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ABSTRACT

A natural candidate model for a gauge theory for the Poincaré group is discussed. It satisfies the usual electric-magnetic symmetry of gauge models and is a contraction of a gauge model for the De Sitter group. Its field equations are just the Yang-Mills equations for the Poincaré group. It is shown that these equations do not follow from a Lagrangean.

1. INTRODUCTION

It is a general belief that gravitation, more than other interactions, is closely related to the detailed texture of the space-time manifold. Einstein went so far as to assert that, were the gravitational field eliminated, there would remain "absolutely nothing, not even a topological space"[1]. His own theory seldon goes to such detailed fine-grained aspects [2], but the successes of General Relativity are surely at the origin of that general opinion. Another widespread surmise, a much more recent one, is that interactions in general are mediated by gauge fields, a conviction supported by the achievements of the electroweak gauge theory and by the successes of Quantum Chromodynamics.

Gauge models for gravitation endeavor to accommodate these two ideas, which point naturally to the investigation of those features of space-time presenting gauge-like characteristics. General Relativity describes the gravitational field essentially as the curvature of a Levi-Civita connexion, which is metric-preserving and torsionless, giving consequently a preeminent role to the metric structure. In alternative theories of the gauge type [3], the central part is played by the connexion itself, metric or not. This is, of course, an imposition of the gauge formalism, in which the basic field is the gauge potential A, a pulled-back connexion on a principal bundle with the gauge group as structural group and space-time as base manifold. This potential is a Lie algebra valued 1- form: taking $\{Z_a\}$ as the set of group generators, with $[Z_a, Z_b] = -i\ f^c{}_{ab} Z_c$, it is written, in a coordinate basis $\{dx^\mu\}$,

$$A = Z_a A^a{}_\mu\, dx^\mu \qquad (1.1)$$

Fields belong to general representations of the group, and to every representation is associated a certain fiber bundle. A connexion defines covariant derivatives on each one of these bundles. A very special case is the adjoint representation, whose carrier space is the vector space of the Lie algebra and to which A itself belongs. The field strength F is the curvature of A, i. e., its own covariant derivative:

$$F = DA = dA - i\, A \wedge A \tag{1.2}$$

or, in components,

$$F = \frac{1}{2} t_a F^a{}_{\mu\nu}\; dx^\mu \wedge dx^\nu \tag{1.3}$$

where

$$F^a{}_{\mu\nu} = \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu - f^a{}_{bc}\, A^b{}_\mu\, A^c{}_\nu \;. \tag{1.3'}$$

The Bianchi identity

$$dF - i\,[A, F] = 0 \tag{1.4}$$

is a direct consequence of (1.2). In words, it says that the covariant derivative of F is zero. We can say that gauge field theories have two distinct facets. One is a geometrical substratum, introducing all the possible connexions on the bundle obtained with the chosen gauge group and summarized in (1.2) and (1.4). Another is the dynamical aspect, allowing to fix, amongst all these connexions, the one really at work in a given physical situation [4]. Introducing the dual of F,

$$*F = \frac{1}{2} t_a \tilde{F}^a{}_{\mu\nu}\; dx^\mu \wedge dx^\nu \tag{1.5}$$

where

$$\tilde{F}^a{}_{\mu\nu} = \frac{1}{2}\sqrt{-\eta}\;\; F^{a\,\rho\sigma}\, \varepsilon_{\rho\sigma\mu\nu} \tag{1.5'}$$

the dynamical prescription is given by the Yang-Mills equation ,

$$d*F - i\left[A, *F\right] = 0 \qquad (1.6)$$

in the sourceless case. This is the Euler-Lagrange equation obtained from the action integral

$$S[A] = \frac{1}{2}\,\mathrm{tr}\int F \wedge *F \qquad (1.7)$$

Source fields,belonging to representations in the associated bundles, contribute with their currents to the right-hand side of (1.6), and obey the additional dynamical equations obtained from their free equations via the minimal coupling prescription.

Going back to the idea of gravitation as a spacetime - rooted effect, we should hope to find structures in the geometry of spacetime analogous to those sketched above, at least the purely geometrical facet . In fact, on any differentiable manifold (which we shall here consider four- dimensional) there is always a bundle naturally defined, the bundle of affine frames [5]. Its structural group is the affine linear group $AL(4,\mathbb{R}) = GL(4,\mathbb{R}) \odot T_4$, the semidirect product of the linear group and the translation group in 4 dimensions. In the case of spacetime, the restriction to Lorentz frames reduces it to the Poincaré group, $P = \mathcal{L} \odot T_{3,1}$. This always-present structure provides the most general gauge-like features one can find which are intimately related to spacetime, and justifies the interest in Poincaré gauge theories. Of course, in order to accommodate the representations to which the known elementary particles belong, the Lorentz group $\mathcal{L}$ is to be viewed as the covering group of $SO(3,1)$, $SL(2,\mathbb{C})$.

A connexion $\tilde{\Gamma}$ on the bundle of affine frames(an "affine connexion") will, just like any connexion on any other bundle, satisfy (1.4) There is, however, something special in this case. Let us write $Z_{\alpha\beta}$, with $\alpha, \beta = 1,2,3,4$, for the generators of the Lorentz group, and I_α for those of the translation group. The Lie algebra is, as a vector space, a direct sum and $\bar{\Gamma}$ decomposes itself into two parts:

$$\bar{\Gamma} = \Gamma + S \tag{1.8}$$

where

$$\Gamma = \tfrac{1}{2} Z_{\alpha\beta} \, \Gamma^{\alpha\beta}{}_{\mu} \, dx^{\mu} \tag{1.9}$$

and

$$S = I_\alpha \, S^{\alpha}{}_{\mu} \, dx^{\mu} \; . \tag{1.10}$$

The same happens to the affine curvature,

$$\bar{F} = d\bar{\Gamma} - i \, \bar{\Gamma} \wedge \bar{\Gamma} = F + T \tag{1.11}$$

with

$$F = d\Gamma - i \, \Gamma \wedge \Gamma \tag{1.12}$$

and

$$T = dS - i \, \Gamma \wedge S - i \, S \wedge \Gamma \; . \tag{1.13}$$

The Bianchi identity

$$d\bar{F} - i \left[\bar{\Gamma}, \bar{F} \right] = 0 \tag{1.14}$$

decomposes in this case into two,

$$dF - i \left[\Gamma, F \right] = 0 \tag{1.15}$$

and

$$dT - i \left[\Gamma, T \right] - i \left[S, F \right] = 0 \; , \tag{1.16}$$

the first with components in the Lorentz sector, the second along the $\{ I_\alpha \}$ Of course, this discussion has been rather succint, but it is a synopsis of the main fundamental properties concerning connexions on spacetime proper. The form Γ given by (1.9) is really a good connexion in the Lorentz sector, which forms an independent sub-structure: it is a connexion on the sub-bundle of Lorentz frames and F its curvature. Now, on the bundle of frames of any differentiable manifold there exists a canonical 1-form, independent of any connexion, the solder form. It is a special characteristic of this bundle and is precisely the form S of (1.10). Unlike Γ, it is not a connexion and as a consequence the translation sector does not form an independent substructure of the affine structure. In reality, an affine connexion like Γ can be defined in a huge number of different ways, with S in (1.8) being "horizontal forms" [6] of a more general type (leading to "generalized affine connexions"); the particular choice of the solder form is convenient because of its simple geometrical properties. We shall use S for the time being, and relax this choise later on. A specific property of S is that its components $S^{\alpha}{}_{\mu}$ are, when S is written on the base manifold with a chosen frame characterized by the four-legs $\{\lambda_{\alpha}{}^{\mu}\}$, just the elements $\{\lambda^{\alpha}{}_{\mu}\}$ of the corresponding dual basis of 1-forms. The connexion Γ defines, as usual, a covariant derivative; its curvature F is its own covariant derivative. The quantity T given by (1.13) is the covariant derivative of S (remember that the detailed expression of the covariant derivative depends on both the degree and the algebraic content of the form), that is, the torsion of Γ . Notice that torsion is always present in the bundle of frames-there is not such a question as introducing it or not. It may be vanishing (as in General Relativity), but it has consequences anyhow. For this reason, equations (1.15) and (1.16) are found in differential geometry textbooks under the names of "first" and "second" Bianchi identities. Even if we started with the homogeneous Lorentz group, the presence of the solder form would force the presence of torsion. Furthermore, the introduction of spinors on a manifold practically enforces (or reveals [18]) the presence of torsion. In any case, the affine formalism takes all that automatically into account. These natural properties of space-time are as near as we can wish to those purely geometrical features of gauge theories incorporated in equations (1. 1-4).

Now, what about dynamics? One main trouble in building up a gauge theory for the Poincaré group rests on its nonsemisimple character [7]. Known gauge models have semisimple groups, whose well-defined Cartan-Killing metric can be used to write an invariant Lagrangean density. This is not the case here and to obtain the dynamical facet one has to resort to other arguments. One could even suspect that a theory for the Poincaré group which is <u>really</u> a gauge theory would have no well-defined Lagrangean (such a suspicion comes out after a few tentatives to build it up) and that the dynamics is to be described by field equations determined in some other way. The requirement we shall be using to get the equations is one of simple coherence with the above ideas of gravitation as a gauge interaction related to the differentiable structure of space - time. We have already shown its purely geometrical aspect; to get its dynamical counterpart, we simply require that the field equations be of the Yang-Mills form. This means that we force ourselves to retain one of the most interesting characteristics which general gauge theories have inherited from electrodynamics: the delicate balance between the "electric" and "magnetic" components of the fields, which is ultimately reflected in the striking similarity of the geometrical equation (1.4) and the dynamical equation (1.6) In effect, the latter is just the former written for the dual of F. This is, of course, the well known discrete duality symmetry of sourceless gauge fields, whose extension to the case of non-vanishing sources has been extensively examined in the study of monopoles but whose import for the sourceless case (where it is undoubtedly present) has only begun to be clarified with the lattice approach [8]. Roughly speaking, (1.6) says that, in the absence of sources, also the covariant coderivative of F vanishes. Field derivatives and coderivatives have, in the lattice formalism, geometrical counterparts as boundaries and coboundaries in the lattice, and much of the highly symmetric properties to which the approach owes its power is a consequence of the duality symmetry. It is highly desirable to preserve this electric-magnetic symmetry of classical gauge fields for yet another reason: it fixes completely the field equations in the Yang-Mills form. We shall later on give some more arguments supporting the resulting equations. If these are simply the Bianchi identities written for the dual field strengths, then (1.14 - 16) tell us that they are

$$d*\bar{F} - i[\bar{\Gamma}, *\bar{F}] = 0 \quad , \tag{1.17}$$

or

$$d*F - i[\Gamma, *F] = 0 \tag{1.18}$$

and

$$d*T - i[\Gamma, *T] - i[S, *F] = 0 \ . \tag{1.19}$$

These equations have been first proposed by Popov and Daikhin [9], who got at them by simply asking what the Yang-Mills equations would be for the Poincaré group. They also pointed out that, if we suppose (which would be rather in contradiction with the above philosophy) Γ to be metric preserving and torsionless, the equations reduce to those of Yang's model for gravitation in empty space. More precisely, only the last term in (1.19) survives, giving Einstein's equation $R_{\mu\nu}=0$, and (1.19) becomes redundant. In this sense, (1.18 - 19) generalize Einstein's theory in empty space. The conditions imposed on Γ , however, have bad consequences: Yang's equations are known to have ill-behaved solutions in the presence of sources, and, of course, all we have said about the symmetry would be lost. The main reason to look for gauge theories for gravitation is not a purely aesthetic or phylosophical one. It comes from the apparent allergy of Einstein's theory to renormalization and to the gauge theories penchant for it. It is not clear just where this affinity comes from [10], but it is a general feeling that conformal invariance is somehow involved in good short-distance behaviour [11]. Gauge models for the conformal group have been proposed [12] in this line of thought. The Yang-Mills equations are conformally invariant and so are equations (1.18 - 19). The reason is simple: the only dependence on the space-time metric appears in the operation of taking the dual of F and T , both of which are 2-forms. In a 4-dimensional space, as can be seen by direct inspection, the dual of a 2-form does not really depend on the metric but only on its conformal class. In the case of a zero-torsion metric preserving connexion, a new dependence on the metric appears inside F (which becomes the Riemann tensor), the argument above holds no more and the resulting Einstein equations are not conformally invariant.

There is another, independent argument favoring the equations above. The Poincaré group P acts on the tangent space at each point of space-time and the union of these spaces constitutes the associated tangent bundle. Each fiber, or each tangent space, is itself a Minkowski space. Now, it is a well known fact that P is a Inonü-Wigner contraction of the De Sitter (DS) group. Unlike P , DS is semisimple and it is not very difficult to build a gauge theory for it. In the associated bundle, to replace $\mathcal{P}$ by DS corresponds to replace each tangent Minkowski space by an osculating De Sitter space [13]. A De Sitter space is characterized by a parameter L, a length related to its (constant)curvature [14]. The contraction process corresponds to making L go to infinity. In the limit, DS becomes ("contracts to") $\mathcal{P}$ and the De Sitter space becomes a Minkowski space. The Lorentz group, a common subgroup of DS and $\mathcal{P}$, remains unscathed in the process, but the four remaining dimensionless parameters of DS get multiplied by L to become the translation parameters of $\mathcal{P}$ [15]. What happens to gauge potentials and fields under contraction has been analysed some time ago [16]. A first remarkable fact is that the Bianchi identities for the DS theory become, after contraction, just the geometrical identities (1.15 - 16) fixing the space-time purely geometrical features. This result suggests that, just as it gives the geometrical setting fixed a priori, a DS contracted theory would give the whole space-time-rooted theory we are looking for. A second remarkable fact is that the contraction process preserves the duality symmetry: the Yang-Mills equations for the De Sitter theory become, after contraction, just the equations (1.18 - 19) above. Maybe this argument is not quite definitive (it can be argued that contraction is not a very healthy process from the mathematical point of view) but it would be difficult to accept as the Poincaré gauge theory one which is not the contraction limit of a De Sitter gauge theory.

The DS - contraction procedure will be of great use in the following, as a guide through the intricacies of the Poincaré theory. It is very important to keep in mind that the results for the $\mathcal{P}$ theory are obtained only under the proviso that all calculations be done first in the DS theory, the contraction limit being taken only as the last step.

In Ref. [16], the behaviour of fields in the joint representation has been examined in some detail and the field equations above were obtained under the assumption that the translational gauge potentials are to be identified with the tetrad fields $h^{\alpha}{}_{\mu}$. There are many difficulties in this interpretation. First, the solder form being a canonical attribute of all frame bundles, their components $h^{\alpha}{}_{\mu}$ are in a sense given a priori[17] and do not participate in the description of any specific field. This is related to the fact that, in reality, the torsion T is an attribute of Γ and not of S . A second difficulty is of a more prosaic nature: the absence of a gauge interaction is characterized by the vanishing (up to a gauge transformation) of the potential field. The tetrad fields cannot vanish or, in other words, the absence of translational gravitation would give place to some "field of singularities". A third problem appears in the presence of source fields: the $h^{\alpha}{}_{\mu}$ will always couple to the kinetic energy term. If they are the fundamental fields, the source fields will have no pure kinetic energy and , so, no propagator. In section II, we shall see how these problems can be avoided. The solution is an old one [19] (the gauge potential is the non-trivial part of the $h^{\alpha}{}_{\mu}$), but in a different context and appears as a consequence of the DS- contraction procedure. In analysing this question, a trait peculiar to theories involving space-time itself, the presence of a " kinematic representation" of the Lie algebra to which all fields belong, will appear as being of fundamental importance. We shall briefly analyse the contraction procedure in this representation, so avoiding a repetition of what was done in Ref. 16. In section III, we prove a rather discouraging result: by using Vainberg's theorem [20,21] of functional calculus, which gives necessary and sufficient conditions for an equation to be derivable from an action principle, we show that there is really no Lagrangean from which (1.18 - 19) follow as the Euler-Lagrange equations. Ways to circunvent the difficulties coming from this fact are briefly discussed in the final section.

II. GENERAL DESCRIPTION OF THE MODEL

A special characteristic of gauge models involving spacetime symmetries is the presence of a "kinematic representation", whose generators are tangent fields. At each point of spacetime, one can choose coordinates $\{x^{\alpha}\}$ for the tangent space [22] and realize the P Lie algebra by the well known generators

$$L_{\alpha\beta} = -i\left(x_{\alpha}\partial_{\beta} - x_{\beta}\partial_{\alpha}\right) \tag{2.1}$$

$$t_{\alpha} = -i\,\partial_{\alpha} \quad . \tag{2.2}$$

The important point is that, as these operators act on source fields through their arguments, all fields will respond to their action. Spinor and vector fields will belong also to other representations and their total response to Lorentz transformations will be governed by

$$J_{\alpha\beta} = L_{\alpha\beta} + S_{\alpha\beta} \quad . \tag{2.3}$$

Scalar fields, however, will be singlets in any other representation, and their "kinematical" response is the only possible explanation for the universality of gravitation in a gauge picture. The presence of this representation is, consequently, a fundamental difference between the present case and that of other groups.

The transformations generated by (2.1,2) will change points in the fiber (i.e. in the tangent space). For an infinitesimal change with parameters ($\delta\omega^{\alpha\beta}$, δa^{α}).

$$\delta x^{\gamma} = \frac{i}{2}\,\delta\omega^{\alpha\beta} L_{\alpha\beta}\, x^{\gamma} + i\,\delta a^{\alpha} Z_{\alpha}\, x^{\gamma}$$

$$= -\,\delta\omega^{\gamma\alpha} x_{\alpha} + \delta a^{\gamma} . \tag{2.4}$$

At a fixed point [22], we shall write the corresponding change of a scalar source field as

$$\delta_0 \phi(x) = \left(\frac{i}{2}\,\delta\omega^{\alpha\beta} L_{\alpha\beta} + i\,\delta a^{\alpha} Z_{\alpha}\right)\phi(x) . \tag{2.5}$$

For the DS Lie algebra, the "kinematic"generators L_{ab} (with a,b= 1,...,5) are [15]:

$$L_{\alpha\beta} = -i\left(x_{\alpha}\partial_{\beta} - x_{\beta}\partial_{\alpha}\right)$$

$$L_{\alpha 5} = -i\,L\left(1 + \frac{x^{2}}{4L^{2}}\right)\partial_{\alpha} + \frac{x^{\beta}}{2L} L_{\alpha\beta} \ , \tag{2.6}$$

where L is the DS length parameter and $x^2 = x_{\alpha} x^{\alpha}$. The 10 group parameters can be grouped as $\{\omega^{ab} = -\omega^{ba}\}$. They are, of course, dimensionless. The contraction is obtained by redefining

$$\delta a_{\alpha} = L\,\delta\omega_{\alpha 5} \tag{2.7}$$

and proceeding to the limit $L \to \infty$. The infinitesimal change under a DS transformation,

$$\delta_0 \phi(x) = \frac{i}{2}\,\delta\omega^{ab} L_{ab}\, \phi(x) \ , \tag{2.8}$$

becomes just (2.5). In this process, generators and parameters change their dimensionalities. The gauge potentials $\Gamma^{ab}{}_{\mu}$ for the DS field will, under contraction, behave in a way analogous to the group parameters [16]: defining

$$\Gamma^{\alpha\beta}{}_{\mu} = A^{\alpha\beta}{}_{\mu} \tag{2.9}$$

and

$$\Gamma^{\alpha 5}{}_{\mu} = L^{-1} B^{\alpha}{}_{\mu} \quad , \tag{2.10}$$

the $A^{\alpha\beta}{}_{\mu}$ and $B^{\alpha}{}_{\mu}$ will, after the limit is taken, appear as the P gauge potentials. The covariant derivative for the DS case is

$$D_{\mu} = \partial_{\mu} + \frac{1}{2} \Gamma^{ab}{}_{\mu} \frac{\delta_{o}}{\delta \omega^{ab}} \ . \tag{2.11}$$

For a scalar field,

$$D_{\mu}\phi = \left(\partial_{\mu} + \frac{i}{2} \Gamma^{\alpha\beta}{}_{\mu} L_{\alpha\beta} + i \Gamma^{\alpha 5}{}_{\mu} L_{\alpha 5}\right)\phi$$

turns into

$$D_{\mu}\phi = \left[\partial_{\mu} - \left(A^{\alpha\beta}{}_{\mu} x_{\beta} - B^{\alpha}{}_{\mu}\right)\partial_{\alpha}\right]\phi \ . \tag{2.12}$$

Notice that we have been using $\{x^{\alpha}\}$ (with the first greek letters) as coordinates in tangent space, and $\{x^{\mu}\}$ (with the second half of the greek alphabet) as coordinates on spacetime. They can be made to coincide but it is more convenient to keep them apart by now. The covariant derivative (2.12) can be written as

$$D_{\mu}\phi = \lambda^{\alpha}{}_{\mu} \partial_{\alpha}\phi \quad , \tag{2.13}$$

where

$$\lambda^{\alpha}{}_{\mu} = \partial_{\mu} x^{\alpha} + B^{\alpha}{}_{\mu} - A^{\alpha\beta}{}_{\mu} x_{\beta} \tag{2.14}$$

can be regarded as a fourleg field. This expression comes out naturally from (2.11) under contraction, using (2.10). In Ref. 16, $\Gamma^{\alpha 5}{}_{\mu}$ had been related to the $\lambda^{\alpha}{}_{\mu}$ which is far less natural. Now, $\lambda^{\alpha}{}_{\mu}$ is the trivial tetrad when the gauge potentials vanish.

For fields belonging also to some other representation, $\mathcal{L}_{\alpha\beta}$ instead of $L_{\alpha\beta}$ has to be used. Let us examine the case of spinor fields. The DS Lie algebra has a beautiful representation in terms of the γ matrices: the generators are

$$\sigma_{ab} = -\frac{i}{2}\left[\gamma_a , \gamma_b\right] \quad ; \tag{2.15}$$

we see that γ_5 acquires the same status of the other γ's .The complete generators are now

$$\mathcal{L}_{ab} = L_{ab} + \frac{\sigma_{ab}}{2} \quad . \tag{2.16}$$

The covariant derivative (2.11) gets extra terms; the part in $\Gamma^{\alpha 5}{}_{\mu} \frac{\sigma_{\alpha 5}}{2} \sim L^{-1} B^{\alpha}{}_{\mu} \frac{\sigma_{\alpha 5}}{2}$ vanishes on contraction and the resulting Poincaré covariant derivative is

$$D_\mu \Psi = \left(h^{\alpha}{}_{\mu} \partial_\alpha + \frac{i}{4} A^{\alpha\beta}{}_{\mu} \sigma_{\alpha\beta} \right) \Psi \quad . \tag{2.17}$$

Notice that, while the factors L and L^{-1} in (2.6) and (2.10) compensate each other, allowing the effect of the translational sector to remain in the kinematical representation, no such effect survives contraction in the spinor representation. A current related to $\sigma_{\alpha 5}$ would possibly lead to anomalies in the quantum case.

The behavior of the gauge potentials under transformations is obtained from that in the DS theory. There,

$$\Gamma'_\mu = U \Gamma_\mu U^{-1} - i U \partial_\mu U^{-1} \quad , \tag{2.18}$$

with

$$U = \exp\left[\frac{i}{2} \omega^{ab} \mathcal{L}_{ab}\right] \quad . \tag{2.19}$$

Of course, the DS group being non-compact, this will not be unitary for the finite representations we are using.

The double index notation allows the use of 5x5 matrices in the adjoint representation and leads to a direct contact with the usual notation (as in (2.30) below). Using

$$(\tau_{ab})_{cd}{}^{ef} = i\, f_{ab,\,cd}{}^{ef}$$

for the matrix elements, a straightforward calculation shows that (2.18) can be written as

$$\Gamma'^{cd}{}_{\mu} = (\Lambda^{-1})^{c}{}_{a}\, \Gamma^{ab}{}_{\mu}\, \Lambda_{b}{}^{d} - (\Lambda^{-1})^{c}{}_{a}\, \partial_{\mu} \Lambda^{ad} \qquad (2.20)$$

where $\Lambda_{b}{}^{d} = (\exp \omega)_{b}{}^{d} = \delta_{b}^{d} + \omega_{b}{}^{d} + \frac{1}{2!}\omega_{b}{}^{c}\omega_{c}{}^{d} + \ldots$, the indices being raised or lowered by the DS metric. In the same line, the field strenght covariance

$$F'_{\mu\nu} = U\, F_{\mu\nu}\, U^{-1}$$

can be put in the form

$$F'^{cd}{}_{\mu\nu} = (\Lambda^{-1})^{c}{}_{a}\, F^{ab}{}_{\mu\nu}\, \Lambda_{b}{}^{d} \; . \qquad (2.21)$$

The field strengths are, in detail,

$$F^{cd}{}_{\mu\nu} = \partial_{\mu}\Gamma^{cd}{}_{\nu} - \partial_{\nu}\Gamma^{cd}{}_{\mu} - \Gamma^{c}{}_{e\mu}\Gamma^{ed}{}_{\nu} + \Gamma^{c}{}_{e\nu}\Gamma^{ed}{}_{\mu} \, . \qquad (2.22)$$

The contraction is done by identifying $\Gamma^{\alpha 5}{}_{\mu} = -\Gamma^{5\alpha}{}_{\mu} = L^{-1} B^{\alpha}{}_{\mu}$;

$\Gamma^{\alpha\beta}{}_{\mu} = A^{\alpha\beta}{}_{\mu}$; $F^{\alpha 5}{}_{\mu\nu} = L^{-1}\, \tau^{\alpha}{}_{\mu\nu}$; after the limit is taken, it remains for the Lorentz sector

$$F^{\alpha\beta}{}_{\mu\nu} = \partial_{\mu} A^{\alpha\beta}{}_{\nu} - \partial_{\nu} A^{\alpha\beta}{}_{\mu} - A^{\alpha}{}_{\gamma\mu} A^{\gamma\beta}{}_{\nu} + A^{\alpha}{}_{\gamma\nu} A^{\gamma\beta}{}_{\mu} \qquad (2.23)$$

(a term $L^{-2}(B^{\alpha}{}_{\mu} B^{\beta}{}_{\nu} - B^{\alpha}{}_{\nu} B^{\beta}{}_{\mu})$ disappears) and, for the translation sector,

$$\tau^{\alpha}{}_{\mu\nu} = \partial_{\mu} B^{\alpha}{}_{\nu} - \partial_{\nu} B^{\alpha}{}_{\mu} - A^{\alpha}{}_{\beta\mu} B^{\beta}{}_{\nu} + A^{\alpha}{}_{\beta\nu} B^{\beta}{}_{\mu} \qquad (2.24)$$

This is just the covariant derivative of $B^{\alpha}{}_{\nu}$, as defined by the connexion A. The same procedure, applied to (2.20) and (2.21), leads to the gauge tranformations for the P theory:

$$A'^{\alpha\beta}{}_{\mu} = (\Lambda^{-1})^{\alpha}{}_{\gamma} A^{\gamma\delta}{}_{\mu} \Lambda_{\delta}{}^{\beta} - (\Lambda^{-1})^{\alpha}{}_{\gamma} \partial_{\mu} \Lambda^{\gamma\beta} \tag{2.25}$$

$$B'^{\alpha}{}_{\mu} = (\Lambda^{-1})^{\alpha}{}_{\gamma} A^{\gamma\delta}{}_{\mu} a_{\delta} + (\Lambda^{-1})^{\alpha}{}_{\gamma} B^{\gamma}{}_{\mu} - (\Lambda^{-1})^{\alpha}{}_{\gamma} \partial_{\mu} a^{\gamma} \tag{2.26}$$

$$F'^{\alpha\beta}{}_{\mu\nu} = (\Lambda^{-1})^{\alpha}{}_{\gamma} F^{\gamma\delta}{}_{\mu\nu} \Lambda_{\delta}{}^{\beta} \tag{2.27}$$

$$\mathcal{T}'^{\alpha}{}_{\mu\nu} = (\Lambda^{-1})^{\alpha}{}_{\gamma} F^{\gamma\beta}{}_{\mu\nu} a_{\beta} + (\Lambda^{-1})^{\alpha}{}_{\gamma} \mathcal{T}^{\gamma}{}_{\mu\nu} . \tag{2.28}$$

Here a note of caution: one might wonder about the inversion of the roles of matrices Λ and Λ^{-1}. The reason for it is that we have been considering affine frame transformations in the tangent spaces, given by

$$e'_{\alpha} = e_{\beta} \Lambda^{\beta}{}_{\alpha} - a_{\alpha} , \tag{2.29}$$

with the matrices acting on the right. This convention, borrowed from the mathematical literature[5], corresponds to the coordinate transformations

$$x'^{\alpha} = (\Lambda^{-1})^{\alpha}{}_{\beta} (x^{\beta} + a^{\beta}) , \tag{2.30}$$

of which (2.4) is the small-parameter version. The above rules for $B'^{\alpha}{}_{\mu}$ and $\mathcal{T}'^{\alpha}{}_{\mu\nu}$ come from (2.20 - 21) by using

$$a^{\alpha} = L \Lambda^{\alpha 5} ,$$

of which (2.7) is the infinitesimal case. The above convention of taking the product of Lorentz transformations and translations instead of the exponential (2.19) makes no difference for the above rules but should be taken into account when comsidering the change of a source field as, for example, the finite version of (2.5).

The torsion $T^{\alpha}{}_{\mu\nu}$ will be the covariant derivative of the tetrad field (the same as (2.24) with $\lambda^{\alpha}{}_{\mu}$ instead of $B^{\alpha}{}_{\mu}$). Using (2.14) and collecting the terms conveniently, we find that

$$T^{\alpha}{}_{\mu\nu} = \tau^{\alpha}{}_{\mu\nu} - F^{\alpha\beta}{}_{\mu\nu}\, x_{\beta} \; . \qquad (2.31)$$

The sourceless field equations for the D S case are

$$\partial_{\mu} F^{\alpha\beta\mu\nu} - \Gamma^{\alpha}{}_{\gamma\mu} F^{\gamma\beta\mu\nu} + F^{\alpha}{}_{\gamma}{}^{\mu\nu} \Gamma^{\gamma\beta}{}_{\mu} + $$
$$+ \Gamma^{\alpha}{}_{5\mu} F^{5\beta\mu\nu} + F^{\alpha}{}_{5}{}^{\mu\nu} \Gamma^{5\beta}{}_{\mu} \qquad (2.32)$$

and

$$\partial_{\mu} F^{\alpha 5\mu\nu} - \Gamma^{\alpha}{}_{\gamma\mu} F^{\gamma 5\mu\nu} + F^{\alpha}{}_{\gamma}{}^{\mu\nu} \Gamma^{\gamma 5}{}_{\mu} = 0 \; . \qquad (2.33)$$

Redefining the fields prior to contraction, we see that the last two terms in (2.32) acquire factors L^{-2} and vanish in the limit. The resulting equation is

$$\partial_{\mu} F^{\alpha\beta\mu\nu} - A^{\alpha}{}_{\gamma\mu} F^{\gamma\beta\mu\nu} + F^{\alpha}{}_{\gamma}{}^{\mu\nu} A^{\gamma\beta}{}_{\mu} = 0 \; . \qquad (2.34)$$

The disappearance of these terms will be responsible for the non-Lagrangean character of the P field equations, as will be seen in the next section. From (2.33), we get

$$\partial_{\mu} \tau^{\alpha\mu\nu} - A^{\alpha}{}_{\gamma\mu} \tau^{\gamma\mu\nu} + F^{\alpha}{}_{\gamma}{}^{\mu\nu} B^{\gamma}{}_{\mu} = 0 \; . \qquad (2.35)$$

A direct computation shows that this set of equations is covariant under the transformations (2.25 - 28)

Equation (2.34) is just (1.18) in components. As to (2.35), it has the same form as (1.19) when written in components, the difference being of course that $\mathcal{T}$ is not the torsion, but simply the covariant derivative of the field B. Had we identified $\Gamma^{\alpha 5}{}_{\mu} = L^{-1} \lambda^{\alpha}{}_{\mu}$, just (1.19) would have resulted. Now comes a surprising result: if we take (2.31) into (2.35), we find that

$$\partial_{\mu} T^{\alpha\mu\nu} - A^{\alpha}{}_{\beta\mu} T^{\beta\mu\nu} + F^{\alpha}{}_{\beta}{}^{\mu\nu} \lambda^{\beta}{}_{\mu} = 0 \quad . \tag{2.36}$$

This is just (1.19) in components and shows that the geometry-dynamics symmetry used as a guiding principle in the previous section is preserved in this formulation.

The behavior of the tetrad (2.14) under P transformations is obtained by using (2.25, 26 and 30). One finds that

$$\lambda'^{\alpha}{}_{\mu} = \partial_{\mu} x'^{\alpha} + B'^{\alpha}{}_{\mu} - A'^{\alpha\beta}{}_{\mu} x'_{\beta} = (\Lambda^{-1})^{\alpha}{}_{\beta} \lambda^{\beta}{}_{\mu} , \tag{2.37}$$

an interesting result: the tetrad field ignores translations, behaving (as it should) as a Lorentz vector field. If we use all the above transformation properties in the relation(2.31) we find also that, under a P transformation,

$$T'^{\alpha}{}_{\mu\nu} = (\Lambda^{-1})^{\alpha}{}_{\beta} T^{\beta}{}_{\mu\nu} \, . \tag{2.38}$$

Looking at the equations and transformation properties for the components in the Lorentz sector, we see that it constitutes a subtheory: all the expressions are those we would find in a gauge theory for the Lorentz group. This is not the case for the translation sector, which clearly is not a subtheory and exhibits rather awkward transformation properties. However, if we look more closely into (2.26 and 28), we find that, for pure Lorentz transformations ($a^{\alpha}=0$), both B^{α}_{μ} and $\mathcal{T}^{\alpha}_{\mu\nu}$ behave as vectors in the algebra indices. The awkwardness comes from the coupling between translations and Lorentz transformations and is just what is necessary to endow those quantities possessing clear geometrical meanings, such as λ^{α}_{μ} and $T^{\alpha}_{\mu\nu}$, with a simple behaviour. The set ($F^{\alpha\beta}_{\mu\nu}$, $\mathcal{T}^{\alpha}_{\mu\nu}$) can be taken as the field strength despite the strange behaviour of $\mathcal{T}^{\alpha}_{\mu\nu}$. In particular, it allows a good, invariant characterization of the vacuum of the model as $F^{\alpha\beta}_{\mu\nu}=0$, $\mathcal{T}^{\alpha}_{\mu\nu}=0$,corresponding to gauge transformations of zero potentials in (2.25 - 26).

Usual gauge potentials have the dimension of mass and field strengths of (mass) ^{2} (in units $\hbar=c=1$). The tetrad fields are dimensionless and, because of the redefinition of fields, B^{α}_{μ} and $\mathcal{T}^{\alpha}_{\mu\nu}$ have dimensions zero and one. If we want to get back the normal dimensions, we must add a length factor ℓ to each B^{α}_{μ} (equivalent to a redefinition $\Gamma^{\alpha 5}_{\mu}=\ell L^{-1} B^{\alpha}_{\mu}$ instead of that previously adopted). Such a problem was to be expected because translations, unlike other transformations, have dimensional parameters. All current densities have dimension 3, except the Noether current associated to translations : the energy-momentum density has dimension 4 and any theory using it as a source current will have to cope with this fact. We shall here prefer to keep B^{α}_{μ} dimensionless and adopt the (equivalent) rule of adjusting the source terms with ℓ factors whenever necessary.

Taking the covariant derivative (2.17) into the usual free Lagrangean for the spinor field (by the minimal coupling prescription), it is easy to check that the variations with respect to B^{α}_{μ} and $A^{\alpha\beta}_{\mu}$ lead to the energy-momentum tensor density $\theta^{\alpha\nu}$ and the total angular momentum density $M^{\alpha\beta\nu}$. The form of (2.14) is enough to ensure the usual relationship between the energy-momentum and the orbital angular momentum , both currents representing the responses of source fields to transformations in the kinematic representation. The equations (2.34 - 35 and 36) have as sources, respectively, $M^{\alpha\beta\nu}$, $\lambda^2\theta^{\alpha\nu}$ and $(\lambda^2\theta^{\alpha\nu} - M^{\alpha\beta\nu}x_\beta)$.

The equations remain covariant under (2.25 - 28), but the coupling between translations and angular momentum imposes on $\theta^{\alpha\nu}$ a peculiar transformation law: for a tranformation corresponding to (2.30),

$$\lambda^2\theta'^{\alpha\nu} = (\Lambda^{-1})^\alpha{}_\beta\left[M^{\beta\gamma\nu}a_\gamma + \lambda^2\theta^{\beta\nu}\right] . \tag{2.39}$$

By taking derivatives of the field equations and combining conveniently the terms, we arrive at the invariant conservation laws

$$\partial_\mu M^{\alpha\beta\mu\nu} - A^\alpha{}_{\gamma\mu}M^{\gamma\beta\mu} + M^\alpha{}_\gamma{}^\mu A^{\gamma\beta}{}_\mu = 0 ; \tag{2.40}$$

$$\partial_\mu(\lambda^2\theta^{\alpha\mu}) - A^\alpha{}_{\beta\mu}(\lambda^2\theta^{\beta\mu}) + J^\alpha{}_\gamma{}^\mu B^\gamma{}_\mu = 0 . \tag{2.41}$$

The angular momentum $M^{\alpha\beta\nu} = \mathcal{L}^{\alpha\beta\nu} + S^{\alpha\beta\nu}$ contains the orbital part $\mathcal{L}^{\alpha\beta\nu}$, which is inconvenient for a field theory. The coordinate x_β appears explicitly in both the field equations and the Lagrangean (notice that the source Lagrangeans are always well defined and there is no problem to obtain the currents, even after contraction). We can follow here the usual procedure [19] to get around this problem: it is enough to use, as the point-dependent Poincaré parameters, the set $\delta\omega^{\alpha\beta}$ and δx^α given by (2.4), instead of the ten original parameters $\delta\omega^{\alpha\beta}$ and δa^α. This stratagem is used without much ado by most authors but it has some consequences deserving discussion even at the price of repeating some apparently trivial things.

Of course, the new parameters are to be considered as functionally independent so that now

$$\frac{\delta_0\Psi}{\delta x^\alpha} = \frac{\delta_0\Psi}{\delta a^\alpha}$$

from (2.4). As

$$\delta_0\Psi = \frac{i}{2}\delta\omega^{\alpha\beta}\,\frac{\sigma_{\alpha\beta}}{2} + \delta x^\alpha\,\partial_\alpha\Psi \quad ,$$

the covariant derivative

$$D_\mu\Psi = \partial_\mu\Psi + \frac{1}{2}A^{\alpha\beta}{}_\mu\,\frac{\delta_0\Psi}{\delta\omega^{\alpha\beta}} + B^\alpha{}_\mu\,\frac{\delta_0\Psi}{\delta a^\alpha}$$

becomes

$$\tilde{D}_\mu\Psi = \left(\partial_\mu x^\alpha + B^\alpha_\mu\right)\partial_\alpha\Psi + \frac{i}{2}A^{\alpha\beta}{}_\mu\,\sigma_{\alpha\beta}\,\Psi \,, \tag{2.42}$$

with a new tetrad field

$$\tilde{\lambda}^\alpha{}_\mu = \partial_\mu x^\alpha + B^\alpha{}_\mu \ . \tag{2.43}$$

For a scalar field, of course, the last term in (2.42)is absent. With the transformations described in terms of the new parameters, the fields B^α_μ will exhibit a behavior different from that given by (2.26). The simplest way to find the new rule is to notice that, if (2.42) is to be covariant, B^α_μ must behave now in the same way the expression $B^\alpha_\mu - A^{\alpha\beta}{}_\mu x_\beta$ behaved in terms of the old parameters. For the infinitesimal case,

$$B'^\alpha{}_\mu = B^\alpha{}_\mu - \delta\omega^\alpha{}_\gamma\,\lambda^\gamma{}_\mu - \partial_\mu\,\delta x^\alpha \ . \tag{2.44}$$

In this parametrization, a pure translation $(\delta\omega^\alpha_\gamma = 0)$ changes $B^\alpha{}_\mu$ in a simpler way:

$$B'^\alpha{}_\mu = B^\alpha{}_\mu - \partial_\mu\,\delta x^\alpha \,, \tag{2.45}$$

whose finite version is

$$B'^\alpha{}_\mu = B^\alpha{}_\mu - \partial_\mu(x'^\alpha - x^\alpha) \ . \tag{2.46}$$

A direct calculation shows that $\tilde{\lambda}^{\alpha}{}_{\mu}$ keeps its behaviour (2.37) and that (2.38) still holds for $\tilde{T}^{\alpha}{}_{\mu\nu}$. However, with (2.43) the relation between the torsion and the field strength $\mathcal{T}$ becomes

$$\tilde{T}^{\alpha}{}_{\mu\nu} = t^{\alpha}{}_{\mu\nu} + \mathcal{T}^{\alpha}{}_{\mu\nu} \qquad (2.47)$$

where

$$t^{\alpha}{}_{\mu\nu} = A^{\alpha}{}_{\beta\nu}\,\partial_{\mu}x^{\beta} - A^{\alpha}{}_{\beta\mu}\,\partial_{\nu}x^{\beta} \qquad (2.48)$$

is a contribution to torsion coming from the Lorentz sector. If now we identify the coordinate systems, so that $\partial_{\mu}x^{\alpha} = \delta^{\alpha}{}_{\mu}$, we see that $t^{\alpha}{}_{\mu\nu}$ measures the asymmetry of the connexion A or, in other words, its non-inertial character. Formally, $t^{\alpha}{}_{\mu\nu}$ is the covariant derivative of the trivial frame $\partial_{\mu}x^{\alpha}$ in the connexion A. One would expect a non-inertial effect in the presence of an angular momentum field density, but the gauge non-linearity may create it even in the absence of sources in eq. (2.34). Another effect of the reparametrization is to hide the duality symmetry for the torsion:eq. (2.36) is no more valid when $T = t + \mathcal{T}$. Notice, however, that the reparametrization, which is essential for a future quantization, keeps $B^{\alpha}{}_{\mu}$ as the fundamental field with the same relation to $\mathcal{T}^{\alpha}{}_{\mu\nu}$ and furthermore preserves the duality symmetry for the dynamical equations. All explicit dependence on the coordinates disappears. The sources in (2.34 - 35) become, respectively, $S^{\alpha\beta\nu}$ and $\lambda^{2}\,\tilde{\theta}^{\alpha\nu}$, where $\tilde{\theta}^{\alpha\nu}$ is the new energy-momentum obtained when the new covariant derivatives are used in the source Lagrangeans.

The reparametrization brings forth a problem in the characterization of the vacuum. Before the change of parameters, the vaccum is given by a gauge transformation of vanishing fields, $\overset{\circ}{B}{}^{\alpha}{}_{\mu} = -(\Lambda^{-1})^{\alpha}{}_{\gamma}\,\partial_{\mu}a^{\gamma}$ or, for infinitesimal transformations, $\overset{\circ}{B}{}^{\alpha}{}_{\mu} = -\partial_{\mu}\delta a^{\alpha}$. This should not change by a reparametrization, but (2.44) wich tell us that the gauge transformation of $B^{\alpha}{}_{\mu} = 0$ is now $\overset{\circ}{B}{}^{\alpha}{}_{\mu} = -\left[\delta\omega^{\alpha}{}_{\gamma}\,\partial_{\mu} x^{\gamma} + \partial_{\mu}\delta x^{\alpha}\right]$ which gives $\overset{\circ}{G}{}^{\alpha}{}_{\mu\nu} \neq 0$. In reality, let us recall that, to obtain (2.44), we used the fact that $B^{\alpha}{}_{\mu}$ should have, in terms of the new parameters, the same transformation properties of $(B^{\alpha}{}_{\mu} - A^{\alpha\beta}{}_{\mu}\,x_{\beta})$ in terms of the old. This is not to say that $B^{\alpha}{}_{\mu}$ has been changed to absorb the term $A^{\alpha\beta}{}_{\mu}\,x_{\beta}$, it is simply a way to fix its transformation properties. If we want to recover the vacuum via the transformation rules, we have to add to it the piece we had extracted, $\overset{\circ}{A}{}^{\alpha\gamma}{}_{\mu}\,x_{\gamma} = -(\partial_{\mu}\delta\omega^{\alpha\gamma})x_{\gamma}$. Once this is done, we obtain the same vacuum as before(although written in terms of the new parameters). An interesting consequence of the change of parameters is that the tetrad of the vacuum fields becomes integrable: the absence of gravitational field is signalled by its holonomy.

As a final remark, it is important to notice that eq. (2.35) comes from (2.33) because, in the field redefinitions prior to contraction, every term gains a common factor L^{-1} In fact, as $\frac{\delta\mathcal{L}}{\delta\Gamma^{\alpha 5}{}_{\mu}} = L\,\frac{\delta\mathcal{L}}{\delta B^{\alpha}{}_{\mu}}$, there is always a factor between DS and P currents. The same factor appears in those conjugate momenta which are well-defined,

$$\Pi^{ab}{}_{j} = \frac{\partial\mathcal{L}}{\partial\partial_0\Gamma^{ab}{}_{j}} \qquad (j = 1,2,3), \tag{2.49}$$

so that the momenta conjugate to $B^{\alpha}{}_{j}$ are $\Pi^{\alpha}{}_{j} = \frac{\partial\mathcal{L}}{\partial\partial_0 B^{\alpha}{}_{j}} = L\,\Pi^{\alpha 5}{}_{j}$. This means that, even if the Lagrangean is not defined, the same Poisson brackets valid for the DS model hold for the Poincaré case, as

$$\left\{\Pi^{\alpha}{}_{i}\,,\,B^{\beta}{}_{j}\right\} = \left\{\Pi^{\alpha 5}{}_{i}\,,\,\Gamma^{\beta 5}{}_{j}\right\} \tag{2.50}$$

In this sense, the road remains open to canonical quantization.

III. NON LAGRANGEAN CHARACTER

In the previous sections we have seen how to put together two well accepted preconceptions: that gravitation stems from spacetime itself, being closely related to the Poincaré group; and that interactions in general are mediated by gauge fields. The approach given above seems to be, of all the possibities allowed by the different degrees of proximity to one or another, the closest possible realization of both. We shall now see that the resulting model is in contradiction with another well accepted idea. Despite the fact that some important equations do not come from a Lagrangean (Navier-Stokes [26], Burger , Korteweg - de Vries [21]), there is a widespread belief that the fundamental equations of Physics should be related to an extremal principle [23]. As a consequence of Feynman's picture of Quantum Mechanics, it has even become a matter of common acceptance that the Action is, in some sense, more "fundamental" than the equations of motion, not the least because it takes into account the global characteristics of the system. There are difficulties in this point of view [24], but we shall not discuss this subject. The model above is the (contraction) limit of a nice Lagrangean DS theory. We shall show that, once the limit is taken, it is no more a Lagrangean theory. More precisely, it will be shown that no action function exists from which the Yang-Mills equations (2. 34 - 35) can be derived as the Euler-Lagrange equations. As we have seen that (2.34) are the Yang-Mills equations for a (Lagrangean) Lorentz subtheory, the trouble will really come from (2.35) In the analyst jargon, the operator acting on the fields $B^{\alpha}{}_{\mu}$ is not a potential operator. Without any pretense to real mathematical rigor, we shall simply state the fundamental Vainberg's theorem involved [20], suitably adapted to the language of field theory and show how it works for gauge theories. In particular, it will become evident that a De Sitter model does satisfy the requirements for a Lagrangean theory. These requirements, however, fail to be observed by (2.34 - 35). To find that a field equation complies to the Lagrangean conditions is in general an easy task. To be sure that it does not is often very difficult. In our case, we shall be able, first, to suspect that the conditions are violated and then, to show indeed that they are ruined by the contraction process.

Suppose that we have an equation

$$\mathcal{D}\varphi(x) = 0\ , \tag{3.1}$$

where $\mathcal{D}$ is a differential operator and $\varphi(x)$ a field belonging to some functional space. The Frĕchet derivative of $\mathcal{D}$ along some field $\eta(x)$ at the point $\varphi(x)$ of the functional space can be calculated by

$$\mathcal{D}'_\varphi\eta = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\mathcal{D}(\varphi+\varepsilon\eta)-\mathcal{D}(\varphi)\right] = \left[\frac{d}{d\varepsilon}\mathcal{D}(\varphi+\varepsilon\varphi)\right]_{\varepsilon=0} \tag{3.2}$$

and is itself a linear operator acting on $\eta(x)$. Given this operator $\mathcal{D}'_\varphi$, its adjoint is the operator $\tilde{\mathcal{D}}'_\varphi$ such that

$$\int d^4x\ \lambda(x)\ \mathcal{D}'_\varphi\eta(x) = \int d^4x\ \eta(x)\ \tilde{\mathcal{D}}'_\varphi\eta(x) \tag{3.3}$$

for any two fields $\lambda(x)$, $\eta(x)$.

Vainberg's theorem says that[21] the necessary and sufficient condition for (3.1) to come by variation from some action functional is that

$$\mathcal{D}'_\varphi = \tilde{\mathcal{D}}'_\varphi \tag{3.4}$$

in a ball around φ. Such a selfadjointness, taken in (3.3), corresponds to a symmetry of the Frĕchet derivative along any two directions $\lambda(x)$ and $\eta(x)$ around $\varphi(x)$,

$$\int d^4x\ \lambda(x)\ \mathcal{D}'_\varphi\eta(x) = \int d^4x\ \eta(x)\ \mathcal{D}'_\varphi\lambda(x) \tag{3.5}$$

and is reminiscent of the integrability condition of calculus[25].

Once this symmetry condition is fulfilled, the action functional can be obtained as

$$S[\Psi] = \int d^4x \; \Psi(x) \int_0^1 d\alpha \; \mathcal{D}(\alpha \Psi(x)) \; . \qquad (3.6)$$

It is an easy exercice to check the statements above for the simplest cases in field theory: for linear equations, they are rather trivial. For a sourceless gauge field, the Yang-Mills equations state the vanishing of

$$\mathcal{D}A^{a\nu} = (\delta^a_c \partial_\mu + f^a{}_{bc} A^b{}_\mu)(\partial^\mu A^{c\nu} - \partial^\nu A^{c\mu} + f^c{}_{de} A^{d\mu} A^{e\nu}) \; . \qquad (3.7)$$

The Fréchet derivative of $\mathcal{D}$ is

$$[\mathcal{D}'_A \Gamma]^{a\nu} = \left[\frac{d}{d\varepsilon} \mathcal{D}(A^{a\nu} + \varepsilon \Gamma^{a\nu})\right]_{\varepsilon=0}$$

$$= [DD\Gamma]^{a\nu} + f^a{}_{bc} \Gamma^b{}_\mu F^{c\mu\nu} \; , \qquad (3.8)$$

where D is the covariant derivative fitted to each case:

$$(D\Gamma)^{a\mu\nu} = \partial^\mu \Gamma^{a\nu} - \partial^\nu \Gamma^{a\mu} + f^a{}_{bc}(A^{b\mu} \Gamma^{c\nu} - A^{b\nu} \Gamma^{c\mu}) \qquad (3.9)$$

$$(D\Psi)^{a\nu} = (\delta^a_c \partial_\mu + f^a{}_{bc} A^b{}_\mu) \Psi^{c\mu\nu} \qquad (3.10)$$

for a field $\Psi^{c\mu\nu}$ $(= -\Psi^{c\nu\mu})$ in the adjoint representation. Now, for any such Ψ and any $\varphi_{a\nu}$,

$$\int d^4x \; \varphi_{a\nu} [D\Psi]^{a\nu} = -\frac{1}{2} \int d^4x \; \Psi^{a\mu\nu} [D\varphi]_{a\mu\nu} \; . \qquad (3.11)$$

This can be found by using (3.10), performing an integration by parts and antisymmetrizing to obtain the covariant derivative $D\varphi$, which has the form (3.9). It follows that

$$\int d^4x \; \varphi_{a\nu} [D(D\Gamma)]^{a\nu} = -\frac{1}{2} \int d^4x \, (D\Gamma)^{a\mu\nu} (D\varphi)_{a\mu\nu} \; . \qquad (3.12)$$

This would be enough to show the symmetry of the first term in (3.8) but we can go a step further. We reverse the roles by putting $\Psi_{a\mu\nu} = [D\Psi]_{a\mu\nu}$ and using again (3.11), arriving at

$$\int d^4x\ \varphi_{a\nu}[D(D\Gamma)]^{a\nu} = \int d^4x\ \Gamma_{a\nu}[D(D\varphi)]^{a\nu}. \qquad (3.13)$$

The condition for the existence of a Lagrangean for the equation $\mathcal{D}A^{a\nu} = 0$ is

$$\int d^4x\ \varphi_{a\nu}\left\{[DD\Gamma]^{a\nu} + f^a{}_{bc}\,\Gamma^b{}_\mu\, F^{c\mu\nu}\right\} =$$

$$= \int d^4x\ \Gamma_{a\nu}\left\{[DD\varphi]^{a\nu} + f^a{}_{bc}\,\varphi^b{}_\mu\, F^{c\mu\nu}\right\} . \qquad (3.14)$$

That the first terms on each side are equal is guaranteed by (3.13). For the remaining terms, it is enough to exchange the indices and use the cyclic property of the structure constants to show that

$$\int d^4x\ \varphi_{a\nu} f^a{}_{bc}\,\Gamma^b{}_\mu\, F^{c\mu\nu} = \int d^4x\ \Gamma_{a\nu} f^a{}_{bc}\,\varphi^b{}_\mu\, F^{c\mu\nu}. \qquad (3.15)$$

So, the requirements are more than satisfied, as the symmetry conditions (3.13) and (3.15) hold separately. Notice that, in gauge theories, it is the summation on the components that makes the symmetrization possible. The Yang - Mills Lagrangean is obtained from (3.6) in the form

$$\mathcal{L} = \frac{1}{2} A_{a\nu}\left(\delta^a{}_c\,\partial_\mu + f^a{}_{bc}\, A^b{}_\mu\right) F^{c\mu\nu} \qquad (3.16)$$

Let us now consider the Yang-Mills equations for the P group, (2.34 - 35). Applied to (2.34) alone, the above treatment would lead to the existence of a good Lagrangean like (3.16) , still a manifestation of the fact that the Lorentz sector constitutes a gauge theory by itself. The problem concerns the whole set of equations. Consider the Fréchet derivative of the differential operator in (2.35) along $\Gamma = (\Gamma^{\alpha}{}_{\beta\mu},\ \eta^{\gamma}{}_{\nu} = L\,\Gamma^{\gamma 5}{}_{\nu})$ at the point $(A^{\alpha}{}_{\beta\mu},\ B^{\gamma}{}_{\nu})$ in the functional space:

$$\mathcal{O}'^{\alpha\nu}_{A,B}[\Gamma,\eta] = -\Gamma^{\alpha}{}_{\epsilon\mu}\,\mathcal{T}^{\epsilon\mu\nu} + \left(\delta^{\alpha}_{\epsilon}\partial_{\mu} - A^{\alpha}{}_{\epsilon\mu}\right)\Big[(D\eta)^{\epsilon\mu\nu} + -\left(\Gamma^{\epsilon}{}_{\gamma}{}^{\mu}B^{\gamma\nu} - \Gamma^{\epsilon}{}_{\gamma}{}^{\nu}B^{\gamma\mu}\right)\Big] + \Big[(D\Gamma)^{\alpha}{}_{\epsilon}{}^{\mu\nu} - \left(\Gamma^{\alpha}{}_{\gamma}{}^{\mu}A^{\gamma}{}_{\epsilon}{}^{\nu} + -\Gamma^{\alpha}{}_{\gamma}{}^{\nu}A^{\gamma}{}_{\epsilon}{}^{\mu}\right)\Big]B^{\epsilon}{}_{\mu} + F^{\alpha}{}_{\epsilon}{}^{\mu\nu}\,\eta^{\epsilon}{}_{\mu} \,. \tag{3.17}$$

For (2.34), $\mathcal{O}'^{\alpha\beta\nu}_{A,B}[\Gamma,\eta]$ is of the form (3.8), the only differences coming from our use of double indices for the Lie algebra components. We take then another direction in the functional space, say $\varphi = (\varphi^{\alpha}{}_{\beta\mu},\ \omega^{\gamma}{}_{\nu})$ and check to see whether or not

$$\int d^4x\ \varphi_{\alpha\beta\nu}\,\mathcal{O}'^{\alpha\beta\nu}[\Gamma,\eta] + \int d^4x\ \omega_{\gamma\nu}\,\mathcal{O}'^{\gamma\nu}[\Gamma,\eta] =$$

$$= \int d^4x\ \Gamma_{\alpha\beta\nu}\,\mathcal{O}'^{\alpha\beta\nu}[\varphi,\omega] + \int d^4x\ \eta_{\gamma\nu}\,\mathcal{O}'^{\gamma\nu}[\varphi,\omega] \,. \tag{3.18}$$

As expected, the Lorentz sector alone satisfies the symmetry condition. Neither η nor ω really appear in the first terms in each side in (3.18). These terms exactly cancel each other, and we have to verify if the second terms, which come from the translational sector, coincide or not. We find that (i) some pieces do allow for symmetrization, such as the last term in (3.17), which contributes with

$$\omega_{\alpha\nu}\,F^{\alpha}{}_{\epsilon}{}^{\mu\nu}\,\eta^{\epsilon}{}_{\mu} \tag{3.19}$$

to the left-hand side; (ii) some other pieces are not symmetrizable. It is always very difficult to be sure that a certain term is not somehow cancelled or symmetrized by some other. The first term in (3.17) is a good suspect: $\omega_{\gamma\nu}\,\Gamma^{\alpha}{}_{\varepsilon\mu}\,\tau^{\varepsilon\mu\nu}$ is not symmetrical by itself. The best way to see that it is not symmetrized by any other is to go back to the DS theory and trace what happens during the contraction process. Let us write (3.15) for the DS case: the left-hand side will be

$$\int d^4x \Big[\tfrac{1}{8}\varphi_{\alpha\beta\nu}\, f^{\alpha\beta}{}_{\gamma\delta,\varepsilon\varphi}\, \Gamma^{\gamma\delta}{}_{\mu}\, F^{\varepsilon\varphi\mu\nu} + \tfrac{1}{2}\varphi_{\alpha\beta\nu}\, f^{\alpha\beta}{}_{\gamma 5,\varepsilon 5}\, \Gamma^{\gamma 5}{}_{\mu}\, F^{\varepsilon 5\mu\nu} +$$

$$+ \tfrac{1}{2}\varphi_{\alpha 5\nu}\, f^{\alpha 5}{}_{\gamma\delta,\varepsilon 5}\, \Gamma^{\gamma\delta}{}_{\mu}\, F^{\varepsilon 5\mu\nu} - \tfrac{1}{2}\varphi_{\alpha 5\nu}\, f^{\alpha 5}{}_{\gamma\delta,\varepsilon 5}\, F^{\gamma\delta\mu\nu}\, \Gamma^{\varepsilon 5}{}_{\mu} \Big] . \qquad (3.20)$$

The f's are the structure constants for the DS group, written in a hopefully clear antisymmetric-double-index notation and the numerical factors account for double counting. The first term above is obviously $\varphi \longleftrightarrow \Gamma$ symmetrical; it is a contribution related only to the Lorentz sector. Also the last term is symmetrical: by contraction, with $L\varphi_{\alpha 5\nu} = \omega_{\alpha\nu}$, $L\Gamma^{\varepsilon 5}{}_{\mu} = \eta^{\varepsilon}{}_{\mu}$ it gives (3.19), related to the last term in (3.17). Now, the second and the third terms are not, each one, symmetrical: they are "symmetrizing companions", they symmetrize each other when we substitute φ for Γ and vice-versa. The third one is precisely that giving by contraction our suspect first term in (3.17), once multiplied by $\omega_{\alpha\nu}$. So, the suspect term would be symmetrized by the term coming from the second term in (3.20). That is where the asymmetry comes from: there is no such a term. If we examine the equations in detail we see that the symmetrizing second term in (3.20) comes from the Fréchet derivative of the last two terms in (2.32). We had called atention to the fact that, in the contraction process leading to (2.32), these terms venish. Summing up: the term, present in the DS theory, which symmetrizes the first term in (3.17), disappears during the contraction process. In this way we can pinpoint how contraction spoils the symmetry necessary for the theory to be Lagrangean: some terms in the field equations disappear and terms like the first one in (3.17) find no more a "symmetrizing companion" in the contracted theory [27]. We could still think that some miracle might occur: we have been analysing terms of (3.17) which correspond to (3.15); there are other non-derivative terms, corresponding to (3.13), which could eventually symmetrize or compensate just the above "ofending" terms. That this is not the case may be verified by a direct term by term comparison.

Consequently, there is no Lagrangean leading to the dynamical equations (2.34 - 35). The argument above can be line-by-line adapted to equations (1.18-19), with the same result [28].

IV. FINAL COMMENTS

A Poincaré gauge theory will be always kept apart from the usual gauge models by two peculiarities: the non semisimple character of the group and the presence of the kinematic representation. Concerning the first of these features, there are two main consequences: the potentials related to the abelian sector have unusual transformation properties and there is no bi-invariant metric on the group. A metric which is only right-invariant may suffice to build up a Lagrangean but the result is anyhow an atypical gauge Lagrangean leading to field equations which are not of the Yang-Mills form [29]. As to the second peculiarity, all local transformations in that representation may ultimately be seen as translations and the trouble is that there is no such a thing as "gauging" translations in the usual way: $\exp[a^\mu \partial_\mu] f(x) = f(x+a)$ becomes false as soon as a^μ becomes point dependent. The simple idea of "gauging" by imposing a local symmetry does not apply. Some of the ideas currently associated to the expression "gauge theory" will have

8. P. Becher and H.Joos, Z.Phys. C15 343 (1982).

9. D.A. Popov and L.I. Daikhin, Soviet Phisics Doklady 20, 818 (1976).

10. The role of discret duality in the short-distance behaviour of gauge fields is not clear. The continuous duality symmetry has been used (S.Deser et al, Phys. Lett 58B, 355 (1975)) to improve renormalizability in the Einstein-Maxwell theory, but it does not exist at a fundamental level in the non-abelian case (S.Deser and C.Teitelboim, Phys. Rev. D13, 1592 (1976)).

11. C.Itzykson and J.B.Zuber, "Quantum Field Theory", Mc Graw-Hill, New York, 1980.

12. C.Fronsdal, Phys. Rev. D30, 2081 (1984).

13. Minkowski space is a homogeneous space under the action of $\mathcal{P}$ the quotient space P/SO(3,1). There are two DS groups, SO(3,2) and SO(4,1) under the action of which DS spaces, their quotients by the Lorentz group, are homogeneous.

14. Intuition can get some help from the analogy with the "completely euclideanized" case, in which the DS group becomes SO(5), $\mathcal{L}$ becomes SO(4) and the homogeneous DS space is $SO(5)/SO(4) \sim S^4$. In this case, L would be the radius of the 4-sphere, which approaches the euclidean 4-space when $L \to \infty$. We are using "contraction" in a broad sense, comprising both Inonü-Wigner and Saletan contractions in the sense of ref. 30.

15. F.Gürsey in "Group Theoretical Concepts and Methods in Elementary Particle Physics", Gordon and Breach, New York, 1964, p.365.

16. R.Aldrovandi and E.Stedile, Intern. J.Theor. Phys. 23, 301 (1984).

17. D.Ivanenko and G.Sardanashvily, Phys. Reports C94,1 (1983).

18. F.W.Hell, "On the Kinematics of the Torsion of Spacetimes", to appear in the Bergmann issue of Foundations of Physics.

19. T.W.B.Kibble, J.Math. Phys. 2, 212 (1961).

20. M.M.Vainberg, "Variational Methods for the Study of Non-Linear Operators", Holden-Day, San Francisco, Calif. 1964.

21. R.W.Atherton and G.M.Homsy, Studies in Applied Mathematics 54, 31 (1975).

22. We prefer, for the time being, to keep separate coordinates $\{x^\alpha\}$ on each tangent space, being understood that they are functions of the corresponding point of spacetime. In this way the analogy with the fibers in gauge theories is more evident . The coordinates will be unified when convenient. One could, alternatively, work directly with a local representation of the Lie algebra in terms of the tangent fields, as is done in Ref. 12 for the conformal case.

23. We are here considering a direct relation, as that found usually in field theory. By conveniently transforming the fields and their derivatives, it is always possible to find some variational principle.

24. See, for instance, S.Okubo, Phys. Rev. D22, 919 (1980), and references therein.

25. The theorem is valid in terms of the far more general Gateau derivative, which only coincides with the Fréchet derivative when it is linear and meets some requirements of uniform

to be forsaken. What is the essential characteristic defining a gauge model worth this name? We have here retained the Yang-Mills field equations and tried to build a model as close as possible to the general scheme of gauge theories. From this point of view, the model above is the most natural candidate for a Poincaré gauge theory for gravitation. Its field equations, beside being the naïve Yang-Mills equations for the group, can be obtained by contraction from a gauge model for the De Sitter group, or from the imposition of the discrete duality symmetry. The last two points are related but only because contraction happens to preserve that symmetry. It could be otherwise, as contraction does not preserve all the good properties of the original model: we have seen how it breaks that subtle symmetry necessary for the theory to be Lagrangean.

Poincaré models have been extensively considered, but almost always with a Lagrangean theory in mind. It is just natural that the one presented here has been missed. A non-Lagrangean theory is not, for sure, a very sympathetic kind of theory. From the privileged point of view of the Yang-Mills equations, however, this is precisely the case for the Poincaré group.

The problem of quantization can be faced in two ways: the canonical approach, whose possibility has been stressed at the end of section II; or the path-integral procedure, with the De Sitter model as an intermediate step. The second approach has been pursued by the authors and will be reported elsewhere. It leads to consistent results at the tree level (such as Newton's law in the static non-relativistic limit) but the question of renormalizability is still unsettled.

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2. Apart from some fine-grained aspects emerging in some particular solutions (Taube-NUT, for instance), exceptions are the extensions to more general spaces of Zeeman's consid erations on the topology of Minkowski space, traceable from R.Göbel, Commun. Math. Phys. 46, 289 (1976) and D.B.Malament, J.Math. Phys. 18, 1399 (1977).

3. The huge literature on the subject can be traced from Refs. 17 and 18 below.

4. In this sketchy description of gauge theories, given because necessary to the exposition of the ideas in the following , there is no place for subtleties such as copies and all that.

5. S.Kobayashi and K.Nomizu, "Foundations of Differential Geometry", vol.I, Interscience, NY (1963).

6. Ref. 5, § III.

7. G.Basombrio, Gen. Relativity and Gravitation 12, 109 (1980).

continuity. We shall suppose it to be case here.

26. B.A.Finlayson, Phys. Fluids 15,963 (1972).

27. First order derivative terms are the most frequent sources of Vainberg's symmetry violation. It is one more miracle of gauge theories that each such term does find a "symmetrizing companion". We can trace back the offending term in (3.17) to the term $A^{\alpha}{}_{\gamma\mu} \mathcal{T}^{\gamma\mu\nu}$ in (2.35), which plays here the same role as the "spoiling" term $\mu \times (\nabla \times \mu)$ in the Navier-Stokes equation.

28. A more involved treatment was formulated by E.Tonti (see Refs. 21 and 26), which allows a direct yes-or-not answer to the whole question. We have prefered the formulation given above because it will allow to trace the symmetry failure back to the contraction process.

29. M.O. Katanaev, Theoretical & Mathematical Physics 54,248 (1983).

30. R.Gilmore, "Lie Groups, Lie Algebras and Some of their Applications", J.Wiley, New York, 1974.