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NUMERICAL ASPECTS CONCERNING A CLASS OF m-SEMIRECURRENT EVENTS AND THEIR APPLICATION TO COUNTER THEORY

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In some problems of the mathematical theory of particle counters[4,5,13], film or filmless measurements of track ionization in high energy physics[2,6,7], queueing theory[2], random walks, etc., a class of semirecurrent and m-semirecurrent events appears. These classes have interesting properties and here we study the numerical estimate of probabilistic formulae corresponding to integer-valued random variables denoting the first occurrence of the experiments at the n th trial, and their asymptotically exponential properties. We present very precise and computationally convenient formulae. The application of m-semirecurrent events to counter theory with prolonging dead time is studied in more detail, and an illustrative numerical example is given.

1. PRELIMINARY RESULTS

We suppose that during the k th experiment, $k = 1,2,\dots$, an event, A^k, either may occur or not. The occurrence of the event at the nth trial, $n = 1,2,\dots$, will be denoted by A^k_n and its non-occurrence by $\bar{A}^k_n$. The events $\{A^k_n : n,k \geq 1\}$ are said to be semirecurrent if for any $k \geq 1$ and i_j with $1 \leq i_0 < i_1 < \dots < i_n$, $n \geq 1$, we have

$$P(A^k_{i_1} \dots A^k_{i_n} \mid A^k_{i_0}) = P(A^{k+i_0}_{i_1-i_0} \dots A^{k+i_0}_{i_n-i_0}) . \tag{1.1}$$

Denote by ν_k, $k \geq 1$ an integer-valued random variable saying that the event A^k occurs in the k th experiment for the first time and put $P^k_n = P(\nu_k = n) = P(\bar{A}^k_1 \dots \bar{A}^k_{n-1} A^k_n)$.

Using (1.1) we may prove that

$$\left.\begin{aligned} P^k_1 &= P(A^k_1), \\ P^k_n &= P(A^k_n) - \sum_{j=1}^{n-1} P(A^k_j)\, P^{k+j}_{n-j}, \quad n \geq 2. \end{aligned}\right\} \tag{1.2}$$

Let us define for any $k \geq 1$ and $|z| < 1$ $U_k(z) = \sum_{n=0}^{\infty} P(A^k_n) z^n$, where $P(A^k_0) \equiv 1$ and $P_k(z) = \sum_{n=1}^{\infty} P^k_n z^n$. Due to (1.2), we have $P_k(z) = \sum_{n=1}^{\infty} P(A^k_n)(1 - P_{k+n}(z))\, z^n$.

An interesting case is obtained when there is an integer m so that $P_m(z) = P_{m+1}(z) = \dots$. Then semirecurrent events are said to be m-semirecurrent. In this paper we shall concentrate ourselves mainly on this class of semirecurrent events.

Remark 1.1. It is clear that if $\{A_n^k: n,k \geq 1\}$ are m-semirecurrent events, then $\{B_n^k: n,k > 1\}$, where $B_n^k = A_n^{k+1}$, $k,n \geq 1$, are (m-1)-semirecurrent.

If $m=1$, then (1.1) and (1.2) do not depend on the superscripts. The m-semirecurrent events are recurrent (for the definition of the recurrent events see, for example, [9,10]) iff $m = 1$. In this case (1.2) reduces to the known formula [9,10] for the recurrent events

$$\left.\begin{aligned} P_1 &= P(A_1), \\ P_n &= P(A_n) - \sum_{j=1}^{n-1} P(A_j) P_{n-j}, \quad n \geq 2, \end{aligned}\right\} \tag{1.3}$$

where $P_n = P_n^1 = P_n^2 = \dots$, $P(A_n) = P(A_n^1) = P(A_n^2) = \dots$ for each n. It is evident that semirecurrent events $\{A_n^k: n,k \geq 1\}$ are m-semirecurrent iff $\{B_n^k: n,k > 1\}$, where $B_n^k = A_n^{k+m-1}$, $n,k \geq 1$, are recurrent events.

If $m = 2$, then (1.1) has the following form: for any $k = 1,2$ and i_j with

$$\begin{aligned} &1 \leq i_0 < i_1 < \dots < i_n, \quad n \geq 1, \\ &P(A_{i_1}^k \dots A_{i_n}^k \mid A_{i_0}^k) = P(A_{i_1-i_0}^2 \dots A_{i_n-i_0}^2). \end{aligned} \tag{1.4}$$

Therefore for (1.2) we conclude that

$$\left.\begin{aligned} P_1^k &= P(A_1^k), \\ P_n^k &= P(A_n^k) - \sum_{j=1}^{n-1} P(A_j^k) P_{n-j}^2, \quad n \geq 2, \end{aligned}\right\} \tag{1.5}$$

for any $k = 1,2$. This class of semirecurrent events is also known in literature as the recurrent events with delay [10,11].

Some basic properties of the semirecurrent and m-semirecurrent events are studied in more detail in [8].

Without ambiguity we shall write $\{A_n^k: n \geq 1,\ k = 1,\dots,m\}$ for m-semirecurrent events $\{A_n^k: n,k \geq 1\}$. We say that for m-semirecurrent events $\{A_n^k: n \geq 1,\ k = 1,\dots,m\}$ the case of periodicity holds, if there is an integer $t > 1$ such that $P(A_n^m) > 0$ if $n = t, 2t, \dots$. In this case $P(A_n^m) = 0$ whenever $n \neq jt$. The greatest integer $t > 1$ with this property is called the period. In the opposite case $\{A_n^m\}_{n=1}^\infty$ is called non-periodic. The sequence $\{A_n^k\}_{n=1}^\infty$, for $k = 1,\dots,m$ is said to be certain or uncertain according to whether $\sum_{n=1}^\infty P_n^k = 1$ or $\sum_{n=1}^\infty P_n^k < 1$.

2. APPROXIMATIVE FORMULAE

In the present paper we shall deal with m -semirecurrent events $\{A_n^k : n \geq 1,\ k = 1,\dots,m\}$ with

$$\left.\begin{array}{ll} \text{(i)} & P(A_n^1) \geq P(A_n^2) \geq \dots \geq P(A_n^m),\quad n \geq 1, \\ \text{(ii)} & P(A_1^k) \geq P(A_2^k) \geq \dots,\quad k = 1,\dots,m, \\ \text{(iii)} & P(A_{n+1}^k) \leq P(A_n^{k+1}),\quad n \geq 1,\ k = 1,\dots,m-1,\ \text{if } m \geq 2. \end{array}\right\} \tag{2.1}$$

Here we derive approximative formulae for $\{P_n^k\}_{n=1}^{\infty}$, $k = 1,\dots,m$. This result will be applied in Part 3 to the modified counter with prolonging dead time.

Define for any $k = 1,\dots,m$

$$a_k(z) = \sum_{n=0}^{\infty} a_n^k z^{n+1},\quad |z| < 1, \tag{2.2}$$

where $a_n^k = P(A_n^k) - P(A_{n+1}^k)$, $n \geq 0$, $(P(A_0^k) = 1)$, and

$$\psi_k(z) = z - a_k(z),\quad |z| < 1, \tag{2.3}$$

We suppose that $p > 0$, where $p = \lim_n P(A_n^m)$. Then $\{A_n^m\}_{n=1}^{\infty}$ is certain and, consequently, due to /8/ Th.4.1 , each $\{A_n^k\}_{n=1}^{\infty}$, $k=1,\dots,m-1$, is so. According to /8/ Th.4.2, $p = \lim_n P(A_n^k)$ for any $k = 1,\dots,m$.

It is clear that $\psi_k(1) = p$, $k = 1,\dots,m$, and from the evident equality $U_k(z) - 1 = \psi_k(z)/(1-z)$, $k = 1,\dots,m$, $|z| < 1$, we conclude

$$P_k(z) = \Big(\psi_k(z) + \sum_{j=1}^{m-k-1} P_j^k(\psi_m(z) - \psi_{k+j}(z))\, z^j\Big)/(1 - z + \psi_m(z)), \tag{2.4}$$
$$k = 1,\dots,m,$$

We recall that here the sum over the empty set is defined as 0. The next result generalizes the analogous one from /2/.

Theorem 2.1. Suppose that for m-semirecurrent events with (2.1) we have

(i) $p > 0$,

(ii) the equation $a_m(z) = 1$ has a solution. Then for any $k = 1,\dots,m$

$$P_n^k = b_k \beta_1 \beta^{-n-1} + r_n^k,\ n \geq 1, \tag{2.5}$$

where

$$\beta = 1 + \sum_{j=1}^{\infty} \frac{1}{j!} \frac{d^{j-1}}{dz^{j-1}}[\psi_m^j(z)]|_{z=1}, \quad \beta_1 = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{d^j}{dz^j}[\psi_m^j(z)]|_{z=1},$$

$$b_m = \beta - 1, \quad b_{m-1} = \psi_{m-1}(\beta), \text{ if } m \geq 2,$$

$$b_{m-i} = \psi_{m-i}(\beta) + \sum_{j=1}^{i-1} P_j^{m-i} \beta^j (\beta - 1 - \psi_{m-i+j}(\beta)), \quad \text{for } 2 \leq i \leq m-1, \text{ if } m \geq 3,$$

and $|r_n^k| \leq C_k(q(1-P(A_1^m)))^n$ (the constants $C_k > 0$ and $q > 0$ do not depend on n).

Proof. If $P(A_1^m) = 1$, then $P(A_n^m) = 1$ for any n and $a_m(z) = 0$ which contradicts with (ii), so that $P(A_1^m) < 1$. According to the Cauchy formula, (2.4) entails

$$P_n^k = \frac{1}{2\pi i} \oint_{|z|=1} \frac{P_k(z)\,dz}{z^{n+1}} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{F_k(z)\,dz}{(1-z+\psi_m(z))\,z^{n+1}},$$

where $F_k(z)$, $k = 1,\dots,m$ denotes the numerator of the right-hand side in (2.4). Since

$$1 - z + \psi_m(z) = 1 - z \sum_{n=0}^{\infty} a_n^m z^n = 1 - a_m(z), \tag{2.6}$$

then its value for $z = 1$ is $p > 0$. The coefficients for $a_m(z)$ are positive, therefore, due to (ii), there is a positive root of the equation $a_m(z) = 1$. Denote by β this minimal one. It is clear that $1 - z + \psi_m(z)$ has no zeros in the circle $|z| < \beta$. If there is $0 < \theta_0 < 2\pi$ such that $a_m(\beta e^{i\theta_0}) = 1$, then this can happen only if $\cos\theta_0 n = 1$ for all $n \geq 1$ for which $a_{n-1}^m \neq 0$. Therefore $a_m(z)$ is a power series in z^t for some integer $t > 1$, and, consequently, $a_0^m = 0$, which is impossible. So, we have shown that β is the unique zero of $1 - z + \psi_m(z)$ in the circle $|z| \leq \beta$. It is clear that it is simple, because $a_m(\beta) > 0$.

Denote by $R > \beta$ the radius of a circle in that (2.6) has the unique zero $z = \beta$. From (iii) of (2.1) we conclude $a_n^k \leq a_{n-m+k}^m + \dots + a_n^m$, $n \geq m - k$. Hence $F(z)$ has no singularities in the circle $|z| < R$. Denote

$$r_n^k = \frac{1}{2\pi i} \oint_{|z|=R} \frac{F_k(z)\,dz}{(1-z+\psi_m(z))\,z^{n+1}} = \frac{F_k(\beta)}{(\psi_m'(\beta)-1)\beta^{n+1}} + P_n^k. \tag{2.7}$$

The integral on the left-hand side of (2.7) can be estimated by the maximum modulus $|r_n^k| \leq C_k R^{-n}$. From (2.6) we have that $(1 - P(A_1^m)) \leq 1$, therefore there is $q > 0$ such that $R = 1/q(1-P(A_1^m))$ and the estimate of the remainder is established.

Putting $\beta_1 = 1/(1-\psi_m'(\beta))$ and $b_k = F_k(\beta)$ we may obtain (2.5) from (2.7). From (2.5) we see that the convergence radius for the power series for $P_k(z)$ is at least β, hence (2.4) implies $b_k \geq 0$.

To establish the explicit expressions for β and β_1, respectively, we consider a function $w = z - \psi_m(z)$ that in a conform way transforms some neighbourhood of the point $w = 1$ to some neighbourhood of the point $z = \beta$. Therefore $w = w(z)$ has an inverse function $z = z(w)$. It is clear that $\beta = z(1)$ and $\beta = z'(1)$. Using the Lagrange expansion formula /16/ we obtain the formulae for β and β_1. Q.E.D.

Remark 2.2. The root of the equation $a_m(z) = 1$ may be evaluated more effectively using the Newton approximation method. In fact, it suffices to take into account the form of (2.6). Then for β_1 we have $\beta_1 = -1/a_m'(\beta)$.

As will be shown in Example 3.4, formulae (2.5) give very precise estimate of P_n^k even for small n. The remainder terms, r_n^k, are, for sufficiently large n, very small with respect to the main factors. Therefore from (2.5) we obtain very precise and computationally convenient formulae $P_n^k \approx b_k \beta_1 \beta^{-n-1}$, $k = 1, \ldots, m$.

Corollary 2.1.1. Let $t > 1$ be an integer and let $\{A_n^k : n \geq 1, k = 1, \ldots, m\}$ be m-semirecirrent events with $P(A_n^k) = 0$ whenever n is not a multiple of t, $k = 1, \ldots, m$. Suppose (2.1) holds whenever n is a multiple of t and

(i) $\lim_n P(A_{nt}^m) > 0$,

(ii) the equation $\tilde{a}_m(z) = \sum_{n=0}^{\infty} (P(A_{nt}^m) - P(A_{nt+t}^m)) z^{n+1} = 1$ has a solution.

Then for each $k = 1, \ldots, m$ $P_{nt}^k = b_k \beta_1 \beta^{-n-1} + r_n^k$, where β, β_1 and b_k are evaluated from Theorem 2.1, replacing $\psi_k(z)$ by

$$z - \sum_{n=0}^{\infty} (P(A_{nt}^k) - P(A_{nt+t}^k)) z^{n+1}, \quad k = 1, \ldots, m.$$

Proof. Defining $A_n^k = A_{nt}^k$, $n \geq 1$, $k = 1, \ldots, m$, we obtain the case described in Theorem 2.1. Q.E.D.

3. APPLICATION TO COUNTERS

An important class of semirecurrent events is obtained if we consider a modified counter with prolonging dead time.

Suppose that particles arrive at the counter at moments $0 = \tau_1 < \tau_2 < \ldots$ according to a recurrent process with the common distribution function $F(t) = P(T_n < t)$, where $T_n = \tau_{n+1} - \tau_n$, $n \geq 1$, are interarrival times. Any arriving particle generates an impulse of a random length (may be constant, too). Due to inertia of the counting device, it is possible that all particles will not be registered. The time during that the device is unable to record is called the dead time. A counter with prolonging dead time is one in which dead time is produced after registration of all impulses of emitted particles. This counter has been studied in [12-16, 5]. For modified counter with prolonging dead time we suppose that any registered particle determines an impulse of a random length with a distribution function, in general, different from the distribution function of a non-registered particle. Since in the present paper we shall study exclusively counters with prologning dead time, we shall call such counters simply counters.

Let $\{\chi_n\}_{n=1}^{\infty}$ be a sequence of impulse lengths, so that, it is a sequence of non-negative independent random variables, independent of $\{T_n\}_{n=1}^{\infty}$, *with distribution functions*

$$H_1(t) = P(\chi_1 < t), \quad H_2(t) = P(\chi_n < t), \quad n \geq 2. \tag{3.1}$$

This counter will be denoted by the triple (F, H_1, H_2). If $H_1 = H_2$, then we obtain a non-modified counter.

Putting

$$A_n^k = \{\chi_k < T_k + \dots + T_{k+n-1},\ \chi_{k+1} < T_{k+1} + \dots + T_{k+n-1}, \dots, \\ \chi_{k+n-1} < T_{k+n-1}\}, \quad n \geq 1, \quad k = 1,2, \tag{3.2}$$

we defined semirecurrent events with delay. The random variable ν_1 may be interpreted as the number of particles arriving at the modified counter (F, H_1, H_2) during the dead time. Similarly, ν_2 denotes the same for the counter (F, H_2, H_2).

It is clear that if for distribution functions of the lengths of impulses we suppose the existence of an integer $m \geq 2$ such that

$$H_k(t) = P(\chi_k < t), \quad k = 1, \dots, m-1, \quad H_m(t) = P(\chi_k < t), \quad k \geq m,$$

then (3.2) defines m-semirecurrent events. Hence all results of this part may be easily modified for any m. We recall that the last above case is a particular one of the semirecurrent events studied in [4], and the same processes may appear in queueing systems with infinitely many servers [2].

It may be easily checked that if $H_1(t) \geq H_2(t)$ $(\geq H_3(t) \geq \dots \geq H_m(t))$ for any t, then $\{A_n^k : n \geq 1,\ k = 1,2\ (,\dots,m)\}$ fulfil (2.1).

Theorem 3.1. Suppose a modified counter (F, H_1, H_2) satisfies:
(i) $H_1(t) \geq H_2(t)$ for any t;

(ii) $\int_0^\infty H_2(t)\,dF(t) > 0$;

(iii) $\sup\{\mu \geq 0 : \int_0^\infty e^{\mu t}\, dH_2(t) < \infty\} = \infty$.

Then Theorem 2.1 holds.

Proof. According to [1], from (ii) we conclude that $p > 0$. In order to prove that the equation $a_2(z) = 1$ has a solution, it suffices to show that for any $\mu > 0$ we have

$$0 \leq a_n^2 \leq P(\chi_{n+2} \geq T_2 + \dots + T_{n+2}) \leq M(e^{\mu \chi_2})(M(e^{-\mu T_2}))^{n+1}.$$

The convergence radius, R, of the series $\sum_{n=0}^{\infty} a_n^2 z^n$ is $R > 1/M(e^{-\mu T_2}) > 1$. If $\mu \to \infty$, then $R = \infty$, and the condition (ii) of Theorem 2.1 is fulfilled. Q.E.D.

For example, if (i) $H_2(t)$ is the distribution function of a positive constant random variable; (ii) $dH_2(t) = a\exp(-bt^c)\,dt$, $t \geq 0$, for some $a > 0$, $b > 0$, $c \geq 2$; then the condition (iii) of Theorem 3.1 is satisfied.

We recall that if $F(t) = 1 - e^{-\lambda t}$, $t \geq 0$, and

$$D = \int_0^\infty t\, dH_2(t) < \infty, \tag{3.3}$$

then, due to [17],

$$p = e^{-\lambda D}. \tag{3.4}$$

Theorem 3.2. Let (F, H_1, H_2) be a modified counter, where $F(t) = 1 - e^{-\lambda t}$, $t \geq 0$. Let $\mu^+ = \sup\{\mu \geq 0 : \int_0^\infty e^{\mu t}\, dH_2(t) < \infty\}$ and $H_1(t) \geq H_2(t)$ for any t. If $\mu^+ > 0$ and $1/\mu^+ \leq D$, then Theorem 2.1 holds. (Here we put $1/\infty = 0$ if $\mu^+ = \infty$).

Proof. If $\mu^+ = \infty$, then the assertion follows from Theorem 3.1. Let now $\mu^+ < \infty$. Choose $0 < \tilde{\mu} < \mu^+$. Then from our assumptions it follows that there is a $c \geq 1$ such that $1 - H_2(t) \leq ce^{-\tilde{\mu} t}$, $t \geq 0$. It is easy to check that

$$0 \leq a_n^2 \leq P(\chi_{n+2} \geq T_2 + \dots T_{n+2}) = \lambda^{n+1} \int_0^\infty (1 - H_2(t_1 + \dots + t_{n+1}))$$

$$\exp(-\lambda(t_1 + \dots t_{n+1}))\, dt_1 \dots dt_{n+1} \leq c(\lambda/(\lambda + \tilde{\mu}))^{n+1}.$$

Hence the convergence radius, R, of the series $\sum_n a_n^2 z^n$ is

$R \geq 1 + \tilde{\mu}/\lambda$, so that $R \geq 1 + \mu^{+}/\lambda$. Using the following simple chain of inequalities, holding for any $x > 0$,

$$e^{x} > 1 + x, \quad 1/(1+x) > e^{-x}, \quad (1 + 1/x)(1 - e^{-x}) > 1,$$

we may show that $(1 + \mu^{+}/\lambda)(1 + e^{-\lambda/\mu^{+}}) > 1$. It is clear that for any $\epsilon > 1$

$$(1 + \epsilon\mu^{+}/\lambda)(1 + e^{-\lambda/\mu^{+}}) > 1, \tag{3.5}$$

so that, due to the continuity, (3.5) holds for some $0 < \epsilon_0 < 1$, too. Then for $z = 1 + \epsilon_0 \mu^{+}/\lambda < R$ we have, according to (3.4),

$$\infty > z_0 \sum_{n=0}^{\infty} a_n^2 z_0^n > z_0 \sum_{n=0}^{\infty} a_n^2 = (1 + \epsilon_0 \mu^{+}/\lambda)(1 - e^{-\lambda D}) \geq$$

$$\geq (1 + \epsilon_0 \mu^{+}/\lambda)(1 - e^{-\lambda/\mu^{+}}) > 1.$$

Hence the equation $a_2(z) = 1$ has a solution. Q.E.D.

Corollary 3.2.1. If in Theorem 3.2 H_2 is the Gamma distribution, especially, if $H_2(t) = 1 - e^{-\mu t}$, $t \geq 0$, for some $\mu > 0$, then Theorem 3.2 is true.

The next theorem was proved in /3/:

Theorem 3.3. Let (i) F be a distribution function of some constant $a > 0$; (ii) $0 < H_2(na) \leq H_1(na)$, $n \geq 1$; (iii) $\int_0^\infty t\,dH_2(t) < \infty$. Moreover, if $H_2(na) < 1$ for any $n \geq 1$, then let (iv)

$$\lim_n (1 - H_2(na))/(1 - H_2((n+1)a)) = R(a) > 1; \quad \text{(v)} \quad \sum_{n=1}^{\infty} (1 - H_2((n+1)a))$$

$$\prod_{i=1}^{n} H_2(ia) R^n(a) > 1.$$

Then Theorem 2.1 holds.

For example, if H_2 is the Gamma distribution, or the geometric one, then the conditions of Theorem 3.3 are fulfilled.

Example 3.4. In Table 1 we give a numerical example of the application of Theorem 2.1 to the counter (F,H,H), where F is the distribution function of the constant equaled to 1, and $H(t) = 1 - e^{-t^2}$, $t \geq 0$. Here the parameters β and β_1 are evaluated by Remark 2.2: $\beta = 2.515773$, $\beta_1 = 2.338680$.

Table

n	P_n	$b_1\beta_1\beta^{-n-1}$	n	P_n	$b_1\beta_1\beta^{-n-1}$
1	6.3212 -01	5.6010 -01	6	5.5578 -03	5.5578 -03
2	2.2097 -01	2.2263 -01	7	2.2092 -03	2.2092 -03
3	8.8531 -02	8.8500 -02	8	8.7814--04	8.7814 -04
4	3.5175 -02	3.5176 -02	9	3.4905 -04	3.4905 -04
5	1.3982 -02	1.3982 -02	10	1.3875 -04	1.3875 -04

From Table we may see that the formula (2.5) yields a very precise estimate for P_n even for small n.

4. ASYMPTOTIC PROPERTIES

We shall continue the study of the properties of m-semirecurrent events. In this part we show that under some conditions ν_k, $k = 1,\dots,m$, is asymptotically exponential when $p \to 0$.

So, we suppose that m-semirecurrent events $\{A_n^k : n \geq 1,\ k = 1,\dots,m\}$ satisfy the condition (2.1). We introduce, for any $k = 1,\dots,m$ a function

$$\phi_k(z) = 1 - \sum_{n=0}^{\infty} a_n^k z^n, \ |z| < 1, \tag{4.1}$$

where $a_n^k = P(A_n^k) - P(A_{n+1}^k)$, $n \geq 0$.

It is obvious that for (2.3) we have

$$\psi_k(z) = z\phi_k(z), \ |z| < 1, \tag{4.2}$$

and if $p \geq 0$, then $\phi_k(1) = p$.

For the generating function, $\phi_k(z)$, we may give the following probabilistic interpretation. Let $p \geq 0$. For any $k = 1,\dots,m$ define the integer-valued random variable, ξ_k, such that

$$P(\xi_k = n) = a_n^k, \ n \geq 0, \qquad P(\xi_k = \infty) = p. \tag{4.3}$$

Then (4.1) and (4.2) entail that

$$\phi_k(z) = 1 - M(z^{\xi_k} I(\xi_k < \infty)), \ |z| < 1, \tag{4.4}$$

where $I(C)$ denotes the indicator function of a measurable set C.

If we put, for any $k = 1, \ldots, m$,

$$\bar{q}_n^k = P(n < \xi_k < \infty),\ n \geq 0,\quad \bar{Q}_k(z) = \sum_{n=0}^{\infty} \bar{q}_n^k z^n,\ |z| < 1,\quad \bar{P}_k(z) = M(z^{\xi_k} I(\xi_k < \infty)), \tag{4.5}$$

then $(1-z)\bar{Q}_k(z) = P(\xi_k < \infty) - \bar{P}_k(z)$, and

$$\bar{P}_k'(1) = \bar{Q}_k(1). \tag{4.6}$$

Therefore from (4.5) and (4.6) we have that

$$\sum_{n=1}^{\infty} n a_n^k = \sum_{n=1}^{\infty} (P(A_n^k) - p). \tag{4.7}$$

Lemma 4.1. Let the m-semirecurrent events $\{A_n^k : n \geq 1,\ k = 1, \ldots, m\}$ fulfil (2.1) and let $p > 0$. Then

(i) $-\phi_k'(1) = \sum_{n=1}^{\infty} (P(A_n^k) - p)$.

(ii) $|\phi_1'(1)| < \infty$ iff $|\phi_2'(1)| < \infty$ iff, etc., iff $|\phi_m'(1)| < \infty$ iff $M_m < \infty$, where $M_m = M(\nu_m(\nu_m - 1))$. (4.8)

In this case

$$\mu_k = M(\nu_k) = (\phi_k'(1) - \phi_m'(1) + 1 + \sum_{j=1}^{m-k-1} P_j^k(\phi_m'(1) - \phi_{k+j}'(1)))/p. \tag{4.9}$$

Proof. (4.8) follows from the above note and (4.7). Hence if $|\phi_1'(1)| < \infty$, then $\infty > |\phi_1'(1)| \geq |\phi_2'(1)| \geq \ldots \geq |\phi_m'(1)|$.

Let now $|\phi_m'(1)| < \infty$. Then, for any $n > m$, we have $P(A_n^1) \leq P(A_{n-m+1}^m)$, which implies that $|\phi_m'(1)| < \infty$. Theorem 5.3 in /8/ and (4.8) prove the equivalence of $|\phi_m'(1)| < \infty$ and $M_m < \infty$. Using (4.2) and taking the derivative of (2.4) we may easily check (4.9). Q.E.D.

Varying the parameter $p \in (0, 1]$ we may obtain, in general, different functions $\phi_k(z)$, $k = 1, \ldots, m$. Taking into account this dependence on p we shall write $\phi_k(z) = \phi_{pk}(z)$. Analogically we write $P(A_n^k) = P(A_n^k(p))$.

Theorem 4.2. Let the m-semirecurrent events $\{A_n^k : n \geq 1,\ k = 1, \ldots, m\}$ satisfy (2.1) and let $p \in (0, 1]$ vary so that (i) $|\phi_{p1}'(1)| < \infty$; (ii) $\lim_{p \to 0^+} \phi_{p1}'(1) = 0$. Then for any $k = 1, \ldots, m$ $\lim_{p \to 0^+} P(a_k(p)\nu_k > t) = e^{-t}$, $t \geq 0$, where $a_k = a_k(p) = 1/M(\nu_k)$.

Proof. Using (4.9) and conditions (i) and (ii) we may show that $\lim_{p \to 0^+} a_k(p) = 0$, $\lim_{p \to 0^+} a_k(p)/p = 1$.

From (2.4) we conclude that

$$P_k(e^{-sa_k\nu_k}) = [e^{-sa_k}(\phi_k(e^{-sa_k}) + \sum_{j=1}^{m-k-1} P_j^k e^{-sa_k j}(\phi_m(e^{-sa_k}) -$$

$$- \phi_{k+j}(e^{-sa_k})))]/[1 - e^{-sa_k} + e^{-sa_k}\phi_m(e^{-sa_k})] =$$

$$= e^{-sa_k}[1 + \sum_{j=1}^{m-k-1} P_j^k e^{-sa_k j}(\phi_m(e^{-sa_k})/\phi_k(e^{-sa_k}) -$$

$$- \phi_{k+j}(e^{-sa_k})/\phi_k(e^{-sa_k}))]/[(1 - e^{-sa_k})/\phi_k(e^{-sa_k}) +$$

$$+ e^{-sa_k}\phi_m(e^{-sa_k})/\phi_k(e^{-sa_k})].$$

First of all, it is clear that $\lim_{p\to 0^+}(1 - e^{-sa_k(p)})/p = s$. Next we show that

$$\lim_{p\to 0^+} \phi_{pk}(e^{-sa_j(p)})/p = 1 \tag{4.10}$$

for any $k, j = 1,\ldots, m$. We note that for any fixed $p \in (0, 1]$ $\phi_k(z)$ is nondecreasing for each $0 \le z \le 1$. Using that and the elementary inequality $e^{-x} \ge 1 - x, x \ge 0$, we obtain

$$p = \phi_{pk}(1) \le \phi_{pk}(e^{-sa_j(p)}) \le \phi_{pk}(1 - sa_j(p)).$$

Due to the inequality $(1 - x)^n \ge 1 - nx$, $n \ge 0$, $|x| \le 1$, we have, for sufficiently small $p > 0$,

$$\phi_{pk}(1 - sa_j(p)) = 1 - \sum_{n=0}^{\infty}(1 - sa_j(p))^n a_n^k \le p + sa_j(p)\sum_{n=1}^{\infty} na_n^k =$$

$$= p - sa_j(p)\phi'_{pk}(1).$$

Hence

$$1 \le \phi_{pk}(e^{-sa_j(p)})/p \le 1 - sa_j(p)/p\phi'_{pk}(1),$$

so that (4.10) holds.

Using this fact we have

$$\lim_{p\to 0^+}\phi_{pi}(e^{-sa_k(p)})/\phi_{pj}(e^{-sa_k(p)}) = 1$$

for each $i, j = 1,\ldots, m$.

Therefore

$$\lim_{p\to 0^+} P_k(e^{-sa_k\nu_k}) = 1/(1+s),\ s \geq 0,$$

and the theorem is completely proved. Q.E.D.

Remark 4.3. The condition (ii) in Theorem 4.2 is equivalent to the following two conditions $\lim_{p\to 0^+} P(A_1^1(p)) = 0$, $\lim_{p\to 0^+} \phi'_{pm}(1) = 0$. Indeed, it suffices to take into account the inequalities $P(A_n^1) \leq P(A^m_{n-m+1})$, $n \geq m$, and $P(A_1^1) \geq P(A_2^1) \geq \ldots \geq P(A_m^1)$.

Example 4.4. Define the recurrent events with delay. Let $n_0 \leq m_0$ be given and $0 < p_1 \leq p_2 \leq \ldots \leq p_{n_0} < 1$, $p_{n_0+1} = 1$, $0 < p_1^* \leq$ $\leq p_2^* \leq \ldots \leq p^*_{m_0} < 1$, $p^*_{m_0+1} = 1$. We put

$$P(A_n^1) = \begin{cases} p_1^* \cdots p_{n-1}^* p_n, & \text{if } 1 \leq n \leq n_0, \\ p_1^* \cdots p_n^*, & \text{if } n_0+1 \leq n \leq m_0, \\ p_1^* \cdots p^*_{m_0}, & \text{if } m_0 < n, \end{cases}$$

$$P(A_n^2) = \begin{cases} p_1^* \cdots p_n^*. & \text{if } 1 \leq n \leq m_0, \\ p_1^* \cdots p_m^*. & \text{if } m_0 < n. \end{cases}$$

For this case Theorem 4.2 holds.

In the rest of this part we apply Theorem 4.2 to the modified counter with prolonging dead time. Similarly as in Part 3 we confine ourselves to a counter (F, H_1, H_2).

Theorem 4.5. Let (F, H_1, H_2) be a counter with (i) $F(t) = e^{-\lambda t}$, $t \geq 0$ $(\lambda > 0)$; (ii) $H_1(t) \geq H_2(t)$ for any t; (iii) $D_2 = \int_0^\infty t^2 dH_2(t) < \infty$; (iv) $H_2(0^+) = 0$. Then Theorem 4.2 holds whenever $\lambda \to \infty$.

Proof. We shall examine conditions of Theorem 4.2 and Remark 4.3. It is easy to check that

$$P(A_1^1) = \lambda \int_0^\infty H_1(t)\, e^{-\lambda t} dt,\ P(A_n^2) = \lambda^{n+1}/n! \int_0^\infty \left(\int_0^t H_2(x)\, dx\right)^n e^{-\lambda t}\, dt,\ n \geq 1, \tag{4.11}$$

$$\phi_2(z) = 1 - \lambda \int_0^\infty \exp(-\lambda \int_0^y (1 - zH_2(u))\, du)(1 - H_2(y)),\, dy, \quad |z| < 1 . \qquad (4.12)$$

Using (3.4) or directly (4.12) when $z = 1$ we have that $p = e^{-\lambda D}$, where D is from (3.3). Since $D_2 < \infty$, then $\int_0^\infty y(1 - H_2(y))\, dy < \infty$ and the integral

$$\int_0^\infty (1 - H_2(y)) \exp(-\lambda \int_0^y (1 - zH_2(u))\, du) \int_0^y H_2(u)\, du\, dy$$

converges uniformly in $z \in [0,1]$. Therefore we may take the derivative of (4.12) with respect to $z = 1$, and obtain

$$\phi_2'(z) = -\lambda^2 \int_0^\infty (1 - H_2(y)) \exp(-\lambda \int_0^\infty (1 - zH_2(u))\, du) \int_0^y H_2(u)\, du\, dy .$$

We have to show that $\lim_{\lambda \to \infty} \phi_2'(1) = 0$.

Let $A > 0$ be arbitrary. Denote

$$I_1(\lambda, A) = \lambda^2 \int_0^A (1 - H_2(y)) \exp(-\lambda \int_0^\infty (1 - H_2(u))\, du) \int_0^y H_2(u)\, du\, dy ,$$

$$I_2(\lambda, A) = \phi_2'(1) - I_1(\lambda, A) .$$

It is obvious that $I_2(\lambda, A) \leq \lambda^2 \exp(-\lambda \int_0^A (1 - H_2(u))\, du)\, D_2$.

Because of $D < \infty$, for an arbitrary $\epsilon > 0$, there is $\Lambda_2(\epsilon, A) > 0$ so that $I_2(\lambda, A) < \epsilon/4$ whenever $\lambda > \Lambda_2(\epsilon, A)$.

Using the per-partes integration method, we conclude

$$I_1(\lambda, A) = I_3(\lambda, A) + I_4(\lambda, A) + I_5(\lambda, A),$$

where

$$I_3(\lambda, A) = -\lambda \exp(-\lambda \int_0^A (1 - H_2(u))\, du) \int_0^A H_2(u)\, du ,$$

$$I_4(\lambda, A) = \exp(-\lambda \int_0^A (1 - H_2(u))\, du) ,$$

$$I_5(\lambda, A) = \lambda \int_0^A \exp(-\lambda \int_0^y (1 - H_2(u))\, du)\, dy - 1 .$$

It is clear that $|I_3(\lambda, A)| \leq \lambda A \exp(-\lambda \int_0^A (1 - H_2(u))\, du)$. Hence there is $\Lambda_3(\epsilon, A) > 0$ so that $|I_3(\lambda, A)| < \epsilon/3$ whenever $\lambda > \Lambda_3(\epsilon, A)$. Similarly there is $\Lambda_4(\epsilon, A) > 0$ so that $|I_4(\lambda, A)| < \epsilon/4$ when $\lambda > \Lambda_4(\epsilon, A)$.

The condition $H_2(0^+) = 0$ entails that for any $\epsilon_1 > 0$ there is $A(\epsilon_1)$ with $1 - \epsilon_1 \leq 1 - H_2(u) < 1$ whenever $0 < u < A(\epsilon_1)$. Therefore

$$\lambda \int_0^{A(\epsilon_1)} e^{-\lambda y}\, dy \le I_5(\lambda, A) \le \int_0^{A(\epsilon_1)} e^{-\lambda(1-\epsilon_1)y}\, dy - 1$$

and

$$-e^{-\lambda A(\epsilon_1)} \le I_5(\lambda, A) \le (1 - e^{-\lambda A(\epsilon_1)})/(1-\epsilon_1) - 1 < \epsilon_1/(1-\epsilon_1).$$

Using the inequality $\epsilon_1/(1-\epsilon_1) < 2\epsilon_1$ which holds for $0 < \epsilon_1 < 1/2$, we get $|I_5(\lambda, A)| \le \max(e^{-\lambda A(\epsilon_1)}, 2\epsilon_1)$. Now, for a given $\epsilon_1 > 0$, we may choose $\Lambda_5(\epsilon_1) > 0$ so that $e^{-\lambda A(\epsilon_1)} < 2\epsilon_1$ whenever $\lambda > \Lambda_5(\epsilon_1)$. From this restriction we may choose ϵ_1 and A so that $\epsilon_1 = \epsilon/8$ and $A = A(\epsilon/8)$. Hence if $\lambda > \max_{2 \le i \le 5} (\Lambda_i(\epsilon, A(\epsilon/8)))$, then $|\phi_2'(1)| < \epsilon$.

Using (4.11) we see that, according to Remark 4.3, the proof of Theorem is finished. Q.E.D.

REFERENCES

1. Афанасьева Л.Г., Михайлова И.В. Изв.АН СССР, Тех.кибернетика, 1978, с.88-96.
2. Двуреченский А., Ососков Г.А. ОИЯИ, P5-82-631, Дубна, 1982.
3. Двуреченский А., Ососков Г.А. ОИЯИ, P5-83-873, Дубна, 1983.
4. Dvurečenskij A., Ososkov G.A. J.Appl.Prob., 1984, 22, No.3.
5. Dvurečenskij A., Ososkov G.A. Applikace mat., 1984, 29, No.3.
6. Dvurečenskij A., Kuljukina L.A., Ososkov G.A. J.Appl.Prob., 1984, 21, p.201-206.
7. Dvurečenskij A. J.Appl.Prob., 1984, 21, p.207-212.
8. Dvurečenskij A. JINR, E5-84-686, Dubna, 1984.
9. Feller W. Amer.Math.Soc., 1949, 67, p.98-119.
10. Феллер В. Введение в теорию вероятностей и ее приложения. "Мир", М., 1967, том 1.
11. Lindvall T. Z.Wahr.verw.Geb., 1979, 48, p.57-70.
12. Pollaczek F. C.R.Acad.Sci.Paris., 1954, 238, p.322-324.
13. Pyke R. Ann.Math.Stat., 1958, 29, p.737-754.
14. Smith W.L. J.Roy.Stat.Soc., 1958, B20, p.243-284.
15. Takâcs L. Teor.Veroj. i Prim., 1956, 1, p.90-102.
16. Takâcs L. Combinatorial Methods in the Theory of Stochastic Processes. J.Wiley, N.Y., 1967.
17. Takâcs L. RAIRO Recher. Opér., 1980, 14, p.109-113.

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Двуреченский А., Ососков Г.А. E5-84-701
Численные аспекты, касающиеся одного класса m-семирекуррентных событий и их применение к теории счетчиков

В некоторых проблемах математической теории счетчиков частиц, фильмовых и бесфильмовых измерениях ионизации треков в физике высоких энергий, а также в теории очередей, случайных блужданий и др. появляются классы семирекуррентных и m-семирекуррентных событий. В работе изучаются численные оценки вероятностных формул, соответствующих вероятности первого появления события на n-ом испытании. Представлены очень точные и удобные для вычислений приближенные формулы. Теория применяется к счетчикам с мертвым временем продлевающегося типа, и один численный пример показан в качестве иллюстрации.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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Dvurečenskij A., Ososkov G.A. E5-84-701
Numerical Aspects Concerning a Class of m-Semirecurrent Events and Their Application to Counter Theory

In some problems of the mathematical theory of particle counters, film or filmless measurements of track ionization in high energy physics, queueing theory, random walks, etc., the classes of semirecurrent and m-semirecurrent events appear. In the paper the numerical estimates of the probabilistic formulae corresponding to the probability of the first occurrence of an event at the nth trial are studied. We present very precise and computationally convenient formulae. The application of the theory to the counters with prolonging dead time and an illustrative numerical example are given.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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