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EXAMPLES OF HARMONIC MAP HEAT FLOW IN DIMENSION TWO

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EXAMPLES OF HARMONIC MAP HEAT FLOW IN DIMENSION TWO *

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ABSTRACT

An example of explosion of the heat flow at infinite time in dimension two and an arbitrary degree is given.

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1. Let $D^2 \subset R^2$ be the unit disk and S^2 be the standard 2–sphere. It is known (see [GH] and [CD]) that there exists a Brouwer degree one map from D^2 to S^2 whose extension domain along the harmonic map heat flow is $D^2 \times [0, \infty)$ and the extension blows up at infinity. In this note, we construct some map with this property and its Brouwer degree is any nonzero integer n. Denote * to be the north pole of the sphere. Let

$$\mathcal{M} = \{u \in C^1(D^2, S^2),\ u^{-1}(-*) = \partial D \quad \text{and} \quad u^{-1}(*) = \{0\}\}$$

let (r, θ) be the polar coordinates at $0 \in D^2$ and let (R, Φ) be the polar coordinates of S^2 at *. We say a map $u : D \to S^2$ is n–symmetric if u is of the form $u(r, \theta) = (R, n\theta)$. Obviously, an n–symmetric map in $\mathcal{M}$ is of degree n. The harmonic map heat flow (h.m.h.f.) of an n–symmetric map u_0 can be reduced to the following equation:

$$\partial_t R = \partial_r^2 R + r^{-1}\partial_r R + \frac{n^2 \sin 2R}{2r^2} \ \forall (r,t) \in [0, \pi] \times (0, T) \tag{1}$$

where T is the maximum extension time. It is easily verified that

$$R_n(r) = 2tg^{-1} a r^n$$

is a stationary solution of h.m.h.f. for any constant $a > 0$. The parameter a can be considered as the blow up speed at 0. Let $B_n(r,t)$ be an n–symmetric map with image in a small neighbourhood and $B_n^{-1}(*) = \partial D^2 \cup \{0\}$. Let $a_n(t)$ satisfy

$$2tg^{-1} a_n(t) \left(\frac{\pi}{2}\right)^n > R_n\left(\frac{\pi}{2}, t\right).$$

Then replacing the $R_1(r), a(t)$ in [GH] by $B_n(r), a_n(t)$ respectively, we can define an upper barrier function

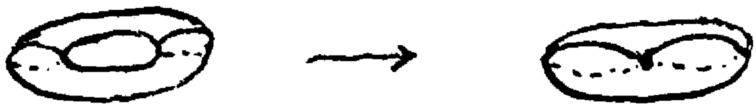
$$B(r,t) = \begin{cases} min(2tg^{-1}a_n(t) r^n,\ B_n(r,t)), & \text{if} \quad r < \frac{\pi}{2}; \\ B_n(r,t) & \text{otherwise}. \end{cases}$$

So we may use the standard argument (see [GH]) to obtain a global extension of any n–symmetric map in $\mathcal{M}$. By a result of L. Lemaire (see [EL]), this global extension must blow up at infinity.

2. Let $T^2 = S^1 \times S^1$ be the flat torus and (r, θ) be the coordinates with ranges $0 \le r \le 2\pi, 0 \le \theta \le 2\pi$. Let

$$g(r) = \begin{cases} r^2, & \text{if} \quad r \le \pi; \\ (2\pi - r)^2, & \text{otherwise} \end{cases}$$

and define a warped torus $\mathcal{T}^2$ with a pseudometric $ds^2 = dr^2 + g(r)d\theta^2$. Since $g(r)$ is Lipschitz continuous, $\mathcal{T}^2$ is a Lipschitz degenerate Riemannian torus. The change from T^2 to $\mathcal{T}^2$ can be expressed as follows:



In particular, it identifies $0 \times S^1$ into one point. Let m,n be two positive integers. We call a map $u : T^2 \to S^2$ (m,n)–type if

$$u(r,\theta) = \begin{cases} (R_1(r), m\theta), & \text{if} \quad r \leq \pi; \\ (R_2(2\pi - r), -n\theta), & \text{otherwise} \end{cases} \tag{2}$$

with $R_1(r), R_2(r) \in \mathcal{M}$. So an (m,n)–type map is of Brouwer degree $m - n$. As in point 1, we can extend any (m,n)–type map

$$u_0(r,\theta) = \begin{cases} (R_{01}(r), m\theta), & \text{if} \quad r \leq \pi; \\ (R_{02}(2\pi - r), -n\theta), & \text{otherwise} \end{cases}$$

to an H^1–solution of the harmonic map heat flow for all positive time by restricting Eq.(1) to R_1 and R_2 with $[0, \pi]$ and $[\pi, 2\pi]$ respectively. The solution defined by (2) blows up at $\{0\} \times S^1$ at infinity if $m - n$ is nonzero.

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