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MULTIDIMENSIONAL LINEAR-LINEAR NODAL TRANSPORT METHODS IN WEIGHTED DIAMOND DIFFERENCE FORM*

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ABSTRACT

Two previously derived approximations to the linear-linear nodal transport method, the linear nodal (LN) and the linear linear (LL) methods, are re-examined together with a new approximation, the bi-linear (BL) method, that takes into account the bilinear nodal flux-moment. The three methods differ in the degree of analyticity retained in the final discrete-variable equations; however, they all possess the very high accuracy characteristic of nodal methods. Unlike previous work, here the final equations are manipulated and cast in the form of the classical weighted diamond difference (WDD) equations. This makes them simple to implement in a computer code, especially for those users who have experience with WDD algorithms. Other algorithms, such as the nodal algorithm, also can be used to solve the WDD-form equations.

A computer program that solves two-dimensional transport problems using the LN, LL, or the BL method was written and was used to solve two simple test problems. The results are used here to confirm our algebraic manipulations of the nodal equations and also to compare the performance of the three methods from the computational, as well as the theoretical, point of view. The three methods are found to have comparable accuracies, especially on meshes that are sufficiently fine. It is apparent that a given method will be more appropriate to use for solving certain problems than the other two methods, depending on the specifications of the problem.

I. INTRODUCTION

Nodal methods have been developed and implemented for the numerical solution of the discrete-ordinates neutron transport equation.[1-8] Numerical testing of these methods and comparison of their results to those obtained by conventional methods have established the high accuracy of nodal methods.[1,2] Furthermore, it has been suggested that the linear-linear approximation is the most computationally efficient, practical nodal approximation.[2] Indeed, this claim has been substantiated by comparing the accuracy of the solution, and the CPU time required to achieve convergence to that solution by several nodal approximations, as well as the diamond difference scheme.[2,3]

Two types of linear-linear nodal methods have been developed in the literature: analytic linear-linear (NLL) methods in which the "transverse-leakage" terms are derived analytically,[1,2] and approximate linear-linear (PLL) methods in which these terms are approximated.[2-6] In spite of their high accuracy, NLL methods result in very complicated discrete-variable equations that exhibit a high degree of coupling, thus requiring special solution algorithms. On the other hand, the sacrificed analyticity in PLL methods is compensated for by simpler discrete-variable equations and diamond difference-like solution algorithm.[3-6]

One PLL scheme which was previously developed and implemented included a bilinear component of the node interior expansion of the flux.[2] For the two test problems considered in Ref. 2, this scheme was found to be very similar to an NLL scheme (also presented in Ref. 2). That is, inclusion of the bilinear moment resulted in a high accuracy of the solution and in the coupling of the discrete-variable equations.[2] Hence, no attempt has been made so far to derive an NLL method that includes the bilinear moment of the flux, as this would have resulted in a very cumbersome set of equations. In addition, it is not clear yet how much can be gained, in terms of better accuracy, at the cost of the anticipated complication of such a method.

The purpose of this work is to derive an NLL method and a PLL method, both of which have been derived before, and to cast them in a simple weighted diamond difference (WDD) form. Also, we derive a new NLL method which includes the bilinear moment of the flux, the BL method, cast in a simple WDD form. It must be emphasized that the final equations for the three nodal approximations presented here are explicit WDD relations, with analytically derived spatial weights, which may be solved via a WDD algorithm. This is in contrast to the nodal equivalent finite difference (NEFD) method[3] in which only the solution algorithm, but not the form of the final equations, resembles WDD. This is an important difference, especially that one of the main criticisms directed towards nodal methods in general is the difficulty of their final forms and the complicated algorithms necessary to solve them. Our ability to write the nodal equations in a simple WDD form makes the high accuracy of such methods accessible to the many users familiar with conventional WDD methods with minor modifications.

In Section II of this paper we will simultaneously derive the discrete-variable equations for the three nodal methods presented here. These equations are shown to form a closed set of algebraic equations (i.e., as many equations as unknowns). In Section III we manipulate the discrete-variable equations to obtain a simple WDD form of the equations, with spatial weights that are analytically derived rather than pre-assigned as in conventional WDD methods. The three linear-linear nodal methods discussed in this paper are compared from a theoretical point of view in Section IV and from a computational point of view in Section V. Also in Section IV we describe the nodal algorithm and the WDD algorithm for each method, as well as the algorithm that we actually used in our computer code, which is a mixture of the nodal and WDD algorithms. A brief summary of the work and the main conclusions are presented in Section VI.

II. THE NODAL FORMALISM FOR THE NEUTRON TRANSPORT EQUATION

In this section we present the nodal formalism through which the discrete-variable equations are derived from the continuum neutron transport equation. In this development we will assume steady-state, and we will consider the case of a monoenergetic external source problem in two-dimensional Cartesian geometry with isotropic scattering. The resulting methods can be extended easily to many energy groups and eigenvalue problems by employing an inner-outer iteration procedure. Addition of a third dimension is straightforward but lengthy.

In order to conserve space, we will derive simultaneously the three approximations of the nodal method considered here. These are: the linear nodal (LN) method[2,4] (which is a PLL method), the linear linear (LL) method,[1,2] and the bilinear (BL) method (both NLL methods).

The discrete-ordinates approximation of the transport equation in two-dimensional Cartesian geometry is given by

$$\mu_k \frac{\partial \psi_k^{\ell+1}}{\partial x} + \eta_k \frac{\partial \psi_k^{\ell+1}}{\partial x} + \sigma \psi_k^{\ell+1} = S^{\ell} \quad , \tag{1}$$

where μ_k and η_k are the x- and y-components of the k-th unit vector, $\hat{\Omega}_k$, $k = 1,\ldots,N$; $\psi_k^{\ell+1}(x,y)$ is the $(\ell+1)$ inner iterate of the k-th angular flux, and σ is the total cross section. The ℓ-th inner iterate of the source, S^ℓ, normally contains the given external source as well as the inscattering source calculated from the ℓ-th inner iteration ($\equiv\sigma_s\phi^\ell$; σ_s is the scattering cross section and ϕ^ℓ is the ℓ-th inner iterate of the scalar neutron flux given by $\phi^\ell \equiv \sum_{K=1}^{N} w_k\psi_k^\ell$) and updated after every inner iteration. The inner iteration procedure is irrelevant to the remainder of the derivation, hence the superscript indicating the iteration index will be suppressed.

First we divide the domain of the problem into M rectangular computational elements, i.e., nodes, of the form $[-a_1, +a_1] \times [-b_1, +b_1]$, $1 = 1,\ldots,M$. We define Legendre polynomials on the interval $[-c,+c]$ by

$$p_n(z) \equiv P_n(z/c),\ n \geq 0 \quad , \tag{2}$$

where P_n is the regular Legendre polynomial of order n, defined on the interval $[-1,+1]$. The normalization relation for p_n then becomes

$$\frac{1}{c}\int_{-c}^{+c} dz\ p_m(z)\ p_n(z) = \frac{2}{(2m+1)}\,\delta_{mn} \quad . \tag{3}$$

Then we define the transverse moments of the angular flux by

$$\psi_{k,xj}(x) \equiv \left(\frac{2j+1}{2b}\right)\int_{-b}^{+b} dy\ \psi_k(x,y)p_j(y) \quad , \tag{4a}$$

$$\psi_{k,iy}(y) \equiv \left(\frac{2i+1}{2a}\right)\int_{-a}^{+a} dx\ \psi_k(x,y)p_i(x) \quad , \tag{4b}$$

where we have suppressed the node index on the node dimensions, a and b. The transverse-moments of the source, $S_{xj}(x)$ and $S_{iy}(y)$ are defined analogously. Also the nodal moments of the angular flux are defined by

$$\psi_{k,ij} \equiv \left(\frac{2i+1}{2a}\right)\left(\frac{2j+1}{2b}\right)\int_{-a}^{+a} dx\int_{-b}^{+b} dy\ \psi_k(x,y)p_i(x)p_j(y) \quad , \tag{5}$$

with analogous expressions for the nodal moments of the source.

Now we operate on the transport equation, Eq. (1), by the transverse moments operator; namely, we multiply Eq. (1) by $(2i+1/2a)\ p_i(x)$ and integrate over the x-dimension of the node, to obtain the x-moments equations

$$\eta_k \frac{d}{dy}\psi_{k,iy} + \sigma\psi_{k,iy} = S_{iy} + \chi_{k,iy},\ i = 0,1 \quad , \tag{6}$$

where the x-leakage moments are given by

$$\chi_{k,oy}(y) \equiv \frac{-\mu_k}{2a}\left[\psi_k(+a,y) - \psi_k(-a,y)\right] \quad , \tag{7a}$$

$$\chi_{k,1y}(y) \equiv \frac{-3\mu_k}{2a}\left[\psi_k(+a,y) - 2\psi_{k,oy}(y) + \psi_k(-a,y)\right] \quad . \tag{7b}$$

The same procedure is repeated in the y-direction to obtain analogous y-moments equations and expressions for the y-leakage moments. This set of equations is exact albeit not closed. Closing the system requires expanding the surface fluxes on the RHS's of Eq. (7) and its y-analogue in a truncated series. Since the order of this expansion is different for each of the three methods considered here, we introduce the coefficients

$$\nu_i = 0,\ \text{if } i=1 \text{ and LN or LL method} \\ = 1,\ \text{otherwise} \quad , \tag{8a}$$

and

$$\lambda_i = 0,\ \text{if } i=1 \text{ and LN method} \\ = 1,\ \text{otherwise} \quad . \tag{8b}$$

Both the zeroth and first order moments of the leakage terms are retained in the LL and BL methods, while in the LN method, the bilinear component of the leakage terms is neglected. That is, for the three methods Eqs. (7) are approximated by

$$\chi_{k,oy}(y) = \frac{-\mu_k}{2a}\left[\psi_{k,xo}(+a)-\psi_{k,xo}(-a)\right] - \frac{\mu_k}{2a}\left[\psi_{k,x1}(+a) - \psi_{k,x1}(-a)\right]\frac{y}{b} \quad , \tag{9a}$$

$$\chi_{k,1y}(y) = \frac{-3\mu_k}{2a}\left[\psi_{k,xo}(+a) - 2\psi_{k,oo} + \psi_{k,xo}(-a)\right] - \lambda_1\frac{3\mu_k}{2a}\left[\psi_{k,x1}(+a) - 2\psi_{k,o1} + \psi_{k,x1}(-a)\right]\frac{y}{b} \quad . \tag{9b}$$

The next step is to substitute Eqs. (9) into the RHS's of Eqs. (6), then solve the latter exactly by using an integrating factor. Evaluating the solution at the node surface, we obtain

$$e^{\epsilon_k}\psi_{k,oy}(+b) - e^{-\epsilon_k}\psi_{k,oy}(-b) = \\ \left(2\sinh\epsilon_k\right)\left[\frac{Soo}{\sigma} - \frac{1}{2\delta_k}\left\{\psi_{k,xo}(+a) - \psi_{k,xo}(-a)\right\}\right] \\ +2\left[\cosh\epsilon_k - \frac{\sinh\epsilon_k}{\epsilon_k}\right]\left[\frac{So1}{\sigma} - \frac{1}{2\delta_k}\left\{\psi_{k,x1}(+a) - \psi_{k,x1}(-a)\right\}\right] \quad , \tag{10a}$$

$$e^{\epsilon_k}\psi_{k,1y}(+b) - e^{-\epsilon_k}\psi_{k,1y}(-b) = \left[2 \sinh \epsilon_k\right] \left[\frac{S_{10}}{\sigma} - \frac{3}{2\delta_k}\left\{\psi_{k,xo}(+a) - 2\psi_{k,oo} + \psi_{k,xo}(-a)\right\}\right] +2\left[\cosh \epsilon_k - \frac{\sinh \epsilon_k}{\epsilon_k}\right] \left[\nu_1 \frac{S_{11}}{\sigma} - \frac{3\lambda_1}{2\delta_k}\left\{\psi_{k,x1}(+a) - 2\psi_{k,o1} + \psi_{k,x1}(-a)\right\}\right] , \tag{10b}$$

where $\delta_k = \sigma a/\mu_k$, and $\epsilon_k = \sigma b/\eta_k$. Two analogous equations can be derived from the y-moments equations. It should be noted that Eq. (10.b) contains the second difference between the three methods considered in this paper. Because neither the LN or the LL methods take into account the bilinear moment of the flux, the bilinear component of the source, S_{11}, arising from inscattering is neglected in these two methods. In contrast, in the BL method ($\nu_1 = 1$) the bilinear component of the source is calculated every iteration.

Equations (10) and their y-moments analogue represent a total of four equations. In addition, there are four incoming flux-moments boundary conditions relating the surface flux-moments in a given node to those in an adjacent node or to the global boundary conditions. On the other hand, the list of discrete-variable unknowns includes eight surface flux-moments and three (four) nodal flux-moments for the LN and LL (BL) methods. The source moments have not been counted as unknowns because in every iteration they are explicitly written in terms of the known nodal flux-moments calculated in the previous iteration. Hence, it is clear that additional relations between the discrete-variable unknowns are necessary in order to solve the set of equations uniquely. These relations are obtained by requiring the nodal balance of all retained flux-moments in each method. That is, we multiply Eq. (1) by

$\left(\frac{2i+1}{2a}\right)\left(\frac{2j+1}{2b}\right)p_i(x)p_j(y)$, $i,j = 0,1$, and integrate over the node to obtain

$$\frac{1}{2\delta_k}\left[\psi_{k,xo}(+a) - \psi_{k,xo}(-a)\right] + \frac{1}{2\epsilon_k}\left[\psi_{k,oy}(+b) - \psi_{k,oy}(-b)\right] + \psi_{k,oo} = \frac{S_{oo}}{\sigma}, \text{ etc.} \tag{11}$$

The equation resulting from the case $i=j=1$ is valid only for the BL method where the bilinear moments are calculated and conserved; thus, the four balance equations close the system of algebraic equations for the BL method. In the LN and LL methods the set of algebraic equations is closed without the $i=j=1$ equation, which is not valid anyway since the bilinear moments are not calculated in these methods.

The set of algebraic equations represented by Eqs. (10), their y-moments analogues, and Eqs. (11) can be solved in their present form using the traditional process of sweeping the mesh. However, this results in complicated expressions and requires the storage of many coefficients for each node type and each distinct angular direction. Instead, in the next section we manipulate these equations in order to cast them in a simple weighted diamond difference (WDD) form which requires the storage of only one (two) spatial weight for each node type per distinct angular direction for the BL (LN and LL) method.

III. THE WEIGHTED DIAMOND DIFFERENCE FORM

Two algorithms have been used previously to solve the discrete-variable equations, Eqs. (10) and (11), for the LN and LL methods. The Nodal algorithm[1] is based on simultaneously solving the equations for the outgoing flux-moments (e.g., for $\mu_k \gtrless 0$, $\psi_{k,xi}(\pm a)$, etc.) first, then using these to calculate the nodal flux-moments, which contribute to the new iterate of the scalar flux. In the Weighted Diamond Difference (WDD) algorithm[3,4] the nodal flux-moments are evaluated first from uncoupled expressions, then used to calculate the outgoing flux-moments. The advantage of the WDD algorithm is that it does not require the simultaneous solution of algebraic equations in each node. This is even more important in three-dimensional problems where the number of coupled equations to be solved simultaneously in the Nodal algorithm is larger. However, the WDD algorithm results in very complicated expressions,[3] and has been limited so far to the LN method.[3,4] Our objective in this section is to rewrite the discrete-variable equations in the form of WDD relations to which either the Nodal or WDD algorithms may be applied. By a WDD relation we mean an expression for the nodal flux-moments in terms of the surface-evaluated transverse flux-moments. The relation is fully specified by means of a spatial weight that, traditionally, was preassigned based on intuition and experience. The spatial weights in the present work are not preassigned, but rather derived analytically from the equations obtained via the nodal formalism. Also, the form of a WDD relation is modified here to accommodate the presence of the first order flux-moments that are not commonly incorporated in conventional WDD methods.

In order to obtain the WDD equations we eliminate the source moments from Eqs. (10) using the full set of Eqs. (11). This yields four expressions, for each method, relating the nodal flux-moments to the transverse flux-moments. The coefficients appearing in these equations can be written in terms of the spatial weights defined by

$$\alpha_{k,i} = \left[\coth \delta_k - \frac{1}{\delta_k}\right] \Big/ \left[1 - \frac{3\nu_i}{\delta_k}\left(\coth \delta_k - \frac{1}{\delta_k}\right)\right] , \tag{12.a}$$

$$\beta_{k,i} = \left[\coth \epsilon_k - \frac{1}{\epsilon_k}\right] \Big/ \left[1 - \frac{3\nu_i}{\epsilon_k}\left(\coth \epsilon_k - \frac{1}{\epsilon_k}\right)\right], \quad i = 0,1 , \tag{12.b}$$

to become,

$$\left(\frac{1 + \beta_{k,o}}{2}\right) \psi_{k,oy}(+b) + \left(\frac{1 - \beta_{k,o}}{2}\right) \psi_{k,oy}(-b) = \psi_{k,oo} + \beta_{k,o}\psi_{k,o1}, \tag{13.a}$$

$$\left(\frac{1 + \beta_{k,1}}{2}\right) \psi_{k,1y}(+b) + \left(\frac{1 - \beta_{k,1}}{2}\right) \psi_{k,1y}(-b) = \psi_{k,1o} + \nu_1\beta_{k,1}\psi_{k,11}$$
$$- \frac{3\tilde{\lambda}}{2\delta_k} \beta_{k,1}\left[\psi_{k,x1}(+a) - 2\psi_{k,o1} + \psi_{k,x1}(-a)\right], \tag{13.b}$$

where

$$\tilde{\lambda} = \lambda_1(1 - \nu_1) - 1, \text{ if LL method,} \quad (14)$$

$$= 0, \text{ otherwise.}$$

Two equations analogous to Eqs. (13), with the x-traverse flux moments appearing on the LHS, can be derived also. This set of four equations now replaces Eqs. (10) and their x-moment analogues, and are solved simultaneously with Eqs. (11), using the Nodal or the WDD algorithm.

In Sec. IV we will compare the equations for the three methods from the theoretical point of view, then in Sec. V we will present numerical results for test problems solved by the three methods to compare their performance. It is worthwhile mentioning that while the LN and LL method formalisms are not new, and have been derived and implemented before,[1-6] writing their final equations in the simple WDD form of Eqs. (13) is new.

IV. COMPARISON OF THE THREE LINEAR-LINEAR NODAL METHODS

In the absence of a complete and rigorous error analysis of nodal transport methods in general, and the three methods considered here in particular, choosing a certain method to solve a given problem is often based on intuition and experience. From the comparisons presented in this and the following section, this author is convinced that none of the methods dominates the other two. That is for each problem to be solved a compromise has to be reached between the available resources (such as the memory size, the computer speed, etc.) and the achieved results (such as accuracy, simplicity of method, and solution algorithm, etc.). The main differences between the three methods are summarized in Table I, and are discussed in the remaining part of this section in more detail.

A. The LN method

For the LN method there are two distinct spatial weights per dimension per distinct discrete-ordinate for each node type. This is the simplest of the three methods because it has the least coupling between the algebraic equations; however, it contains the least analytical information. In fact as in the NEFD algorithm[3], the LN method equations can be manipulated to obtain uncoupled expressions for the nodal flux-moments in terms of the incoming transverse flux-moments.

An interesting fact about the expressions for the spatial weights, Eqs. (12) is that they are odd functions of the discrete ordinate. For example, $\alpha_{k,i}(-\mu_k) = -\alpha_{k,i}(\mu_k)$. This significantly reduces the storage requirement, and simplifies the solution procedure by allowing us to write the final set of equations in terms of incoming and outgoing transverse flux-moments which is required by the sweep methodology. For example, for $\mu_k \gtrless 0$, the outgoing transverse flux-moment $\psi^o_{k,x,i} = \psi_{k,x,i}(\pm a)$ and the incoming transverse flux-moment $\psi^i_{k,x,i} = \psi_{k,x,i}(\mp a)$. The balance equations then become,

$$\frac{1}{2\delta_k}\left[\psi^o_{k,xo} - \psi^i_{k,xo}\right] + \frac{1}{2\epsilon_k}\left[\psi^o_{k,oy} - \psi^i_{k,oy}\right] + \psi_{k,oo} = S_{oo}/\sigma \ , \quad (15.a)$$

$$\frac{3s_x}{2\delta_k}\left[\psi^o_{k,xo} - 2\psi_{k,oo} + \psi^i_{k,xo}\right] + \frac{1}{2\epsilon_k}\left[\psi^o_{k,1y} - \psi^i_{k,1y}\right] \quad (15.b)$$

$$+ \psi_{k,1o} = S_{1o}/\sigma \ ,$$

$$\frac{1}{2\delta_k}\left[\psi^o_{k,x1} - \psi^i_{k,x1}\right] + \frac{3s_y}{2\epsilon_k}\left[\psi^o_{k,oy} - 2\psi_{k,oo} + \psi^i_{k,oy}\right] \tag{15.c}$$

$$+ \psi_{k,o1} = S_{o1}/\sigma .$$

The WDD relations become,

$$\left(\frac{1+\alpha_{k,o}}{2}\right)\psi^o_{k,xo} + \left(\frac{1-\alpha_{k,o}}{2}\right)\psi^i_{k,xo} = \psi_{k,oo} + s_x\alpha_{k,o}\psi_{k,1o} , \tag{16.a}$$

$$\left(\frac{1+\alpha_{k,1}}{2}\right)\psi^o_{k,x1} + \left(\frac{1-\alpha_{k,1}}{2}\right)\psi^i_{k,x1} = \psi_{k,o1} , \tag{16.b}$$

and two analogous equations for the y-moments, where $s_x \equiv \text{sign}(\mu_k)$, $s_y \equiv \text{sign}(\eta_k)$, and $\delta_k, \epsilon_k, \alpha_{k,i}, \beta_{k,i}$ represent the absolute values of the corresponding quantities.

The Nodal algorithm is obtained by using Eqs. (16) and their analogues to replace the nodal flux-moments in Eqs. (15). This results in a set of three coupled equations that must be solved simultaneously for the outgoing flux-moments in terms of the incoming flux-moment. Alternatively a WDD algorithm (equivalent to the NEFD[3]) is obtained by solving Eqs. (16) and their analogue for the outgoing flux-moments, then substituting the result in Eqs. (15). This yields three equations which can be solved for the nodal flux-moments before they are coded, or can be solved numerically for each node. In the code we developed, and for which the results presented in Sec. V were obtained we chose to implement the second option for its simplicity.

B. The LL method

The LL method also has two distinct spatial weights per dimension, per distinct ordinate, per node type. It has a higher degree of coupling than the LN and BL methods, which may explain why a WDD algorithm has never been implemented for it yet. The LL method retains more analytical information in the discrete-variable equations than the LN method, manifested in the higher order representation of the leakage terms.

The nodal balance equations are the same as Eqs. (15). The WDD equations are,

$$\left(\frac{1+\alpha_{k,o}}{2}\right)\psi^o_{k,xo} + \left(\frac{1-\alpha_{k,o}}{2}\right)\psi^i_{k,xo} = \psi_{k,oo} + s_x\alpha_{k,o}\psi_{k,1o} , \tag{17.a}$$

$$\left(\frac{1+\alpha_{k,1}}{2}\right)\psi^o_{k,x1} + \left(\frac{1-\alpha_{k,1}}{2}\right)\psi^i_{k,x1} = \psi_{k,o1} \tag{17.b}$$

$$- \frac{3\alpha_{k,1}s_x}{2\epsilon_k s_y}\left[\psi^o_{k,1y} - 2\psi_{k,1o} + \psi^i_{k,1y}\right] ,$$

and two analogous equations in the y-moments. Just like the LN method, a Nodal algorithm, and a WDD algorithm are possible here. To obtain the Nodal algorithm, Eq. (17.b) and its analogue are solved for $\psi_{k,o1}$ and $\psi_{k,1o}$, then $\psi_{k,oo}$ is obtained from the analogue of Eq. (17.a) to obtain four coupled equations in the four outgoing flux-moments. On the other hand, the WDD algorithm is obtained by individually solving Eq. (17.a) and its analogue for $\psi^o_{k,xo}$ and $\psi^o_{k,oy}$, and simultaneously solving Eq. (17.b) and its analogue for $\psi^o_{k,x1}$ and $\psi^o_{k,1y}$. The resulting expressions are substituted in the balance equations yielding three coupled equations which must be solved simultaneously for the nodal flux-moments. In our implementation of this method we analytically eliminated $\psi^o_{k,xo}$ and $\psi^o_{k,oy}$ using Eq. (17.a) and its analogue, then numerically solved the remaining five equations in each node.

C. The BL method

The BL method is unique in that it requires the calculation and storage of only one spatial weight per dimension per distinct discrete-ordinate for every node typed. Of course, it contains the most analytical information in the algebraic equations. It has more coupling than the LN method, but less than the LL method. However, it requires the calculation of an additional angular flux-moment, and the storage of an additional scalar flux-moment. In the context of the two test problems of Sec. V, there seems to be very little gain in accuracy due to using the BL method as opposed to the LN or LL methods. Problem configurations may occur, however, which lead to bilinear moments of relatively large magnitude that would cause the BL method to be more accurate than the, otherwise almost identical, LN and LL methods.

For the BL method, the balance equations, Eqs. (15) must be augmented by a conservation equation for the bilinear nodal flux-moment,

$$\frac{3s_x}{2\delta_k}\left[\psi^o_{k,x1} - 2\psi_{k,o1} + \psi^i_{k,x1}\right] + \frac{3s_y}{2\epsilon_k}\left[\psi^o_{k,1y} - 2\psi_{k,1o} + \psi^i_{k,1y}\right] + \psi_{k,11} = S_{11}/\sigma \,. \tag{18}$$

The WDD relations are given by,

$$\left(\frac{1+\alpha_k}{2}\right)\psi^o_{k,xo} + \left(\frac{1-\alpha_k}{2}\right)\psi^i_{k,xo} = \psi_{k,oo} + s_x\alpha_k\psi_{k,1o}\,, \tag{19.a}$$

$$\left(\frac{1+\alpha_k}{2}\right)\psi^o_{k,x1} + \left(\frac{1-\alpha_k}{2}\right)\psi^i_{k,x1} = \psi_{k,o1} + s_x\alpha_k\psi_{k,11}\,, \tag{19.b}$$

with analogous equations in the y-moments. By similar arguments as before, one can derive both a Nodal and a WDD algorithm for the BL method. The algorithm we adopted in our code was derived by solving Eqs. (19) and their analogue for the outgoing flux moments, which are then eliminated from the balance equations, Eqs. (15) and (18). This yields four coupled equations in four nodal flux-moment, which are solved simultaneously in every node.

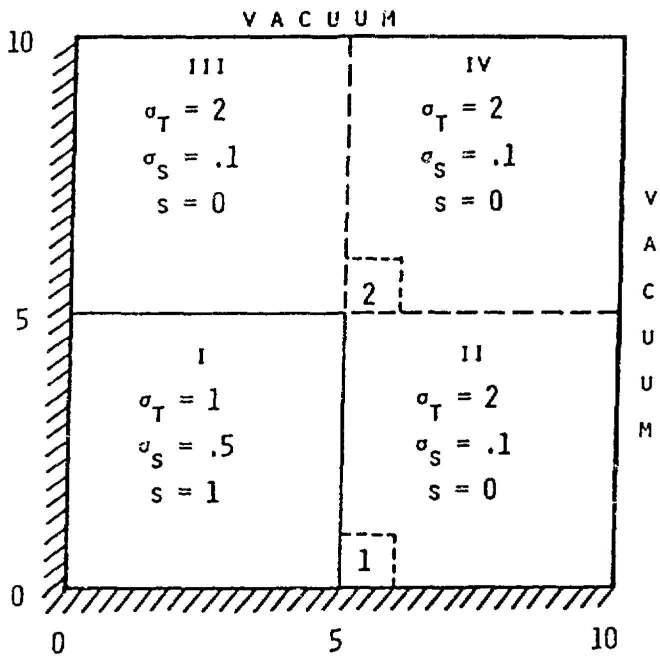
V. NUMERICAL COMPARISON OF THE THREE LINEAR-LINEAR NODAL METHODS

We have implemented the equations for the LN, LL, and BL methods derived above in a computer code using the algorithms described in Sec. IV to solve them numerically. These are, in some sense, a combination of the Nodal and WDD algorithms; the final equations are coupled (like the Nodal algorithm) and are solved simultaneously for the nodal flux-moments (like the WDD algorithm). This was preferred to a pure WDD algorithm because it is much simpler and can be extended easily to three-dimensional geometry. In order to monitor easily the memory size required to solve a problem, we used a container array methodology, in which all arrays have variable lengths and are contained in one array of fixed length, the container array, A.

Two test problems were solved using the three methods in order to compare several of their computational aspects, and also to validate the correctness of our derivation of the WDD form.

The first test problem is shown in Fig. 1. This problem was solved on a sequence of meshes using an S-4 EQN type angular quadrature, and a pointwise, relative convergence criterion of 10^{-4} on each one of the calculated nodal flux-moments. The converged solution was used to calculate the quadrant-averaged scalar fluxes, and flux averages over the two regions denoted 1 and 2 shown in Fig. 1. This was to make possible comparisons of both global quantities as well as relatively local quantities.

Fig. 1. Geometry and nuclear data for test problem 1.



The results of the calculations are shown in Table II. Also shown is the length of the container array, the number of iterations and CPU time (in seconds) required to achieve convergence, and the number of calculated negative fluxes. In general, the LN method takes the shortest CPU time to converge, followed by the BL method, while the LL method takes longest.

This is not surprising since in the algorithm we adopted, three, four, and five equations, respectively, are solved simultaneously for each direction per node. Of course, the BL method consumes more memory than the other two methods because it requires the storage of two additional arrays containing the old and new bilinear moments of the scalar flux. Therefore, in problems with only a few material regions, such as the two cases presented here, the BL method requires more storage; however, for problems with many material compositions, or high order angular quadrature, and hence many distinct spatial weights, the BL method may require a smaller storage area. The calculated quantities seem to change only very little with the method, especially at locations where the flux is relatively large. In fact, several quantities calculated by the three methods on the same mesh are identical. Moreover, all three methods yield, approximately, the same number of negative scalar fluxes.

The second test problem is a modification of Khalil's steel and water problem[9] with a smaller average scattering ratio to avoid excessively slow convergence since our code is not equipped with acceleration methods yet. The geometrical configuration and nuclear data are shown in

Fig. 2. Geometry and nuclear data for test problem 2.

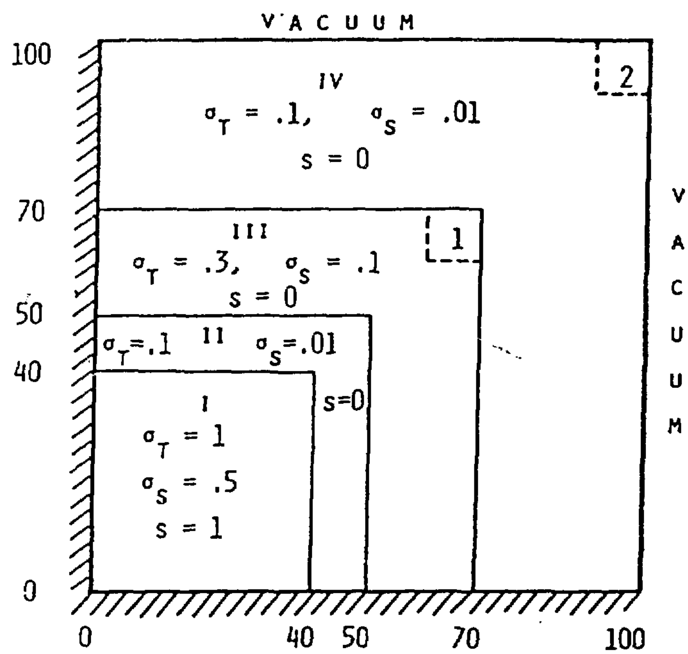


Fig. 2. We used an S-4, EQN type angular quadrature with a 10^{-4} pointwise relative convergence criterion on all the calculated nodal moments of the scalar flux by each method. The results for this problem presented in Table III suggest conclusions identical to those reached from the results of the first test problem. Here, the difference between the solutions obtained by the three methods is more pronounced especially at low flux regions when coarse meshes are used. However, the solutions seem to converge to one another when finer meshes are used.

VI. SUMMARY AND CONCLUSIONS

We have rederived the Linear Nodal (LN) and the Linear Linear (LL) methods for solving the neutron transport equations, which have been shown previously to possess very high accuracies on coarse meshes. Our main objective was to simplify the final equations for these methods whose difficulty limited the wide spread practical use of high order nodal methods. We were able to write the final equations for these two methods, as well as a third new method, the Bi-Linear (BL) method, in a simple Weighted Diamond Difference (WDD) form. Unlike traditional WDD methods, here the spatial weights are derived analytically from the equations rather than preassumed.

The LN method is the simplest of the three methods. The bilinear term in the scattering source is neglected and the expansion of the transverse leakage term in the linear moments equation is truncated at the zeroth order. Yet, the LN has been successfully applied by other authors to obtain accurate solutions to transport problems. In the LL method, the transverse leakage is expanded in a linear series in the equations for the constant and linear moments. Hence, it contains more analytic information but is far more complex than the LN method. The accuracy of the LL method also has been demonstrated elsewhere. The BL method, which differs from the LL method only in that it retains the bilinear moments of the scattering source, has not been considered before, except in an approximate sense. The BL method is the most complete analytically, at the expense of calculating and storing an additional nodal flux-moment, the bilinear moment. On the other hand its WDD form is written in terms of only one spatial weight (versus two for LN and LL) per distinct discrete ordinate for each node type. Furthermore, the BL final equations are less coupled than the LL equations, which makes their numerical solution faster, in spite of the additional effort expended in calculating the additional moment.

We coded the three methods in one computer program, and used them to solve two test problems in order to verify the derivations and to compare the computational performance of the three methods. The numerical results indicate that the three methods solutions are very close to one another especially on fine meshes. The three methods seem to differ most at locations of very low flux levels. It is very difficult to reach rigorous conclusions from the results of two test problems. Additional work on the error analysis of these methods will be necessary in order to identify classes of problems to which each method is more suitable to apply.

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Table I. Summary of Comparison Between the Three Linear-Linear Methods.

	LN	LL	BL
Number of weights/ordinate/dimension/node	2	2	1
Number of discrete-variable unknowns/ordinate/node	7	7	8
Include bilinear flux moment	No	No	Yes
Include leakage bilinear component	No	Yes	Yes
Coupling of WDD equations in different dimensions	No	Yes	No
WDD algorithm possible	Yes	Yes	Yes
Number of equations solved simultaneously in WDD algorithm	1	4	4
Max coupling* in WDD equations	4	6	4
Min coupling* in WDD equations	3	4	4
Max coupling* in Balance equations	6	6	7
Min coupling* in Balance equations	5	5	5

*Max (min) coupling in a set of equations is the max (min) number of unknown variables appearing in any equation in the set.

Table II. Comparison of the numerical solutions by the three methods for test problem 1 (see Fig. 1). Calculations were performed on ORNL's Vax 8600.

Mesh	I	II	IV	1	2	Length of A	# of its	CPU (sec)	Neg. fluxes
				LN Method					
10x10	1.676	.4170E-1	.1986E-2	.2151	.4148E-1	724	19	19	8
20x20	1.676	.4161E-1	.1990E-2	.2145	.4006E-1	2564	21	83	16
40x40	1.676	.4159E-1	.1992E-2	.2144	.4003E-1	9844	19	286	52
				LL Method					
10x10	1.676	.4170E-1	.1989E-1	.2151	.4207E-1	724	17	26	6
20x20	1.676	.4161E-1	.1991E-1	.2145	.4001E-1	2564	19	115	18
40x40	1.676	.4159E-1	.1992E-1	.2144	.4004E-1	9844	19	474	40
				BL Method					
10x10	1.676	.4169E-1	.1996E-1	.2151	.4218E-1	898	24	30	6
20x20	1.676	.4160E-1	.1992E-1	.2145	.4004E-1	3338	23	113	18
40x40	1.676	.4159E-1	.1992E-1	.2144	.4004E-1	13018	23	463	40

Table III. Comparison of the numerical solutions by the three methods for test problem 2 (see Fig. 2). Calculations were performed on ORNL's Vax 8600.

Mesh	I	II	III	IV	1	2	Length of A	# of its	CPU (sec)
					LN Method				
10x10	1.953	.3650	.1626E-1	.2222E-4	-.3909E-4	.3948E-6	940	23	23
20x20	1.955	.3504	.1556E-1	.2727E-4	.7959E-5	.2789E-7	2780	23	84
40x40	1.957	.3388	.1516E-1	.2696E-4	.9325E-5	.2761E-7	10060	30	428
80x80	1.957	.3339	.1504E-1	.2682E-4	.9408E-5	.2729E-7	39020	30	1684
					LL Method				
10x10	1.953	.3651	.1623E-1	.2376E-4	-.3037E-4	-.1989E-7	940	23	37
20x20	1.955	.3504	.1556E-1	.2727E-4	.8524E-5	.2693E-7	2780	23	138
40x40	1.957	.3388	.1516E-1	.2696E-4	.9579E-5	.2744E-7	10060	29	691
80x80	1.957	.3339	.1504E-1	.2682E-4	.9403E-5	.2725E-7	39020	29	2748
					BL Method				
10x10	1.953	.3651	.1623E-1	.2231E-4	-.2170E-4	-.4675E-7	1042	23	28
20x20	1.955	.3503	.1556E-1	.2726E-4	.8935E-5	.2707E-7	3482	24	112
40x40	1.957	.3388	.1516E-1	.2696E-4	.9584E-5	.2744E-7	13162	31	559
80x80	1.957	.3339	.1504E-1	.2682E-4	.9407E-5	.2725E-7	51722	30	2275