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SURVIVAL OF GRAND UNIFIED MONOPOLES

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Abstract

If a Grand Unified Theory with a compact unifying group G is spontaneously broken to H , magnetic monopoles are created. The fate of such an H-monopole under a subsequent breaking to $K \subset H$ is shown to depend on the behaviour of its non-Abelian charge Q introduced by Goddard, Nuyts, Olive: if Q belongs to the Lie algebra k of K , the monopole survives: if Q can be H-rotated to k, it can be converted. A necessary condition for an H-monopole to survive is that its Higgs charge satisfy a topological constraint.

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In Grand Unified Gauge Theories (GUT's) magnetic monopoles arise naturally as everywhere-regular, finite-energy, static, purely-magnetic solutions to the coupled Yang-Mills-Higgs (YMH) equations. [1]. What happens to these monopoles if the symmetry is broken subsequently to smaller and smaller subgroups? This question has been asked and partially answered before [2-4]. Our approach is hoped to be more direct and it allows us to derive the previous knowledge as a consequence.

Let us consider a GUT with a compact, connected and simply connected unifying group G. At some energy scale the G-symmetry is broken to a closed subgroup H of G by the v.e.v. of a Higgs field ϕ (transforming according to some representation $\phi \to g\cdot\phi$, $g \in G$ of G). The field equations admit magnetic monopole solutions [1]. These solutions satisfy boundary conditions: on S^2, the "2-sphere at infinity", (i) ϕ takes its values in an orbit $G\cdot\phi_0 = G/H$; (ii) ϕ is parallel, $D_j\phi = 0$ $j = 1,2,3$. It follows from (ii) that on S^2 the YMH equations decouple and the YM potential A_j satisfies the pure YM equation

$$(1) \qquad D_i F^{ij} = 0 \quad \forall j.$$

The point is that any solution of (1) is characterized by a vector Q in h, the Lie algebra of H, called the "non-Abelian magnetic charge" [5,6] : there exist gauges over $U_+ = S^2 \setminus \{\text{south pole}\}$ and $U_- = S^2 \setminus \{\text{north pole}\}$ respectively such that (1) is solved by

(2) $$A_r^\pm = 0 \, , \quad A_\theta^\pm = 0 \, , \quad A_\varphi^\pm = \pm Q(1 \mp \cos\theta)$$

for some fixed $Q \in h$. For a given solution Q is unique up to a constant global gauge transformation. In particular, it can always be rotated to any given Cartan algebra of h . Q must be *quantized*

(3) $$\exp 4\pi Q = 1 \, .$$

(1) implies that ϕ defines a class $[\phi]$ in $\pi_2(G/H)$ which we call its Higgs charge. The Higgs fields are gauge-related if and only if they have the same Higgs charge. $\delta_H : \pi_2(G/H) \to \pi_1(H)$ is an isomorphism since G is simply connected. $\delta_H[\phi]$ is represented by the loop

(4) $$\gamma(t) = \exp 4\pi Q t \qquad 0 \le t \le 1 \quad .$$

To have topological non-triviality, we require $[\phi] \neq 0$.

Conversely, for any quantized $Q \in h$, (2) provides us with a solution of (1) which can be extended to a YMH solution on S^2 simply by putting $\phi = \phi_0$ in $U_\pm$. Observe that this is a well-defined Higgs field since $Q \in h$ means that $Q \cdot \phi_0 = 0$, and so $\phi_0 = \exp 2Q\varphi \cdot \phi_0$ in $U_+ \cap U_-$. The solution (A_j,ϕ) constructed in this way is interpreted as an H-monopole (i.e. a monopole created in the "phase" when the symmetry group is H) assuming $[\phi] \in \pi_2(G/H)$ is non zero.

Let us now assume that the symmetry is broken spontaneously to a closed subgroup K of H . In the K "phase" monopoles are pairs (A_j,X) , where X is a new Higgs field engineering the symmetry breaking. In particular, $X:S^2 \to G/K$ where K is the stability *subgroup of a* X_0 *in a suitable representation;* X *must be parallel*, and $\pi_2(G/H) \ni [X] \neq 0$. The theory above tells us that any K-monopole is characterized by a K-magnetic charge $Q' \in k$ (the Lie algebra of K).

Let us now consider an H-monopole associated to a $Q \in h$. This same Q defines a K-monopole as well as soon as

$$(5) \qquad Q \in k \quad \text{and} \quad \pi_1(K) \ni [\exp 4\pi Qt]_K \neq 0 \ .$$

Saying that an H-monopole survives means hence that its K- and H magnetic charges are in fact the same vector Q .

Observe that for surviving monopoles $\delta_H[\phi]$ and $\delta_K[X]$ are represented by the same loop (4), so, if $i : K \hookrightarrow H$ is the inclusion, (6a) $\delta_H[\phi] = i_* \, \delta_K[X]$

where $i_*: \pi_1(K) \to \pi_1(H)$ is the homomorphism induced by i. Alternatively, the natural projection $\sigma : G/K \to G/H$ induces $\sigma_* : \pi_2(G/K) \to \pi_2(G/H)$ and (6a) means

(6b) $[\phi] = \sigma_*[X]$.

Consequently, an H-monopole can survive only if

(7) $[\phi] \in \mathrm{Im}\,\sigma_*$ i.e. $\delta_H[\phi] \in \mathrm{Im}\, i_*$.

Let us now assume that (7) holds. Then $[\exp 4\pi Qt]_K \neq 0$ automatically. If there exists an $h \in H$ such that $h\,Q\,h^{-1} \in k$ - and this may or may not be the case depending on how the Cartan algebra of k is related to that of h (see examples below) - then the monopole can save itself by reorienting somehow its field's direction and so survive converted [4]. But if (7) is violated it must disappear. The proposed scenario for this is confinement in a flux tube [3,4]. Observe that (7) depends only on how the generators of $\pi_1(K)$ sit in H. In particular, if they are contractible in H, $\mathrm{Im}\, i_* = 0$ and no H-monopole can survive. Those K-monopoles whose K-charges are in $\mathrm{Ker}\, i_*$

cannot be surviving H-monopoles. They are hence newly created, light monopoles.

An important special case is when $\pi_1(K) \simeq \mathbb{Z}$. $\pi_1(K)$ has then a generator of the form $\gamma_0(t) = \exp 2\pi\zeta t$, $0 \le t \le 1$. (7) requires $[\phi] = r\cdot[\gamma_0]$ for some integer r . If furthermore, Q and $r\cdot\zeta/2$ are conjugate then the monopole can be converted by reorienting its field. If $Q = r\zeta/2$, then the monopole survives unchanged.

In some cases, these relations can be translated into numbers. For example, if ϕ is in the adjoint representation of G , then $\pi_1(H) \simeq \mathbb{Z}^p$ where p is the dimension of $Z(h)$, the centre of the Lie algebra h . $[\phi]$ is then simply a p-tuple of integers, $\phi = (m_1,\ldots,m_p)$ [7]. On the other hand, γ_0 above has $(k_1,\ldots,k_p)$ as homotopy class in $\pi_1(H)$, so (7) means

(8) $$m_j = r\cdot k_j\ ,\quad j = 1,\ldots,p \quad \text{for some integer } r\ .$$

For example, if K is a $U(1)$ subgroup, γ_0 is K itself; if K is a $U(3)$ subgroup - locally $SU(3)_c \times U(1)_{em}$ - then

$\zeta = i\,(\mathrm{diag}(0,0,\sqrt{-1}))$ is a generator for $\pi_1(K)$. (i is the inclusion $i : k = su(3) \times u(1) \hookrightarrow h$ of Lie algebras).

If $k \subset Z(h)$ and Q is not in $Z(h)$, then no conjugate of Q belongs to $Z(h)$, and so no such H-monopole can survive, not even converted. For example, if $h = su(2)_W + u(1)_Y$, $k = z(h) = u(1)_{em}$, then only pure electromagnetic monopoles can survive the transition $H \to K$ [2].

We just mention that the formulae above have a nice geometric interpretation. YM fields are connection forms on principal bundles. Spontaneous symmetry breaking means bundle reduction. Higgs charges classify the principal bundles over S^2 , so (7) means that H-bundle reduces to a K-bundle. (ii) is the condition for the connection to reduce. (2) - the content of the theorem of Goddard, Nuyts and Olive [5,6] - is expressed by saying that monopoles have 1-dimensional holonomy generated by Q . If an H-bundle reduces to a K-bundle, the H-connection reduces to a K-connection if and only if k contains the holonomy, i.e. (5). Full details are given in [9].

As a first illustration, let us consider G = SU(3) broken to H = U(2) by a Higgs 8 ϕ. The base point is $\phi_0 = (\lambda\sqrt{-1})$ diag $(1,1,-2)$, $\lambda \in \mathbb{R}$. The non-Abelian charge Q reads, after diagonalization,

(9) $$Q = \frac{n}{2}\,\mathrm{diag}\,(n_1,\ n_2-n_1,\ -n_2)\cdot i$$

where n, n_1, n_2 are integers and n_1, n_2 have no common divisor. The form (9) is unique if we require $2n_1 \geq n_2 \geq n_1$. The U(2)-Higgs charge is then $m = n.n_2$.

Let K be a U(1) subgroup $K = \exp 2\pi\zeta t$, $\zeta \in U(2)$. ζ can be diagonalized by a suitable U(2)-conjugation:

(10) $$h\zeta h^{-1} = \sqrt{-1}\,\mathrm{diag}\,(k_1,\ k_2-k_1,\ -k_2)$$

The integers k_1, k_2 here have no common divisor; they are unique if $2k_1 \geq k_2$ $2k_2 \geq k_1$. Conjugate vectors generate homotopic loops, so $[K]_{U(2)} = k_2$.

(7) requires hence $m = rk_2$ for some r. In particular [8], if $k_2 = 0$ all H-monopoles must disappear when the symmetry breaks down to K ; if $k_2 = 2$, m must be even, etc.

It is easy to see that Q and some half-integer multiple of (10) are U(2)-conjugate if and only if $n_2 = k_2$ and $n_1 = k_1$ provided $2k_1 > k_2$.*

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* If $2k_1 = k_2$, ζ is in the centre and so ζ and Q cannot be conjugate unless they are equal, which is the third case. Otherwise, the monopole is again confined in a flux tube [2] .

In this case the monopole can be converted by reorienting its field's direction.

Finally, the H-monopole survives unchanged if Q and $r^{\zeta}/2$ are not only conjugate but actually equal: $r = n$, $n_1 = k_1$ and $n_2 = k_2$ where now K itself is diagonal, and has the form (10).

If we choose instead $\phi_0 = \sqrt{-1}\ \mathrm{diag}\ (\lambda_1, \lambda_2, -\lambda_1\ \ -\lambda_2)$ with $\lambda_1 > \lambda_2$ as base point, the residual symmetry group is $U(1) \times U(1)$, so $\pi_1(H) \simeq \mathbb{Z}^2$. If we choose $n_1 = \sqrt{-1}(1, -1, 0)$ and $n_2 = \sqrt{-1}(0, 1, -1)$ as generators for $\mathfrak{h}$, the Higgs charge of the monopole defined by Q in (9) is $[\phi] = (nn_1, nn_2)$.

$H = U(1) \times U(1)$ is Abelian, so any subgroup K is already of the form (10). (7) requires $n_1 = k_1$ and $n_2 = k_2$, so $Q \in k$. Hence all monopoles satisfying the topological constraint survive unchanged, while all others are destroyed.

As a second example let us consider $G = SU(3)$ and ϕ transforming according to the representation 6 [4 13] The representation space consists of 3×3 symmetric matrices with SU(3) acting as $\phi \to g\cdot\phi = g\phi g^T$. If we choose $\phi_0 = \mathbb{1}_3 = \mathrm{diag}\ (1,1,1)$ as base point, we get $H = SO(3)$ as unbroken symmetry group. H is generated in fact by the Gell-Mann matrices

$$\lambda_7 = \sqrt{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad -\lambda_5 = \sqrt{-1}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \sqrt{-1}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{11}$$

Let us choose our Cartan algebra T to contain λ_2. The non-Abelian magnetic change, after rotation to T, reads

(12) $Q = m\lambda_2$, m an integer.

$\pi_1(SO(3)) = \mathbb{Z}_2$; the Higgs charge of the SO(3)-monopole defined by (12) is $[\phi]_{SO(3)} = m \pmod 2$. So we get topologically non-trivial solutions only for m odd.

Let us consider a $U(1)$ subgroup $K = \exp 2\pi t\xi \subset H \subset SU(3)$. ξ can be brought to T by a suitable SO(3)-rotation. $\xi \to h\xi h^{-1} = \zeta \in T$ for some $h \in H$. ζ is a minimal generator since ξ is minimal. Hence $\xi = \pm\lambda_2$. Consequently $[K]_{SO(3)} = [\exp 2\pi\zeta t]_{SO(3)} = 1 \pmod 2$. It follows that all topologically non-trivial (m odd) H-monopoles satisfy the topological constraint (7). $i_* : \mathbb{Z} \ni m \to m \pmod 2 \in \mathbb{Z}_2$ so the light, newly created $K = U(1)$ monopoles are evenly charged [4].

Let us assume m odd in (12). Q can be $H = SO(3)$-rotated to $k = \mathbb{R}.\xi$, so all odd-charged SO(3)-monopoles can save themselves by reorienting their fields and hence be converted. Their new U(1) charge will be simply m. They survive however <u>unchanged</u> if Q and ξ are actually <u>parallel</u>, i.e. $\xi = \pm \lambda_2$.

Let us consider the prototype GUT based on G = SU(5) [10,11]. The SU(5) symmetry is broken first by a Higgs <u>24</u> (adjoint) ϕ. If $\phi_0 = v\sqrt{-1}\,\mathrm{diag}\,(1,1,1,-3/2,-3/2)$, we get $H = S(U(3) \times U(2)) = [SU(3)_c \times SU(2)_W \times U(1)_Y]/\mathbb{Z}_6$ of those matrices $\begin{bmatrix} A & \\ & B \end{bmatrix}$ $A \in U(3)$, $B \in U(2)$ det A. det B = 1.

$$(13) \qquad Q = \frac{n}{2}\sqrt{-1}\,\mathrm{diag}\,(n_1, n_2, -n_1, n_3-n_2, n_1-n_3, -n_4)$$

where n, n_1, n_2, n_3, n_4 are integers, the last four has no common divisor, and $2n_1 \geq n_2 \geq n_3/2 \geq n_4/4$. $\pi_1(H) \simeq \mathbb{Z}$. The Higgs charge of the monopole which corresponds to Q is calculated as [7, 12].

$$(14) \qquad m = [\phi] = \frac{tr_3}{\sqrt{-1}}\,(2Q) = n.n_3$$

(trace on the first 3 entries).

At a lower energy scale H is further broken to $K = U(3) \simeq SU(3)_c \otimes U(1)_{em}^*$ by $X = (\phi, \psi)$, where ψ is in <u>5</u>, having base point $v(0,0,0,1)$.

$$K = \begin{bmatrix} A & & \\ & \det A^{-1} & \\ & & 1 \end{bmatrix}, \quad A \in U(3).$$

* $\otimes$ means local product, $U(3) = SU(3)_c \times U(1)_{em}/\mathbb{Z}_3$.

A generator of $\pi_1(K)$ is found as

(15) $\zeta = \sqrt{-1}\,\mathrm{diag}\,(0, 0, 1, -1, 0)$

so $[K]_H = tr_3(\zeta)/\sqrt{-1} = 1$.

Im $i_* = \mathbb{Z}$, the <u>topology constraint is always satisfied.</u>

Q as given in (13) can belong to k if and only if $n_4 = 0$. If so, the H-monopole survives unchanged and has K-charge $[X] = \mathrm{Tr}\,(2Q)/\sqrt{-1} = nn_2 = m$.

$Q \in H$ can be H-rotated into $k = u(3)$ if and only if $n_3 = n_4$. If so, the H monopole reorients its field and survives thus converted. No new monopoles are now produced when the symmetry breaks to K since i_* is <u>injective</u>, Ker $i_* = 0$.

In some versions of the SU(5) theory [10] the base point in the first breaking is rather $\phi_0 = v\sqrt{-1}\,\mathrm{diag}\,(1, 1, 1, -3/2 + \epsilon, -3/2 - \epsilon)$ leading to $H' = S(U(3) \times U(1) \times U(1)) = U(3) \times U(1)$ as residual symmetry group. $\pi_1(H) = \mathbb{Z}^2$ now. One can choose $\zeta_1 = \sqrt{-1}\,\mathrm{diag}\,(0, 0, 1, -1, 0)$ and $\zeta_2 = \sqrt{-1}\,\mathrm{diag}\,(0,0,0,1,-1)$ as generators for $\pi_1(H')$. We have two Higgs charges, m_1 and m_2. Q as given by (13) belongs to h'. The charges of the corresponding monopole are found as

(16)
$$m_1 = \frac{tr_3}{\sqrt{-1}}(2Q) = nn_3$$
$$m_2 = \frac{tr_4}{\sqrt{-1}}(2Q) = nn_4$$

K = U(3) as above is a subgroup of H'; its two Higgs charges in $\pi_1(H')$ are

$$k_1 = \frac{tr_3}{\sqrt{-1}}(\zeta) = 1 \text{ and } k_2 = \frac{tr}{\sqrt{-1}}(\zeta) = 0. \tag{17}$$

Consequently Im i_* is generated by $(1,0) \in Z^2$, and so only such H'-monopoles satisfy the topological constraint (7) which have no m_2-charge, i.e. $n_4 = 0$. Those with $n_4 \neq 0$ are destroyed under the transition H' → K. If $n_4 = 0$, Q belongs to k, so it survives unchanged. Its K-charge is $m_1 = nn_3$. No new K-monopoles are produced since i_* is still injective.

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