut-off . For $g_{4\parallel,\perp}$, $g_{2\parallel,\perp} \ll v_F$ we have

$$u_{\varsigma\sigma} \cong u^0_{\varsigma\sigma} - (g_{2\parallel} \pm g_{2\perp})^2/8\pi^2 v_F \; , \; u^0_{\varsigma\sigma} = v_F + (g_{4\parallel} \pm g_{4\perp})/2\pi \; , \; \varphi^2_{\varsigma\sigma}(k) \cong (g_{2\parallel} \pm g_{2\perp})^2/16\pi^2 v_F^2 \sim e^{-rk} . \tag{3.8}$$

The non-interacting one-particle Green's function is given by

$$G^0_{1s}(x,t) = -i\langle 0| T\, \psi_{1s}[x + i\alpha(t)/2, t]\, \psi^+_{1s}[i\alpha(t)/2, 0]\,|0\rangle \; , \tag{3.9}$$

where the Jordan's cut-off procedure has been used ; $\alpha(t) = \alpha\,\mathrm{sgn}(t)$, $|0\rangle$ is the non-interacting ground-state of the of the hamiltonian H_0 (Eq. (2.50)) and the operators are written in the Heisenberg picture. By straightforward calculation we get

$$G^0_{1s}(x,t) = \frac{1}{2\pi} \; \frac{e^{ik_F[x - v_F t + i\alpha(t)]}}{x - v_F t + i\alpha(t)} \; . \tag{3.10}$$

and $G^0_{2s}(x,t) = -i\langle 0| T\, \psi_{2s}[x - i\alpha(t)/2, t]\, \psi^+_{2s}[-i\alpha(t)/2, 0]\,|0\rangle = G^0_{1s}(-x,t)$

For the interacting system the exact ground-state $|\tilde{0}\rangle = \exp(-S_\sigma)\exp(-S_\varsigma)|0\rangle$ of the hamiltonian H (Eq.(3.1)) appears in Eq.(3.9) . By using the Jordan's boson representation (2.59)) as well as Eqs. (2.57) , (2.61) , (3.2) and (3.6) we get for $t>0$

$$G_{1s}(x,t>0) = -i\exp\left[i(k_F + i\pi)(x - v_F t + i\alpha)\right]\exp\Big\{-(w_s^2+w_\sigma^2)f(r) + \qquad (3.11)$$
$$+\tfrac{1}{2}f[-i(x-u_s t)+\alpha] + \tfrac{1}{2}f[-i(x-u_\sigma t)+\alpha] + \tfrac{1}{2}w_s^2 f[-i(x-u_s t)+r] +$$
$$+\tfrac{1}{2}w_s^2 f[i(x+u_s t)+r] + \tfrac{1}{2}w_\sigma^2 f[-i(x-u_\sigma t)+r] +$$
$$+\tfrac{1}{2}w_\sigma^2 f[i(x+u_\sigma t)+r]\Big\}$$

where the function $f(z)$ **is given by Eq. (2.34) and the** k **-dependence of** $u_{s,\sigma}$ **(Eqs. (3.8)) has explicitly been used. Making use of the fact that the limit** $\alpha \to 0$ **should be taken while** r **is kept finite we may write in Eq. (3.11) for small values of the coupling constants**

$$f[-i(x-u_{s,\sigma}t)+\alpha] = f[-i(x-u^0_{s,\sigma}t)+\alpha] + \left\{f[-i(x-u_{s,\sigma}t)+r] - f[-i(x-u^0_{s,\sigma}t)+r]\right\}. \qquad (3.12)$$

For $t<0$ **the Green's function is given by Eq.(3.11) where** $\alpha \to -\alpha$ **and** $r \to -r$ **, so that , making use of the expansion (2.35) of the function** $f(z)$ **we obtain**

$$G_{1s}(x,t) = \frac{1}{2\pi}\frac{\exp\{ik_F[x-v_F t+i\alpha(t)]\}}{\{[x-u^0_s t+i\alpha(t)][x-u^0_\sigma t+i\alpha(t)]\}^{1/2}}\frac{\{[x-u^0_s t+ir(t)][x-u^0_\sigma t+ir(t)]\}^{1/2}}{\{[x-u_s t+ir(t)][x-u_\sigma t+ir(t)]\}^{1/2}}\cdot \qquad (3.13)$$
$$\cdot\{r^{-2}[x-u_s t+ir(t)][x+u_s t-ir(t)]\}^{-\alpha_s}\{r^{-2}[x-u_\sigma t+ir(t)][x+u_\sigma t-ir(t)]\}^{-\alpha_\sigma}\cdot$$
$$\cdot\exp\left\{\frac{it}{2}\left[g_{4\parallel} + w_s^2(g_{4\parallel}+g_{4\perp}) + w_\sigma^2(g_{4\parallel}-g_{4\perp})\right]\right\},$$

where $r(t) = r\,\mathrm{sgn}(t)$ **and**

$$\alpha_{s,\sigma} = \frac{1}{2}w^2_{s,\sigma} = \frac{v^0_{s,\sigma}-v_{s,\sigma}}{4u_{s,\sigma}} \cong (g_{2\parallel} \pm g_{2\perp})^2/32\pi^2 v_F^2 . \qquad (3.14)$$

In the limit of $g_{4\parallel,\perp} \to 0$ **we get**

$$G_{1s}(x,t) = G^{o}_{1s}(x,t)\,\frac{x-v_F t + i r(t)}{\left\{[x-v_\rho t + i r(t)][x-v_\sigma t + i r(t)]\right\}^{1/2}}\cdot \tag{3.15}$$
$$\cdot\left\{r^{-2}[x-v_\rho t + i r(t)][x+v_\rho t - i r(t)]\right\}^{-\alpha_\rho}\left\{r^{-2}[x-v_\sigma t + i r(t)][x+v_\sigma t - i r(t)]\right\}^{-\alpha_\sigma}.$$

Similarly we obtain $G_{2s}(x,t) = G_{1s}(-x,t)$. One can see that the Green's function (3.15) calculated by means of Jordan's boson representation and the correct cut-off procedure reproduces the results obtained by direct diagram summation[14, 15] in which the two cut-offs parameters α and r appear. The parameter α may be associated to a bandwidth cut-off while r corresponds to a momentum transfer cut-off. The same is true for the charge - and spin-density response functions as well as for the singlet - and triplet-superconductor response functions . The calculation of these functions is qarried out in the same way as for the one-particle Green's function . We confine ourselves to give the results of this calculation ::

$$N(x,t) = -i\,\langle\tilde{0}|\,T\,\psi^{+}_{21}[x+i\alpha(t)/2,t]\,\psi_{11}[x+i\alpha(t)/2,t]\,\psi^{+}_{11}[i\alpha(t)/2,0]\,\psi_{21}[i\alpha(t)/2,0]\,|\tilde{0}\rangle =$$
$$= -i\,G_{11}(x,t)\,G_{21}(-x,-t)\left\{r'^{-2}[x-v_\rho t + i r'(t)][x+v_\rho t - i r'(t)]\right\}^{\beta_\rho}\cdot$$
$$\cdot\left\{r'^{-2}[x-v_\sigma t + i r'(t)][x+v_\sigma t - i r'(t)]\right\}^{\beta_\sigma},$$

$$\chi(x,t) = -i\,\langle\tilde{0}|\,T\,\psi^{+}_{21}[x+i\alpha(t)/2,t]\,\psi_{1-1}[x+i\alpha(t)/2,t]\,\psi^{+}_{1-1}[i\alpha(t)/2,0]\,\psi_{21}[i\alpha(t)/2,0]\,|\tilde{0}\rangle =$$
$$= -i\,G_{11}(x,t)\,G_{21}(-x,-t)\left\{r'^{-2}[x-v_\rho t + i r'(t)][x+v_\rho t - i r'(t)]\right\}^{\beta_\rho}\cdot$$
$$\cdot\left\{r'^{-2}[x-v_\sigma t + i r'(t)][x+v_\sigma t - i r'(t)]\right\}^{-\beta_\sigma},$$

$$\Delta_s(x,t) = -2i \langle \tilde{0}| T\, \psi_{21}[x-i\alpha(t)/2, t]\, \psi_{1-1}[x+i\alpha(t)/2, t]\, \psi^+_{1-1}[i\alpha(t)/2, 0]\, \psi^+_{21}[-i\alpha(t)/2, 0] |\tilde{0}\rangle =$$
$$= 2i\, G_{11}(x,t)\, G_{21}(x,t) \left\{ r^{-2}[x - v_s t + i r'(t)][x + v_s t - i r'(t)] \right\}^{-\beta_s} \cdot$$
$$\cdot \left\{ r^{-2}[x - v_\sigma t + i r'(t)][x + v_\sigma t - i r'(t)] \right\}^{\beta_\sigma} , \qquad (3.16)$$

$$\Delta_t(x,t) = -2i \langle \tilde{0}| T\, \psi_{21}[x-i\alpha(t)/2, t]\, \psi_{11}[x+i\alpha(t)/2, t]\, \psi^+_{11}[i\alpha(t)/2, 0]\, \psi^+_{21}[-i\alpha(t)/2, 0] |\tilde{0}\rangle =$$
$$= 2i\, G_{11}(x,t)\, G_{21}(x,t) \left\{ r^{-2}[x - v_s t + i r'(t)][x + v_s t - i r'(t)] \right\}^{-\beta_s} \cdot$$
$$\cdot \left\{ r^{-2}[x - v_\sigma t + i r'(t)][x + v_\sigma \tau - i r'(t)] \right\}^{-\beta_\sigma}$$

where $r'(t) = \frac{1}{2} r\, \mathrm{sgn}(t)$ and $\beta_{s,\sigma} = (g_{21} \pm g_{21}) / 4\pi v_{s,\sigma}$. Similar results are obtained for the $4k_F$ -response function. We may conclude that Jordan's boson representation and the correct cut-off procedure allow us to obtain the same expressions of the correlation functions of the TLM as those obtained by direct diagram summation [14,15]. In these expressions the cut-off parameter α corresponds to a bandwidth cut-off while the cut-off parameter r corresponds to the momentum transfer cut-off.

IV. BACKSCATTERING AND UMKLAPP SCATTERING HAMILTONIAN.

The backscattering hamiltonian of the JFM is

$$H_b = H - g_{1\parallel} \sum_{s,k>0} [\rho_{1s}(-k)\rho_{2s}(k) + \rho^+_{2s}(k)\rho^+_{1s}(-k)] - g_{1\perp} \int dx [h_\sigma(x) + h^+_\sigma(x)] , \qquad (4.1a)$$

$$h_\sigma(x) = \psi^+_{11}(x)\, \psi_{1-1}(x)\, \psi^+_{2-1}(x)\, \psi_{21}(x) , \qquad (4.1b)$$

where H is given by Eq. (3.1) and $h_{\downarrow}(x)$ has been introduced by Luther and Emery[3] in order to simulate the backscattering interaction in the FGM, where a fermion near $+k_F$ (fermion of the first type in the TFM) is scattered near $-k_F$ (fermion of the second type in the TFM) and conversely, the spin being not affected by this interaction process. On the analogy with the FGM we set the point $p=0$ in the TFM at $\pm k_F$ and measure the number of particles and the energy in the FGM relative to $\pm k_F$ and μ (the chemical potential), respectively. Therefore the non-interacting ground-state $|0\rangle$ of the hamiltonian H_0 (Eq. (2.50)) is filled with fermions of the first type from $p=-\infty$ to $p=0$ and with fermions of the second type from $p=0$ to $p=+\infty$. It follows that $B_{js}|0\rangle = 0$ and $H_0|0\rangle = 0$

We extend the (s,σ) -representation given by Eq. (3.2) to all the operators which enter into the boson representation (2.59) by defining[36]

$$B_{js} = \frac{1}{\sqrt{2}}\left(B_{j1} + B_{j-1}\right), \quad S_{js} = \left(S_{j1} S_{j-1}\right)^{1/\sqrt{2}} \quad C_{1s} = C^{+}_{2-1}, \quad C_{2s} = C_{21}, \tag{4.2}$$

$$B_{j\sigma} = \frac{1}{\sqrt{2}}\left(B_{j1} - B_{j-1}\right), \quad S_{j\sigma} = \left(S_{j1} S^{-1}_{j-1}\right)^{1/\sqrt{2}} \quad C_{1\sigma} = C_{11}, \quad C_{2\sigma} = C_{1-1}.$$

The kinetic hamiltonian H given by Eq. (2.62) becomes in the (s,σ) - representation

$$H_0 = \pi v_F \sum_j \left(B_{js}^2 + B_{j\sigma}^2\right) + 4\pi v_F \sum_{j,k>0}\left[S_j^{-1}(\pm k)\, S_j(\pm k) + \sigma_j^{-1}(\mp k)\, \sigma_j(\mp k)\right], \tag{4.3}$$

where the upper (lower) sign corresponds to $j = 1(2)$. Turning back to the field operators we may write

$$\begin{aligned} H_0 = {} & v_F \sum_{p>0} p\, a^{+}_{1ps} a_{1ps} + v_F \sum_{p\le 0} p\left(a^{+}_{1ps} a_{1ps} - 1\right) + v_F \sum_{p>0} p\, a^{+}_{1p\sigma} a_{1p\sigma} + v_F \sum_{p\le 0} p\left(a^{+}_{1p\sigma} a_{1p\sigma} - 1\right) - \\ & - v_F \sum_{p<0} p\, a^{+}_{2ps} a_{2ps} - v_F \sum_{p\ge 0} p\left(a^{+}_{2ps} a_{2ps} - 1\right) - v_F \sum_{p<0} p\, a^{+}_{2p\sigma} a_{2p\sigma} - v_F \sum_{p\ge 0} p\left(a^{+}_{2p\sigma} a_{2p\sigma} - 1\right). \end{aligned} \tag{4.4}$$

so that we have introduced this way the field operators $\psi_{js,\sigma}(x) = \sum_{k} \ldots_{jks} e^{ikx}$. One can easily verify that the operators $\rho_j(k)$, $\sigma_j(k)$, $S_{js\sigma}$, $B_{j\sigma}$, $\psi_{js\sigma}(x)$ and H_0 given by Eq. (4.4) possess all the properties listed in Sec. II, among which

$$S_{js\sigma} B_{js\sigma} S_{js\sigma}^{-1} = B_{js\sigma} + 1 \,, \quad S_{js\sigma} H_0 S_{js\sigma}^{-1} = H_0 \mp 2\pi v_F \left(B_{js,\sigma} \mp 1/2 \right) . \tag{4.5}$$

Therefore the (ς,σ) -transformation is a canonical one and the boson representation (2.59) is valid in this representation providing the spin index s in Eq. (2.59) is replaced by ς or σ. The hamiltonian H_b, given by Eqs. (4.1a) and (4.3) reads in the (ς,σ) - representation

$$H_b = H_{1\varsigma} + H_{1\sigma} + g_{1\perp} \int dx \left[h_\sigma(x) + h_\sigma^{+}(x) \right] \tag{4.6a}$$

$$\begin{aligned} H_{1\varsigma} = {} & \pi v_F \sum_j B_{j\varsigma}^2 + (g_{4\parallel} + g_{4\perp} + 2\pi v_F) \sum_{k>0} \left[\rho_1^{+}(-k)\rho_1(-k) + \rho_2^{+}(k)\rho_2(k) \right] + \\ & + (g_{2\parallel} - g_{1\parallel} + g_{2\perp}) \sum_{k>0} \left[\rho_1(-k)\rho_2(k) + \rho_2^{+}(k)\rho_1^{+}(-k) \right] , \end{aligned} \tag{4.6b}$$

$$\begin{aligned} H_{1\sigma} = {} & \pi v_F \sum_j B_{j\sigma}^2 + (g_{4\parallel} - g_{4\perp} + 2\pi v_F) \sum_{k>0} \left[\sigma_1^{+}(-k)\sigma_1(-k) + \sigma_2^{+}(k)\sigma_2(k) \right] + \\ & + (g_{2\parallel} - g_{1\parallel} - g_{2\perp}) \sum_{k>0} \left[\sigma_1(-k)\sigma_2(k) + \sigma_2^{+}(k)\sigma_1^{+}(-k) \right] , \end{aligned} \tag{4.6c}$$

and

$$\begin{aligned} h_\sigma(x) = {} & \ldots_{1\sigma} \ldots_{2\sigma} \ldots_{1\varsigma} \ldots_{2\varsigma} S_{1\sigma}^{\sqrt{2}} S_{2\sigma}^{\sqrt{2}} \exp\left[-i\pi\sqrt{2} \left(B_{1\sigma} + B_{2\sigma} \right) \right] \\ & \exp\left[2\pi \sum_{k>0} \sqrt{2}\, k^{-1} e^{-ikx} \sigma_1^{+}(-k) \right] \exp\left[-2\pi \sum_{k>0} \sqrt{2}\, k^{-1} e^{ikx} \sigma_1(-k) \right] \\ & \exp\left[-2\pi \sum_{k>0} \sqrt{2}\, k^{-1} e^{ikx} \sigma_2^{+}(k) \right] \exp\left[2\pi \sum_{k>0} \sqrt{2}\, k^{-1} e^{-ikx} \sigma_2(k) \right] \end{aligned} \tag{4.7}$$

Taking the projection of $h_\sigma(x)$ on $|\phi\rangle_{\varphi_1 - \varphi_2 = 0}$ (see Appendix) the product $C_{1\varsigma} C_{2\varsigma}$ can be replaced by 1, so that $h_\sigma(x)$ depends only on σ -degrees of freedom which are completely decoupled from the

ϱ -degrees of freedom.

Let us focus our attention on the hamiltonian $H_{1\sigma}$ given by Eq. (4.6c). We define the canonical transformation $\exp(S_\sigma)\exp(T_\sigma)$ with S_σ given by Eq. (3.4b) and $T_\sigma = -T_\sigma^+$ given by[37]

$$\tilde{B}_{j\sigma} = e^{T_\sigma} B_{j\sigma} e^{-T_\sigma} = \sqrt{2} B'_{j\sigma}, \quad \tilde{S}_{j\sigma} = e^{T_\sigma} S_{j\sigma} e^{-T_\sigma} = S_{j\sigma}^{1/\sqrt{2}}, \quad [T_\sigma, \sigma_j(\mp k)] = 0 . \tag{4.8}$$

The hamiltonian $H_{1\sigma}$ becomes

$$\tilde{H}_{1\sigma} = \exp(S_\sigma)\exp(T_\sigma) H_{1\sigma} \exp(-T_\sigma)\exp(-S_\sigma) = 2\pi v_F \sum_j B_{j\sigma}^2 + \tag{4.9}$$

$$+ 2\pi v_\sigma \sum_{k>0} [\sigma_1^+(-k)\sigma_1(-k) + \sigma_2^+(k)\sigma_2(k)],$$

$$v_\sigma^2 = [v_F + (2\pi)^{-1}(g_{4\parallel} - g_{4\perp})]^2 - [(2\pi)^{-1}(g_{2\parallel} - g_{2\perp} - g_{1\parallel})]^2$$

and

$$\tanh 2\varphi_\sigma = \frac{g_{1\parallel} - g_{2\parallel} + g_{2\perp}}{g_{4\parallel} - g_{4\perp} + 2\pi v_F}, \tag{4.10}$$

where a weak k -dependence is assumed for $g_{1\parallel}, g_{2\parallel,\perp}$ of the form $e^{-rk/2}$, r being the small, positive parameter of the momentum transfer cut-off. Using Eqs. (2.50) and (2.62) we get at once

$$\tilde{H}_{1\sigma} = \pi(2v_F - v_\sigma)\sum_j B_{j\sigma}^2 + v_\sigma \sum_{p>0} p\, a_{1p\sigma}^+ a_{1p\sigma} + v_\sigma \sum_{p\le 0} p\,(a_{1p\sigma}^+ a_{1p\sigma} - 1) - \tag{4.11}$$

$$- v_\sigma \sum_{p<0} p\, a_{2p\sigma}^+ a_{2p\sigma} - v_\sigma \sum_{p\ge 0} p\,(a_{2p\sigma}^+ a_{2p\sigma} - 1) .$$

One can easily verify that the transformation (4.8) is a canonical one. In particular we have

$$\tilde{S}_{j\sigma} \tilde{B}_{j\sigma} \tilde{S}_{j\sigma}^{-1} = \tilde{B}_{j\sigma} \mp 1, \quad \tilde{S}_{j\sigma} \tilde{H}_{1\sigma} \tilde{S}_{j\sigma}^{-1} = \tilde{H}_{1\sigma} \mp 2\pi v_F (\tilde{B}_{j\sigma} \mp 1/2) . \tag{4.12}$$

The effect of this transformation on $h_\sigma(x)$ is

$$h_\sigma(x) = c^{+}_{1\sigma} c_{2\sigma} S_{1\sigma} S_{2\sigma} \exp\left[-i\pi (B_{1\sigma}+B_{2\sigma})x\right] \exp\left[-i\pi (B_{1\sigma}+1/2)x\right] \cdot \tag{4.13}$$
$$\cdot \exp\left[-i\pi (B_{2\sigma}-1/2)x\right] \cdot \exp\left[2\pi \sum_{k>0} \sqrt{2}\, k^{-1} e^{-ikx} (v_\sigma + w_\sigma)\sigma_1^{+}(-k)\right] \cdot$$
$$\cdot \exp\left[-2\pi \sum_{k>0} \sqrt{2}\, k^{-1} e^{ikx} (v_\sigma + w_\sigma)\sigma_1(-k)\right] \exp\left[-2\pi \sum_{k>0} \sqrt{2}\, k^{-1} e^{ikx} (v_\sigma + w_\sigma)\sigma_2^{+}(k)\right] \cdot$$
$$\cdot \exp\left[2\pi \sum_{k>0} \sqrt{2}\, k^{-1} e^{-ikx} (v_\sigma + w_\sigma)\sigma_2(k)\right] \cdot \exp\left[-8\pi \sum_{k>0} k^{-1} w_\sigma (v_\sigma + w_\sigma)\right],$$

where $v_\sigma = \cosh \varphi_\sigma$ **and** $w_\sigma = \sinh \varphi_\sigma \sim e^{-rk/2}$ **are the parameters given by Eqs. (3.5) . For small values of** r **we may take the limit** $r \to 0$ **in the sums of the type** $\sum_{k>0} \sqrt{2}\, k^{-1} e^{-ikx} (v_\sigma + w_\sigma)\sigma_1^{+}(-k)$ **, etc. , in Eq. (4.13) . Setting** $\sqrt{2}(v_\sigma + w_\sigma) = 1$ **we obtain the Luther-Emery condition**[3]

$$\sqrt{2}\, e^{\varphi_\sigma} = 1, \quad v_\sigma = \frac{3}{2\sqrt{2}}, \quad w_\sigma = -\frac{1}{2\sqrt{2}}, \quad \tanh 2\varphi_\sigma = -\frac{3}{5}, \tag{4.14}$$

so that

$$u_\sigma = \frac{4}{5}\left[v_F + (2\pi)^{-1}(g_{4\parallel} - g_{4\perp})\right] . \tag{4.15}$$

The last exponential factor in Eq. (4.13) yields

$$\exp\left[-8\pi \sum_{k>0} k^{-1} w_\sigma (v_\sigma + w_\sigma)\right] = \exp\left[3\pi \sum_{k>0} k^{-1} e^{-rk/2}\right] \exp\left[-\pi \sum_{k>0} k^{-1} e^{-rk}\right] = \frac{\sqrt{2}}{\pi r} .$$

It follows that in the limit of small r **,** $\tilde{h}_\sigma(x)$ **becomes**

$$\tilde{h}_\sigma(x) = \frac{\sqrt{2}}{\pi r} \exp\left[-i\pi (B_{1\sigma}+B_{2\sigma})x\right] \psi^{+}_{1\sigma}(x)\, \psi_{2\sigma}(x) , \tag{4.16}$$

where Jordan's boson representation has been used to recover the field operators $\psi_{j\sigma}(x)$ **in Eq. (4.13) . As** $[B_{1\sigma} + B_{2\sigma}, H_b] = 0$ **we may take** $B_{1\sigma} + B_{2\sigma} = 0$ **in Eq. (4.16). The full backscattering hamiltonian becomes**

$$\tilde{H}_b = H_{1\rho} + H_\sigma \,, \qquad (4.17)$$

$$H_\sigma = \tilde{H}_{1\sigma} + \sqrt{2}(\pi r)^{-1} g_{1\perp} \int dx \left[\psi_{1\sigma}(x)\,\psi^+_{2\sigma}(x) + \psi_{2\sigma}(x)\,\psi^+_{1\sigma}(x)\right] ,$$

where $H_{1\rho}$ and $\tilde{H}_{1\sigma}$ are given by Eqs. (4.6b) and (4.11), respectively. The hamiltonian H_σ differs from that diagonalized by Luther and Emery[3] by the term $\pi(2v_F - v_\sigma)\sum_j B^2_{j\sigma}$ which comes from the complete form (2.62) (zero-mode contribution included) of the bosonized kinetic hamiltonian. The effect of this term is not trivial and will be investigated elsewhere. In order to get the Luther-Emery solution we impose here the additional condition $2v_F = v_\sigma$ which leads to

$$(g_{4\parallel} - g_{4\perp})/2\pi v_F = \tfrac{3}{2}\,; \qquad (g_{1\parallel} - g_{2\parallel} + g_{2\perp})/2\pi v_F = -\tfrac{3}{2} \,. \qquad (4.18)$$

Under this additional condition H_σ is diagonalized by the canonical transformation $\exp(R_\sigma)$, $R_\sigma = \sum_p \theta^\sigma_p \left(a_{1p\sigma} a^+_{2p\sigma} - a_{2p\sigma} a^+_{1p\sigma}\right)$,

$$\tan 2\theta^\sigma_p = -\sqrt{2}\, g_{1\perp}/\pi r v_\sigma p = -g_{1\perp}/\sqrt{2}\,\pi r v_F p \,; \qquad (4.19)$$

$$\tilde{H}_\sigma = \exp(R_\sigma) H_\sigma \exp(-R_\sigma) = \sum_p \lambda_\sigma(p)\left(a^+_{1p\sigma} a_{1p\sigma} - a^+_{2p\sigma} a_{2p\sigma}\right) .$$

$$\lambda_\sigma(p) = \mathrm{sgn}(p)\left[4v_F^2 p^2 + \Delta_\sigma^2\right]^{1/2}, \quad \Delta_\sigma = \sqrt{2}\,|g_{1\perp}|/\pi r$$

One can see that the gap Δ_σ which appears in the spectrum of this model at $p = 0$ (that is at $p = \pm k_F$ in the FGM) is no longer proportional to α^{-1} as it is in Ref. 3, but it is proportional to r^{-1}, which has a finite value. The parameter α of the bandwidth cut-off introduced in the present approach does not appear in the diagonalisation of H_b at all. This parameter helps us only to make the products of two field operators finite ; as indicates the prescription (2.60) . Therefore, by using the present cut-off procedure which allows two cut-off parameters α and may safely take the limit $\alpha \to 0$, as it is required by the exact boson representation, while r is kept finite in the diagonalization of the backscattering hamiltonian.

The same is true for the umklapp scattering hamiltonian[4] which is given by

$$H_u = H_b + 2g_3 \int dx \left[h_\varrho(x)\, e^{-iGx} + h_\varrho^+(x)\, e^{iGx} \right] , \qquad (4.20)$$

$$h_\varrho(x) = \psi^+_{1,1}(x)\, \psi^+_{1,-1}(x)\, \psi_{2,-1}(x)\, \psi_{2,1}(x) ,$$

where $G = 4k_F$ is a reciprocal lattice vector of the FGM. By using the (ϱ,σ)-representation and the canonical transformation $\exp(S_\varrho)\cdot \exp(T_\varrho)$, with S_ϱ given by Eq. (3.5a) and $T_\varrho = -T_\varrho^+$ defined by

$$\tilde{B}_{j\varrho} = e^{T_\varrho} B_{j\varrho}\, e^{-T_\varrho} = \sqrt{2}\, B_{j\varrho} , \quad \tilde{S}_{j\varrho} = e^{T_\varrho} S_{j\varrho}\, e^{-T_\varrho} = S_{j\varrho}^{1/\sqrt{2}} , \quad [T_\varrho , \varrho_j(\mp k)] = 0 , \qquad (4.21)$$

we get similarly $\tilde{h}_\varrho(x) = \sqrt{2}(\pi r)^{-1} \psi^+_{1\varrho}(x)\, \psi_{2\varrho}(x)$ provided that

$$\tanh 2\varphi_\varrho = \frac{g_{4\perp} + g_{2\perp} - g_{1\parallel}}{g_{4\parallel} + g_{4\perp} + 2\pi v_F} = -\frac{3}{5} .$$

The hamiltonian H_u becomes

$$\tilde{H}_u = H_\varrho + H_\sigma ,$$

$$H_\varrho = \tilde{H}_{1\varrho} - 2\sqrt{2}(\pi r)^{-1} g_3 \int dx \left[\psi_{1\varrho}(x)\, \psi^+_{2\varrho}(x)\, e^{iGx} + \psi_{2\varrho}(x)\, \psi^+_{1\varrho}(x)\, e^{-iGx} \right] .$$

$$\tilde{H}_{1\rho} = \pi(2v_F - v_\rho)\sum_{q} B_{q\rho}^2 + v_\rho \sum_{p>0} p\, a^+_{1p\rho} a_{1p\rho} + v_\rho \sum_{p\le 0} p\,(a^+_{1p\rho} a_{1p\rho} - 1) - \tag{4.22}$$
$$- v_\rho \sum_{p<0} p\, a^+_{2p\rho} a_{2p\rho} - v_\rho \sum_{p\ge 0} p\,(a^+_{2p\rho} a_{2p\rho} - 1)$$

$$v_\rho = \frac{4}{5}\left[v_F + (2\pi)^{-1}(g_{4\parallel} + g_{4\perp}) \right] .$$

In order to get the solution given by Emery , Luther and Peschel[4] we put $2v_F = v_\rho$, that is

$$(g_{4\parallel} + g_{4\perp})/2\pi v_F = (g_{2\parallel} + g_{2\perp} - g_{1\parallel})/2\pi v_F = \frac{3}{2} . \tag{4.23}$$

The hamiltonian H_ρ can then be diagonalized by the canonical transformation $\exp(R_\rho)$, $R_\rho = \sum_p \theta_p^\rho (a_{1p-G/2\rho} a^+_{2p+G/2\rho} - a_{2p+G/2\rho} a^+_{1p-G/2\rho})$

$\tan 2\theta_p^\rho = \sqrt{2} g_3 / \pi v_F r p$:

$$\tilde{H}_\rho = \exp(R_\rho) H_\rho \exp(-R_\rho) = \sum_p \left[\lambda_{1\rho}(p)\, a^+_{1p\rho} a_{1p\rho} + \lambda_{2\rho}(p)\, a^+_{2p\rho} a_{2p\rho} \right] ,$$

$$\lambda_{j\rho}(p) = -v_F G \pm \mathrm{sgn}(p \pm G/2) \left[4v_F^2 (p \pm G/2)^2 + \Delta_\rho^2 \right]^{1/2} ,$$

$$\Delta_\rho = 2\sqrt{2}|g_3|/\pi r$$

and again the gap Δ_ρ is proportional to r^{-1} . The gap appears at $p = \mp G/2 = \mp 2k_F$ which corresponds to $p = \mp k_F$ in the FGM . We note that the simultaneous diagonalization of H_ρ and H_σ requires, from Eqs. (4.18) and (4.23) , $g_{4\perp} = g_{2\perp} = 0$, $g_{4\parallel} = g_{2\parallel} - g_{1\parallel} = 3\pi v_F$.

V. CHARGE-DENSITY RESPONSE FUNCTION OF THE TFM WITH BACSCATTERING.

It is well known that Grest[25] calculated perturbationally the zeroth and first order contributions to the charge-density response function of the TFM with backscattering by using the boson representation and cut-off procedure introduced by Luther and Peschel[22] and found that the cut-off parameter α does not apply in the same way to the $g_{1\parallel}$ and $g_{1\perp}$ terms. Obviously this result can not be accepted as the two terms differ only by their spin indices, and consequently, these two contributions should be the same . We perform here Grest's calculation by using the Jordan bosonization technique and find that the aforementioned inconsistency does not longer subsist. The charge-density response function of the TFM with bacscattering is given by

$$N(x,t) = N_1(x,t) + N_2(x,t) \tag{5.1}$$

$$N_1(x,t) = -i \langle\tilde{0}| T\, \psi^+_{21}(x,t)\, \psi_{11}(x,t)\, \psi^+_{11}(0,0)\, \psi_{21}(0,0) |\tilde{0}\rangle$$

$$N_2(x,t) = -i \langle\tilde{0}| T\, \psi^+_{21}(x,t)\, \psi_{11}(x,t)\, \psi^+_{1-1}(0,0)\, \psi_{2-1}(0,0) |\tilde{0}\rangle ,$$

where $|\tilde{0}\rangle$ is the exact ground-state of the TFM with bacscattering defined by the hamiltonian given by Eqs. (4.1a,b). The calculation is carried out up to the first order and the hamiltonian is written in the (ρ,σ) -representation. The zeroth order contribution to $N_1(x,t)$ is straightforwardly obtained by using the boson representation (2.59) and the cut-off procedure (2.60) . The result is

$$N_1^0(x,t) = -i(2\pi)^2 \left\{ [x - v_F t + i\alpha(t)][x + v_F t - i\alpha(t)] \right\}^{-1/2} , \tag{5.2}$$

where $\alpha(t) = \alpha\, \mathrm{sgn}(t)$ and $g_{4\parallel,\perp}$ have been taken equal to zero (these terms are included in the free hamiltonian). One can see that Eq. (5.2) can be obtained from $N(x,t)$ given by Eqs. (5.16) by setting all the coupling constants zero. The first order contributions to $N_1(x,t)$ are given by those terms of the hamiltonian that contain only ρ_σ -operators. For calculating these contributions we use the commutators of the ρ_σ -operators with the field operators and then we replace the $\psi_{\pm s}(x,t)$ operators by their boson representations. Doing so we get

$$N_1(x,t) = N_1^0(x,t)\left[1 + \frac{g_2 - g_{1\parallel}}{v_F} f(x,t)\right] \tag{5.3}$$

$$f(x,t) = \sum_{k>0} k^{-1} e^{-rk/2}\left[2 - e^{ik(x+v_Ft)} - e^{ik(x-v_Ft)}\right]$$

where $g_1 = g_{2\parallel} = g_{2\perp}$ and the k -dependence of the g_2 and $g_{1\parallel}$ has explicitly been introduced through the factor $e^{-rk/2}$. The first non-vanishing contribution to $N_2(x,t)$ comes from the first-order theoretical perturbation calculation and is given solely by the $g_{1\perp}$ -term of the hamiltonian (Eq. (4.1b)). By using the boson representation this contribution is easily obtained :

$$N_2(x,t) = -2\, g_{1\perp}(2\pi)^{-4}\int dx_1 dt_1 \{[x_1 + v_F t_1 - i\alpha(t_1)][x - v_F t_1 + i\alpha(t_1)]\cdot [x - x_1 + v_F(t-t_1) - i\alpha(t-t_1)][x - x_1 - v_F(t-t_1) + i\alpha(t-t_1)]\}^{-1} \tag{5.4}$$

The Fourier transform of the function $N(x,t)$ has the expression

$$N(\omega) = \frac{1}{\pi v_F}\ln\left(\frac{\alpha\omega}{v_F}\right)\left[1 - \frac{g_2 - g_{1\parallel} - g_{1\perp}}{2\pi v_F}\ln\left(\frac{\alpha\omega}{v_F}\right)\right] \tag{5.5}$$

in the limit $\alpha\omega/v_F \to 0$. One can see that the cut-off parameter α applies in the same way to both $g_{1\parallel}$ and $g_{1\perp}$ in contrast to the result reported by Grest[25, 28]. We should remark here that the same result could be obtained much easier by using the Fourier representation of the fermion field operators and the Jórdan's cut-off procedure (2.60).

Finally we should like to comment on the response function $N_1(x,t)$ calculated by Gutfreund and Klemm[24(b)] for the exactly soluble TFM with backscattering by using the Luther and Peschel bosonization technique. We calculate here the same response function by making use of the Jordan cut-off procedure. After somewhat lengthy algebra we get $N_1(x,t) = -2i\, N_1^{\sigma}(x,t)\, N_1^{\rho}(x,t)$

$$N_1^{\rho}(x,t) = \frac{1}{2\pi}\,\frac{\left\{[x-v_F t+ir(t)]\,[x+v_F t-ir(t)]\right\}^{1/2}}{\left\{[x-v_F t+i\alpha(t)][x+v_F t-i\alpha(t)]\,[x-v_\rho t+ir(t)][x+v_\rho t-ir(t)]\right\}^{1/2}}\,\cdot \tag{5.6}$$

$$\cdot\left\{(r/2)^{-2}\,[x-v_\rho t+ir(t)/2]\,[x+v_\rho t-ir(t)/2]\right\}^{g/2},$$

where $g = g_2/\pi v_F + 3/5$ $\quad g_{1\parallel} = -g_{4\parallel} \cong -3\pi v_F$, $g_{4\perp} \cong 0$, $|g| \ll 1$ (see Eqs. (4.18)) and $v_\rho^2 = [v_F + (2\pi)^{-1}(g_{4\parallel}+g_{4\perp})]^2 - [(2\pi)^{-1}(2g_2 - g_{1\parallel})]^2$. This expression is identical to that reported by Gutfreund and Klemm[24(b)] provided that r is replaced by α . The spin degrees of freedom are included in $N_1^{\sigma}(x,t)$ whose leading term is

$$N_1^{\sigma}(x,t) \cong (\sqrt{2}/\pi r)^{1/2} \sum_{p>0} e^{-\alpha p}\,\frac{\Delta_\sigma}{|\lambda_\sigma(p)|} \tag{5.7}$$

Δ_σ and $\lambda_\sigma(p)$ being given by Eqs. (4.19). The Fourier transform of $N_1(x,t)$ for small values of ω is $N_1(\omega) \propto -r^{-1/2}\ln(2\pi\alpha)\,(r\omega)^{-1-g}$ which agrees with the result reported by Gutfreund and Klemm[24(b)] except for the factors in the front of $(r\omega)^{-1-g}$ and provided that r is replaced by α . Similar results can be obtained for the other response functions of the exactly soluble TFM with backscattering by using Jordan's cut-off procedure.

VI. SUMMARY.

The boson representation and cut-off procedure introduced by Jordan[31] for describing a single fremion field in one dimension have been generalized to the four fermion operators of the one-dimensional TFM. It has been shown that the hermitean-conjugate fermion fields at the same space-point satisfy a certain relationship (Jordan's commutator) that has been overlooked so far by the theory of the TFM. In order to satisfy the Jordan commutator the cut-off parameter α should be used in a well-defined way (Jordan's cut-off procedure) that differs from that introduced by Luther and Peschel[22] and Haldane[21(a), 26] It has been shown that the exact solutions of the TFM with backscattering as well as with umklapp scattering are valid only if the zero-mode terms are absent in the kinetic hamiltonian. This requires a further condition on the coupling constants ($g_{4\parallel} \mp g_{4\perp} = 2\pi v_F$, respectively). It has been shown that all the inconsistencies reportes for the previous cut-off procedure are removed when one works with the Jordan technique . The one-particle Green's function and response functions of the TLM have been calculated and found to coincide with those obtained by direct diagram summation . The gap parameters appearing in the exactly soluble TFM with backscattering and umklapp scattering are proportional to γ^{-1} , γ being the parameter of the momentum cut-off. It follows that one may take $\alpha \to 0$ (Jordan's boson representation being exact only in this limit) and keep γ finite in diagonalizing these hamiltonians. Under exactly the same conditions the anticommutation relations and Jordan's commutator are preserved by the canonical transformation on the boson operators that diagonalises the TLM. The charge-density response function of the TFM with backscattering has perturbationally been calculated up to the first order. It has been found that the cut-off parameter α applies in the same way to both $g_{1\parallel}$ and $g_{1\perp}$ terms of this function. The same response function has been calculated for the exactly soluble TFM with backscattering at low frequencies. There is no major difference in the infrared behaviour of this function, except for Υ replacing α. The parameter α corresponds to the bandwidth cut-off while Υ^{-1} is a momentum transfer cut-off.

APPENDIX.

Let us consider four types of fermions labeled by $(js) = i = 1,2,3,4$ so that $(1,+1) = 1$, $(1,-1) = 2$, $(2,+1) = 3$ and $(2,-1) = 4$, each with the energy levels p = integer. The ground-state $|0\rangle$ of this system is filled with particles from $p = -\infty$ to $p = 0$ (or any other constant, not necessarily the same for all particles ; in this case the definition of b_i below should be changed correspondingly). Let us define the "charge" operators

$$b_i = \sum_{p>0} n_p^i + \sum_{p\leq 0} (n_p^i - 1) ,$$

where n_p^i is the occupation number of the p-level with i-type particles, $n_p^i = 0,1$. All the b_i yield zero when acting upon the ground-state, $b_i|0\rangle = 0$. We consider the states $|b_1 b_2 b_3 b_4\rangle$ characterized by specified eigenvalues b_i (integers) of the "charge" operators and define the operators F_i by

$$c_i |b_1 b_2 b_3 b_4\rangle = (-1)^{\sum_{0\le j<i} b_j} |b_1 \dots b_i - 1 \dots b_4\rangle ,$$
$$c_i^+ |b_1 b_2 b_3 b_4\rangle = (-1)^{\sum_{0\le j<i} b_j} |b_1 \dots b_i + 1 \dots b_4\rangle$$

where $i = 1,2,3,4$ and $b_0 = 0$. It is easily to check that the commutation relations (2.61) are satisfied by the operators $c_i = c_{js}$ defined on the space spanned by the states $|b_1 b_2 b_3 b_4\rangle$. In Sec.IV we introduced the operators c_{js} and $c_{j\sigma}$ by $c_{1s} = c^+_{2-1}$, $c_{2s} = c_{21}$, $c_{1\sigma} = c_{11}$ and $c_{2\sigma} = c_{1-1}$. Taking the superposition

$$|\varphi\rangle_{\varphi_1\varphi_2} = \sum_{b_1 b_2 b_3 b_4} e^{i(b_1\varphi_1 + b_3\varphi_2)} (-1)^{\frac{1}{2}[b_1(b_1-1) + b_3(b_3-1)]} |b_1 b_2 b_3 b_4\rangle ,$$

where φ_i are real parameters one can easily verify the relations

$$c^+_{1s} c^+_{2s} |\varphi\rangle_{\varphi_1\varphi_2} = c_{1s} c_{2s} |\varphi\rangle_{\varphi_1\varphi_2} = e^{i\varphi_2} |\varphi\rangle_{\varphi_1\varphi_2} , \quad c_{1\sigma} c_{2\sigma} |\varphi\rangle_{\varphi_1\varphi_2} = c_{1\sigma} c^+_{2\sigma} |\varphi\rangle_{\varphi_1\varphi_2} = -e^{i\varphi_1} |\varphi\rangle_{\varphi_1\varphi_2} ,$$

$$c_{1s} c^+_{2s} |\varphi\rangle_{\varphi_1\varphi_2} = c^+_{1s} c_{2s} |\varphi\rangle_{\varphi_1\varphi_2} = -e^{-i\varphi_2} |\varphi\rangle_{\varphi_1\varphi_2} , \quad c^+_{1\sigma} c_{2\sigma} |\varphi\rangle_{\varphi_1\varphi_2} = c^+_{1\sigma} c^+_{2\sigma} |\varphi\rangle_{\varphi_1\varphi_2} = e^{-i\varphi_1} |\varphi\rangle_{\varphi_1\varphi_2} ,$$

which are the additional conditions imposed on c_{js} in order to diagonalize the hamiltonian with backscattering and umklapp scattering[35]

expression is used for the energy of these states , $\varepsilon_p = \mu + v_F(|p| - k_F)$ where μ is the Fermi level and v_F is the Fermi velocity , thus obtaining two linear branches of the fermion spectrum as p lies near $+k_F$ or $-k_F$. The dynamics of the low excited states is governed by two interaction processes. The first one is the forward

REFERENCES.

1. For a recent review see, for example, High-Conducting One-Dimensional Solids, edited by J.T.Devreese, R.P.Evrard and V.E.van Doren (Plenum , New York, 1979) ; Physics in One Dimension, Proceedings of the International Conference, Fribourg , Switzerland, edited by J.Bernasconi and T.Schneider (Springer, Berlin , 1981).

2. J.M.Luttinger, J.Math.Phys. 4, 1154 (1963).

3. A.Luther and V.J.Emery, Phys.Rev.Lett. 33, 589 (1974).

4. V.J.Emery, A.Luther and I.Peschel, Phys.Rev. B13, 1272(1976).

5. Yu.A.Bychkov, L.P.Gorkov and I.E.Dzyaloshinsky, JETF 50 ,738 (1966) (Sov. Phys. - JETP 23, 489 (1966)) ; I.E.Dzyaloshinsky and A.I.Larkin, JETF 61, 791 (1971) (Sov. Phys. - JETP 34, 422 (1972)).

6. T.Chmi, K.Nakajima and T.Tsuneto, Prog. Theor. Phys. (Osaka) 55, 1396 (1976).

7. N.Menyhárd and J.Sólyom, J.Low Temp. Phys. 12, 529 (1973) ; 21, 431 (1975) ; J.Sólyom , ibid 12, 547 (1973) ; H.Fukuyama , T.M.Rice, C.M.Varma and B.I.Halperin, Phys. Rev. B10, 3775 (1974); M.Kimura, Prog. Theor. Phys. (Osaka) 49, 697 (1973).

8. S.T.Chui, T.M.Rice and C.M.Varma, Solid State Commun. 15, 155 (1974).

9. G.S.Grest, E.Abrahams, S.T.Chui, P.A.Lee and A.Zawadowski, Phys Rev. B14, 1225 (1976) ; J.Sólyom , G.Szabó in Lecture Notes on Physics, vol. 65, edited by L.Pál, G.Grüner, A.Jánossy and J.

Sólyom (Springer, Berlin , 1977).

10. J.Sólyom, Solid State Commun. 17. 63 (1975) ; M.Kimura, Prog. Theor. Phys. (Osaka) 53, 955 (1975).

11. J.Sólyom, Adv. Phys. 28, 201 (1979).

12. M.Apostol, Solid State Commun. 37, 257 (1981) ; M.Apostol, A. Corciovei and S.Stoica, phys. stat. sol.(b) 103, 411 (1981).

13. S.Tomonaga Prog. Theor. Phys. (Osaka) 5, 544 (1950).

14. I.E.Dzyaloshinsky and A.I.Larkin, JETF 65, 411 (1973) (Sov.Phys. - JETP 38, 202(1974)).

15. H.U.Everts and H.Schulz , Solid State Commun. 15, 1413 (1974) see also Ref. 11, p. 230 pp. 234-235 , and M.Apostol , Rev.Roum. Phys. 26, 119 (1981) ; M.Apostol and V.Bârsan, ibid. 26, 375 (1981).

16. H.C.Fogedby, J.Phys. C 9, 3757 (1976) ; R.A.Klemm and A.I.Larkin, Phys. Rev. B19, 6119 (1979).

17. D.C. Mattis and E.H.Lieb, J.Math. Phys. 6, 304 (1965).

18. H.Gutfreund and M.Schick, Phys. Rev. 168, 418 (1968).

19. (a) A.Theumann, J.Math.Phys. 8, 2460 (1967) ; (b) Phys. Lett. 59A 99(1976).

20. T.Bohr, preprint Nordita - 81/4, January 1981.

21. (a) F.D.M.Haldane, J.Phys. C 14, 2585 (1981); Phys. Rev. Lett. 45, 1358 (1980) ; Phys. Lett. 81A , 153 (1981).

22. A.Luther and I.Peschel, Phys. Rev. B9 , 2911 (1974).

23. D.C.Mattis, J.Math. Phys. 15, 609 (1974).

24. (a) P.A.Lee, Phys. Rev. Lett. 34, 1247 (1975) ; (b) H.Gutfreund

and R.A.Klemm, Phys. Rev. B14, 1075 (1976).

25. G.S.Grest, Phys.Rev. B14, 5114 (1976).

26. F.D.Haldane, J.Phys. C12 , 4791 (1979).

27. G.Grinstein, P.Minnhagen and A.Rosengren , J.Phys. C12, 1271 (1979).

28. See Ref.11, pp.259-260.

29. A.Theumann, Phys.Rev. B15, 4524(1977).

30. See, for example , R.Heidenreich, B.Schroer, R.Seiler and D. Uhlenbrok, Phys. Lett. 54A , [illegible]19 (1975) .

31. P.Jordan, Z.Physik 93, 464 (1935) ; 99, 109 (1936) ; 102 , 243 (1936) ; 105, 114, 229 (1937).

32. It is indeed surprising that the significance of Jordan's boson representation for the theory of the one-dimensional TFM has passed unnoticed until now, although Mattis and Lieb[17] refer to it.

33. See Ref. 11, p.250; also Ref. 4.

34. Strictly speaking we may not replace the sum $\sum_{p} [illegible]$ by the integral $L(2\pi)^{-1}\int_0^{\pi L^{-1}} [illegible]$ as we have done in deriving Eq.(2.21). However this apparent inaccuracy leads to the correct result which can be rigorously obtained as follows. Let us introduce the set S_n of unitary operators defined by $S_n a_p S_n^{-1} = a_p + \pi L^{-1}\alpha_n$ etc., $\alpha_n = n$, n integer. We have $S_n\psi(x)S_n^{-1} = \exp(-i\pi L^{-1}\alpha_n x)\psi(x)$, $S_n B S_n^{-1} = B - \alpha_n$, $S_n H_0 S_n^{-1} = H_0 - 2\pi L^{-1} v_F \alpha_n B + \pi L^{-1} v_F \beta_n$, where $\beta_n = n(n+1)$. The operator S will be defined by $S a_p S^{-1} = a_p + \pi L^{-1}\alpha$, $S\psi(x)S^{-1} = \exp(-i\pi L^{-1}\alpha x)\psi(x)$, $SBS^{-1} = B - \alpha$, $SH_0S^{-1} = H_0 - 2\pi L^{-1} v_F \alpha B + \pi L^{-1} v_F \beta$ where $\alpha = \lim_{n\to\infty}(\alpha_n)^{1/n} = 1$ and $\beta = \lim_{n\to\infty}(\beta_n)^{1/n} = 1$. It follows

that S defined in this way has the same effect as that of S given by Eqs. (2.18) provided that the sum $\sum_{0 \le p \le ...} p$ is replaced by the integral $L(\omega) \int_0^{...} p\,dp = ...$. It is noteworthy that this definition of S allows us to introduce real powers of this operator, S^{μ} μ-real, by simply changing α and β in u_α and u_β.

35. The conditions (2.61) are satisfied by the Dirac matrices as well as by operatorial representations of the coefficients c_{js} in terms of the "charge" operators $\hat{S}_{js}$ (see Ref. 30 and Ref.11, p.24o). However in order to diagonalize the Luther-Emery hamiltonian as well as the umklapp scattering hamiltonian the coefficients c_{js} are ~~further~~ subjected to additional condition (see Sec. IV) which are satisfied neither by the Dirac matrices nor by these operatorial representations.

36. As regards the real powers of the operators S_{js} see Ref.34.

37. One can easily verify that the anticommutation relations and the Jordan commutator are also preserved by this extended transformation which affects the "charge" operators $\hat{S}_{js,\sigma}$ and the operators $S_{js,\sigma}$ as well. The proof of this statement is identical with that given at the end of Sec. II and requires the limit $\alpha \to 0$ to be taken firstly while r is kept finite.