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Abstract

The slow relaxation of isolated toroidal plasmas towards their thermodynamical equilibrium is studied in an Onsager framework based on the entropy metric. The basic tool is a variational principle, equivalent to the kinetic equation, involving the profiles of density, temperature, electric potential, electric current. These profiles enter in two functionals reflecting respectively the entropy of the field plasma-plasma system and the entropy production rate. These functionals are symmetrical. By themselves, they would drive an Onsager evolution of the system. However the variational principle contains also an antisymmetrical functional reflecting the trajectory effects. The latter is eliminated, so that the Onsager relaxation is automatically established in the situations of low collisionality where the trajectories are integrable and close to the magnetic surfaces (e.g. in axisymmetric tokamaks). In such situations the Onsager character of the slow relaxation is a mere consequence of the hamiltonian nature of the field-plasma system. In the collisional or non integrable cases, an Onsager evolution may be still derived from the variational principle, but the plasma layers around the successive magnetic surfaces must be independent enough, in the sense that unconfined trapped particles are forbidden unless they are detrapped long before they depart significantly from the magnetic surfaces. New minimization procedures are proposed to obtain entropy and entropy production rate functionals expressed in terms of the profiles of density, etc., which drive the Onsager relaxation of the profiles. Onsager relaxations are possible in the presence of a turbulent field, either in an integrable situation (e.g. well separated magnetic islands) or in a non integrable case (overlapping islands). The variational principle then involves the characteristic frequencies of the turbulent field, on the same grounds to the profiles of density, etc.

1. INTRODUCTION

Onsager (Onsager 1931) has proposed on the basis of time reversibility arguments that the relaxation of an isolated system close to its thermodynamical equilibrium state, if it is very slow, conforms to the following scheme: the small deviation from the thermodynamical equilibrium is specified by parameters Λ^a, making the vector $\Lambda = (\Lambda^a)$, which determine the entropy S of the system with respect to the equilibrium as a negative quadratic form

$$S_{ab}\Lambda^a\Lambda^b = S \tag{1}$$

where of course $S_{ab} = S_{ba}$; the relaxation of the vector $\Lambda(t)$ towards the equilibrium state $\Lambda = 0$ is expressed by an equation of the form

$$2S_{ab}\frac{d\Lambda^b(t)}{dt} = \dot{S}_{ab}\Lambda^b \tag{2}$$

with the key symmetry relations

$$\dot{S}_{ab} = \dot{S}_{ba} \tag{3}$$

The equation (2) implies that the variation in the course of time of the entropy $\frac{dS}{dt} = 2S_{ab}\frac{d\Lambda^a}{dt}\Lambda^b$ is equal whatever Λ to the quadratic form $\dot{S}_{ab}\Lambda^a\Lambda^b$. The latter is therefore identified with the positive rate of entropy production within the system due to the irreversible collisional process. It is usual to use the entropy $S_{ab}\Lambda^a\Lambda^b$ as a metric in the space Λ and to introduce the covariant coordinates Λ_a of Λ, namely $\Lambda_a = S_{ab}\Lambda^b$. The equation (2), written $2\frac{d\Lambda_a(t)}{dt} = \dot{S}_{ab}\Lambda^b$, then gives the "fluxes" $2\frac{d\Lambda_a}{dt} = 2S_{ab}\frac{d\Lambda^b}{dt}$ in terms of the "forces" Λ^a, the dot product (fluxes. forces) being equal to $\frac{dS}{dt}$. Following the ideas of Prigogine (Glansdorff and Prigogine 1971), one may put the scheme (1,2,3) in a variational form. Let us use the two symmetric tensors S_{ab} and $\dot{S}_{ab}$ to generate the two symmetric bilinear forms $S(\Lambda,\underline{\Lambda}) = S(\underline{\Lambda},\Lambda) = S_{ab}\Lambda^a\underline{\Lambda}^b$ and $\dot{S}(\Lambda,\underline{\Lambda}) = \dot{S}(\underline{\Lambda},\Lambda) = \dot{S}_{ab}\Lambda^a\underline{\Lambda}^b$ in Λ and $\underline{\Lambda}$. Here $\underline{\Lambda}$ is a working version of Λ to be varied around the physical Λ. The equations (2,3) are equivalent to state that, at each time, one or the other of the two functions of $\underline{\Lambda}$

$$-2S(\frac{d\Lambda}{dt},\underline{\Lambda}) + \dot{S}(\Lambda,\underline{\Lambda}) \quad \text{or} \quad -4S(\frac{d\Lambda}{dt},\underline{\Lambda}) + \dot{S}(\underline{\Lambda},\underline{\Lambda}) \tag{4}$$

is an extremum for all the variations of $\underline{\Delta}$ around Λ. The vector $\Lambda(t)$ is a sum of modes $\lambda\exp(-\gamma t)$, where each vector λ minimizes the entropy production $\dot{S}(\lambda,\lambda)$ under the constraint of performing a constant entropy $S(\lambda,\lambda)$, and $\gamma=\frac{\dot{S}}{2S}$. The goal of this article is to show, starting from the kinetic and the Maxwell equations, that the magnetically confined toroidal plasmas conform under some conditions to the scheme (1-4).

The Onsager scheme typically applies in two kinds of situation. A first possibility is that the isolated system is made of independent particles exhibiting stationary states, and relaxes towards its thermodynamical equilibrium because a small resonant hamiltonian perturbation induces a diffusion among the stationary states. For instance the perturbation consists of weak hamiltonian interactions between the particles. In its principle such a situation is easy to understand in a quantum mechanical framework. The system formed by the unperturbed, independent particles exhibits eigenstates N of energy h_N. A small deviation from the thermodynamical equilibrium is specified at a given time by the vector Λ representing the deviation of P_N, the probability of finding the system in the various states N, with respect to the maxwellian. Such a vector Λ determines of course the entropy S of the system with respect to the thermodynamical equilibrium. A slow relaxation of Λ takes place when a weak hamiltonian perturbation induces small transition probabilities per unit of time $W_{N\to M}$ from a state N to a state M. Of course $h_N = h_M$ if the perturbation is not oscillating in time, i.e. is static or quasi static. Under some conditions, $W_{N\to M}$ may be deduced from the classical perturbation theory for the systems depending on time (Kemble 1937). A consequence of that theory is the detailed balance equations $W_{N\to M}=W_{M\to N}$ which reflects the hermiticity of the interaction hamiltonian. Van Kampen (Van Kampen 1954, 1957) establishes a master equation giving the evolution in time of the probability P_N in terms of the transition probabilities $W_{N\to M}$. That equation gains of course a degree in symmetry if one takes into account that $W_{N\to M}=W_{M\to N}$. It results, if the perturbation is quasi static, in an Onsager relaxation (1-4) for the vector Λ (see Appendix 2). The obtained Onsager relaxation relies on the hamiltonian behaviour of the system, without reference to the reversibility in time of its trajectories which is the basis of the original argument of Onsager. The Van Kampen line may therefore justify such a relaxation in a system which does not enjoy that reversibility, for instance in a magnetized plasma. It must be stressed that it has a counterpart in classical mechanics, namely the quasilinear diffusion theory. Let us consider a classical hamiltonian system formed by N quasi independent particles whose individual motion is integrable. The unperturbed system exhibits stationary states specified by $3N$ action variables $\boldsymbol{J}$, playing now the role of the quantum label N. A resonant hamiltonian perturbation, if it is weak enough but however satisfies the Chirikov

criterium, drives a quasilinear diffusion of the system in space $\boldsymbol{J}$ (Rosenbluth et al 1966, Zaslavsky and Chirikov 1972, Rechester et al 1981). The quasilinear diffusion equation is the continuous classical version of the quantum Van Kampen master equation when the symmetries $W_{N\to M} = W_{M\to N}$ have been duly taken into account. If the perturbation is static or quasi static, a relaxation (1-4) applies to the vector Λ representing the deviation from maxwellian of the probability density of the system in the $\boldsymbol{J}$ space, or as well of the distribution of the individual particles in their 3D action space $\mathbf{J}$ (see Appendix 2).

However, the most usual situations of Onsager relaxation have a different character. The isolated system is a medium formed of particles which deviates from its thermodynamivcal equilibrium because for instance the temperature and the densities of the various particle species depend on the position. One may consider that the medium consists of small cells which are nearly independent and individually close to a thermodynamical equilibrium, and that it relaxes as a whole to its thermodynamical equilibrium because of the exchanges between neighbouring cells (Kreuzer 1981, p 5). The key of the situation is that these exchanges are small so that the relaxation of the system is slow. Strong interactions between the particles are therefore necessary to prevent a fast migration of the particles from a cell to its neighbours. This appparently opposes the present situation to the situation of nearly independent particles considered above. Nevertheless the two situations present a deep similarity, namely, the nearly independence of the cells on one side and of the particles on the other. The vector Λ specifying the deviation from thermodynamical equilibrium of the medium represents the spatial variations of temperature, densities, etc, which indeed determine the entropy of the system. The classical demonstration of the Onsager scheme (1-4) in that case relies on the reversibility in time of the fluctuations of the vector Λ. The probability of finding the system in the state Λ at a given time is determined by the entropy $S_{ab}\Lambda^a\Lambda^b$ of the system with respect to the thermodynamical equilibrium, namely, it is proportional to $\exp(S_{ab}\Lambda^a\Lambda^b)$. The averages $<\Lambda_a(t)\Lambda_b(t)>$ along the fluctuating trajectory $\Lambda(t)$ are then proportional to S_{ab}. The first element of the demonstration is that the correlations $<\Lambda_a(t)\Lambda_b(t+\tau)>$ can be calculated for large positive τ as if $\Lambda(t+\tau)$ could be derived from $\Lambda(t)$ by simply applying the relaxation equation (2). That point, curiously presented as obvious in many textbooks (Landau and Lifshitz 1958), is in fact not so easy to establish (Kreuzer 1981, p 44, Krommes and Hu 1993). At that stage, it appears that the symmetry (3) is insured if one may state that $<\Lambda_a(t)\Lambda_b(t+\tau)> = <\Lambda_b(t)\Lambda_a(t+\tau)>$. One then introduces the second element of the demonstration: at thermodynamical equilibrium, the fluctuating trajectories $\Lambda(t)$ and $\Lambda(-t)$ are equally probable, so that $<\Lambda_a(t)\Lambda_b(t+\tau)> = <\Lambda_a(t)\Lambda_b(t-\tau)> = <\Lambda_a(t+\tau)\Lambda_b(t)>$. This is justified, in the absence of magnetic field, by the reversibility of the trajectories $\boldsymbol{x}(t),\boldsymbol{V}(t) \Rightarrow \boldsymbol{x}(-t), -\boldsymbol{V}(-t)$, where $\boldsymbol{x}$ represents the set of the particle positions and $\boldsymbol{V}$ the set of the particle velocities, under

the condition that all the components of Λ reflects observables $O(\boldsymbol{x},\boldsymbol{V})$ exhibiting the same parity in $\boldsymbol{V}$. It is no longer justified when the vector Λ mixes observables with odd and even parity, or in presence of a magnetic field since the trajectory $\boldsymbol{x}(t),\boldsymbol{V}(t)$ does not imply the trajectory $\boldsymbol{x}(-t)$, $-\boldsymbol{V}(-t)$. In those cases, the symmetry (3) cannot be justified by arguments drawn only from the fluctuation analysis. Those arguments allow however to relate the components $\dot{S}_{ab}$ of the relaxation tensor, as it is defined by the equation (2), for opposite magnetic configurations $\mathbf{B}$ and $-\mathbf{B}$: assuming that each covariant component Λ_a reflects observables with a definite parity $\varepsilon_a = \pm 1$ in $\boldsymbol{V}$, one has $\dot{S}_{ab}(\mathbf{B}) = \varepsilon_a\varepsilon_b\dot{S}_{ba}(-\mathbf{B})$ (Casimir 1945, Onsager and Machlup 1953, Fitts 1962). Those relations, outside the scheme (1-4), are often considered as forming the Onsager scheme itself. To establish the scheme (1-4), i.e. the relations $\dot{S}_{ab}(\mathbf{B}) = \dot{S}_{ba}(\mathbf{B})$, via the Λ fluctuation analysis, one must introduce by a way or another further informations allowing to relate the values of $\dot{S}_{ab}(\mathbf{B})$ and $\dot{S}_{ab}(-\mathbf{B})$. This line has been followed by Boozer for the tokamak case (Boozer 1992).

The two lines - nearly independent particles or nearly independent cells - correspond to actual Onsager situations in magnetically confined toroidal plasmas. The weakly collisional plasmas in axisymmetric tokamak configurations belong to the first category with nearly independent particles, since each particle of the confined plasma has individually an integrable motion close to the magnetic surfaces, described by 3 action variables $\mathbf{J}$. Weak hamiltonian interactions are then expected to induce a quasilinear diffusion of the particle system in its action space, resulting in an Onsager relaxation (1-4) for the deviation Λ from maxwellian of the distribution of the particles in the space $\mathbf{J}$. That line is underlying in the work of Mynick and Duvall in the framework of a generalized Balescu-Lenard collision operator (Mynick and Duvall 1989). In the present article, we derive a variational principle equivalent to the usual kinetic equation, which determines the evolution in time t of the deviation U with respect to the maxwellian of the particle distribution in the usual phase space $\mathbf{x},\mathbf{p}$. In its final expression that principle determines the relaxation of the field-plasma system formed by the plasma and the confining field determined by the Maxwell equations. In a weakly collisional axisymmetric tokamak, our general principle involving U directly results in the principles (4) for the vector Λ representing the deviation with respect to the maxwellian of the distribution of the particles in their $\mathbf{J}$ space. One may then show that the principles (4) apply as well to a reduced vector Λ representing the variations of density and temperature, the electric currents, etc, which determine the entropy of the isolated system. The considered Onsager relaxations appear to be a mere consequence of the weakly collisional regime, of the integrability of the individual trajectories and of the basic symmetry of the Fokker Planck collision operator in the kinetic equation, which is itself guaranteed as soon as the weak interaction between the particles is hamiltonian (see Appendix 2). The situation is different in the general case of toroidal plasmas which are

in a collisional regime, or which involves class of trapped particles that do not exhibit an integrable individual motion close to the magnetic surfaces. These plasmas behave as systems formed of quasi independent cells, which of course are now thin plasma layers around the succesive magnetic surfaces. We still derive from our general variational principle involving U the principles (4) for the vector Λ representing the density and temperature variations, the electric currents etc (Nguyen 1992), but it appears that a minimum collisionality is necessary when a class of trapped particles experience a transverse drift across the whole of the configuration: these trapped particles must be detrapped by collisions before they significantly depart from the magnetic surfaces. That condition is precisely that which prevents a strong coupling through the unruly particles between the layers around the successive magnetic surfaces. A nearly independence of these layers is therefore necessary to insure an Onsager relaxation. Our demonstration of the Onsager relaxations from the kinetic equation is not so general as in the weakly collisional situations with integrable individual trajectories: we are now restricted to interaction hamiltonians which are symmetric with respect to an inversion of the guiding centre velocities along the flux lines. This recalls the basic role plaid by the reversibility of the trajectories $\boldsymbol{x}$ (t),$\boldsymbol{V}$ (t) and $\boldsymbol{x}$ (-t), -$\boldsymbol{V}$(-t) in the Λ fluctuation analysis. However it is not at all proved that the restriction corresponds to a physical necessity.

At this point, our work complements the important studies of Balescu (Balescu 1991) and of Sugama and Horton (Sugama and Horton 1996). In these studies one derives from the drift kinetic equation, at first order in the ratio $\frac{\rho_c}{L}$ of the Larmor radius to the plasma scale, the structure of the particle distribution functions, being given the gradients ∇n, ∇T of the density n and temperature T as well as the inductive electric field E. Such a structure then leads to the average particle and energy fluxes Γ, Γ_E across the magnetic surfaces and to the electric currents I along these surfaces. It is shown that the transport matrix linking those elements exhibits Onsager symmetries. The weakly collisional situations with integrable individual trajectories are not considered. Because the drift kinetic equation is treated only at first order in $\frac{\rho_c}{L}$, the necessity of a collisional detrapping of the unconfined trapped particles is not met. Interestingly, the demonstration of the Onsager symmetries by Sugama and Horton clearly relies on the symmetry of the collisions with respect to the inversion of the parallel motion. On the other hand our work is consistent with the study of Boozer who has accurately pointed out the thermodynamic constraints which apply to the Onsager forces and fluxes in toroidal plasmas (Boozer 1992). Within the scheme (1-4), the forces Λ^a are defined by the prescription that they determine the entropy $S = S_{ab}\Lambda^a\Lambda^b$ of the isolated system, the fluxes being then

$2S_{ab}\frac{d\Lambda^b}{dt}$. In our toroidal plasmas, materially and electromagnetically isolated, the forces Λ^a must then represent the density and temperature variations, i.e. typically the gradients ∇n and ∇T, and also the currents I along the magnetic surfaces: indeed those currents control through the Ampere law and the boundary conditions the magnetic energy, and thereby the entropy of the isolated field-plasma system, since any increase or decrease of the magnetic energy decreases or increases the part of the fixed total energy which remains available for thermal motions. Since the forces Λ^a involve the currents I, the fluxes $2S_{ab}\frac{d\Lambda^b}{dt}$ must involve, in addition to the particle and heat fluxes Γ, Γ_E representing the time derivatives of the density and temperature profiles, the time derivative of I, i.e. the time derivative of the relaxing magnetic field, i.e. finally the inductive electric field E. Boozer has indeed proposed forces and fluxes with that organization. On the contrary the matrix displayed by Balescu and by Sugama and Horton links "conventional" forces and fluxes, namely, forces proportional to ∇n, ∇T and E, and fluxes to Γ, Γ_E and I. The normalization is of course such that the dot product (fluxes. forces) produces the time derivative $\frac{dS}{dt}$ of the entropy S whatever ∇n, etc, but it is known that if a series of characteristics X^a, Φ_a of the state of the system at each time satisfy the identity $(\Phi_a X^a) = \frac{dS}{dt}$, this is not sufficient to guarantee that the quantities X^a and Φ_a form a system of forces Λ^a conform to the prescription (1) and fluxes $2S_{ab}\frac{\partial\Lambda^b}{\partial t}$ (Coleman and Truesdell 1960). In fact the matrix linking ∇n, ∇T and E to Γ, Γ_E and I is not symmetrical and considerations involving the parity in the particle velocities are necessary to reveal the Onsager symmetries. Let us remark that in a tokamak where the plasma is rotating around the major axis, the corresponding rotational energy plays the role of the magnetic energy above: the scheme (1-4) then designates the electric voltage across the magnetic surfaces, which controls that energy, among the forces Λ^a; the time derivative of the rotational velocity participates in the fluxes $2S_{ab}\frac{d\Lambda^b}{dt}$. Generally it will appear in this article that, through a variational principle of type (4), it is possible to cover complex aspects of the collisional relaxation of toroidal plasmas, for instance the role of the various electric fields, the deformation in time of the magnetic surfaces, etc.

Beyond the collisions, one may wonder whether the toroidal plasmas experience an Onsager relaxation when the confinement triggers an electromagnetic turbulence. It has been pointed out by Shaing (Shaing 1988) that, in weakly collisional regimes, if the turbulent field induces a quasilinear diffusion of the particles, the symmetries of that diffusion imply an Onsager matrix between the fluxes Γ, Γ_E and appropriate Onsager forces, consisting of linear combinaisons of $\nabla n, \nabla T$ and of the mode frequencies ω.

Similar conclusions may be driven from the works of Mynick and Duvall (Mynick and Duvall 1989) and Morozov and coworkers (Morozov et al 1988). The fluxes Γ, Γ_E are no longer determined, as in the collisional transport case, by the friction forces between the various class of particles, proportional to the differences between the corresponding diamagnetic velocities, proportional themselves to ∇n, ∇T: they are now determined by the friction F exerted by the turbulent modes on the various classes of particles, proportional to the differences between the phase velocities $\frac{\omega}{k}$ of the modes and the diamagnetic velocities. This explains the dependence on ω of the Onsager forces considered by Shaing. Sugama and coworkers (Sugama and Horton 1995, Sugama et al 1996) incorporate among the Onsager fluxes the heating power $W = \frac{\omega}{k} F$ which is produced by the modes for finite ω values. By introducing an Onsager force of type $\frac{1}{T}$ as conjugate of W, they display an elegant Onsager matrix between the Onsager fluxes Γ, Γ_E, W and Onsager forces of type ∇n, ∇T, $\frac{1}{T}$, independent of the frequencies ω. In fact, in the studied situation where the turbulent field at a finite ω is imposed by external means independent of the plasma, the latter is submitted to a constant heating power and formally relaxes towards an equilibrium state where $\frac{1}{T} = 0$. In this article we take another point of view, again inspired by the scheme (1-4). The slow relaxation at constant energy of an isolated field-plasma system will be studied by treating the turbulent field as a simple component of the consistent electromagnetic field. As that turbulent field is due to the confinement, its frequencies ω relaxes to 0 when the thermodynamical equilibrium is reached. Indeed our variational formalism leads us to base the Onsager relaxations upon a vector Λ representing the deviation U with respect to the maxwellian of the particle distribution, together with the frequencies ω. For a weak collisionality, a first possibility is that the trajectories are integrable in presence of the turbulent field. This may be the case for instance if the latter produces small, well separated magnetic islands. We then find that the principle (4) applies to a vector Λ representing the density and temperature variations, the electric currents, etc, and also the frequencies ω representing the island rotation velocities, which typically minimize the collisional entropy production rate. If the integrability of the trajectories is destroyed since the magnetic islands overlap, one is led to Onsager relaxations based upon a quasilinear expression of the entropy production rate. In all cases, the various irreversible process induced by the turbulent field - friction forces, heating power, etc - are naturally taken into account by the principle (4).

2. PLASMAS IN AN IMPOSED CONFINING FIELD

2.1 VARIATIONAL PRINCIPLE EQUIVALENT TO THE KINETIC EQUATION

The basic tool for the study of the slow relaxation of a magnetically confined plasma is the kinetic equation which gives the evolution in time t of the distribution function F(x,p,s,t) of the particles of each species s in the six dimensional phase space $\mathbf{x},\mathbf{p} = (x_1,x_2,x_3,p_1,p_2,p_3)$ (see Appendix 1)

$$\frac{\partial F(\mathbf{x},\mathbf{p},s,t)}{\partial t} + \{H,F\} = C(F)$$

$$\{H,F\} = \frac{\partial H}{\partial \mathbf{p}} \cdot \frac{\partial F}{\partial \mathbf{x}} - \frac{\partial H}{\partial \mathbf{x}} \cdot \frac{\partial F}{\partial \mathbf{p}}$$

where $H(\mathbf{x},\mathbf{p},s,t) = \frac{(p - e(s)\mathbf{A}(\mathbf{x},t))^2}{2m(s)} + e(s)\Psi(\mathbf{x},t)$ is the hamiltonian created at time t by the confining field $\mathbf{B} = \nabla \times \mathbf{A}(\mathbf{x},t), \mathbf{E} = -\nabla\Psi(\mathbf{x},t) - \frac{\partial \mathbf{A}(\mathbf{x},t)}{\partial t}$, the Poisson bracket $\{H,F\}$ express the variation rate of F along the trajectories and C(F) is the Fokker-Planck collision term. We assume that the system is isolated by proper particle and field barriers and relaxes towards a well defined thermodynamical equilibrium, characterized by the relaxed field $\mathbf{B}_0 = \nabla \times \mathbf{A}_0(\mathbf{x}), \mathbf{E}_0 = -\nabla\Psi_0(\mathbf{x})$, corresponding to the hamiltonian $H_0(\mathbf{x},\mathbf{p},s) = \frac{(\mathbf{p} - e(s)\mathbf{A}_0(\mathbf{x}))^2}{2m(s)} + e(s)\Psi_0(\mathbf{x})$, and by the maxwellian distribution function

$$F_0(\mathbf{x},\mathbf{p},s) = \frac{\nu_0(s)}{(2\pi T_0 m(s))^{3/2}} \exp\left(\frac{-H_0(\mathbf{x},\mathbf{p},s)}{T_0}\right) \qquad (5)$$

The basic constants $\nu_0(s)$ and T_0 are determined by the initial content in particles of each species and in energy of the isolated system. It appears convenient to characterize at each time t the small deviation of the system with respect to that thermodynamical equilibrium by the dynamical variable $U(\mathbf{x},\mathbf{p},s,t) << T_0$ such that

$$F(\mathbf{x},\mathbf{p},s,t) = \frac{\nu_0(s)}{(2\pi T_0 m(s))^{3/2}} \exp\left(\frac{-H(\mathbf{x},\mathbf{p},s,t) + U(\mathbf{x},\mathbf{p},s,t)}{T_0}\right) \qquad (6)$$

The average velocity of the particles of species s at the position $\mathbf{x}$ is given by

$$\mathbf{V}_{average}(\mathbf{x}, s, t) = \frac{1}{n}\int F\frac{\partial H}{\partial \mathbf{p}}d_3p = \frac{1}{n}\int F\frac{\partial U}{\partial \mathbf{p}}d_3p \ ; \ n = \int F d_3p$$

It is worth noticing that the average at given $\mathbf{x}$ and s of the velocities $\mathbf{V}(\mathbf{x},\mathbf{p},s,t) = \frac{\partial H}{\partial \mathbf{p}}$ is also the average of the derivatives $\frac{\partial U}{\partial p}$. These derivatives will be met below as macroscopic velocities, for instance as diamagnetic velocities, exhibited by the various species or subspecies. The collision term C(F) in the kinetic equation is obviously proportional to the deviation U at first order. At a given t we have C(F) = D(U) where D is a well defined linear operator acting in the space formed by the dynamical variables function of x,p,s: namely, D($\mathbf{U}$(x,p,s,t)) is obtained by replacing the differences $\frac{\partial F(\mathbf{x},\mathbf{p},s,t)}{F(\mathbf{x},\mathbf{p},s,t)\partial \mathbf{p}} - \frac{\partial F(\mathbf{x},\mathbf{p}',s',t)}{F(\mathbf{x},\mathbf{p}',s',t)\partial p}$ which are at the heart of the Landau collision integrals by $\frac{\partial \mathbf{U}(\mathbf{x},\mathbf{p},s,t)}{T_0\partial p} - \frac{\partial \mathbf{U}(\mathbf{x},\mathbf{p}',s',t)}{T_0\partial p}$. On the other hand the variation rate $\{H,F\}$ may be written $\frac{F}{T_0}\{H,U\}$. We introduce a working dynamical variable $\underline{U}(\mathbf{x},\mathbf{p},s,t)$, to be varied around the physical deviation U($\mathbf{x}$,$\mathbf{p}$,s,t). The kinetic equation is exactly equivalent to the following principle: at each time the linear functional in $\underline{U}$

$$\sum_s\int \frac{1}{T_0}\frac{\partial F}{\partial t}\underline{U}d_3xd_3p + \sum_s\int \frac{F}{T_0^2}\{H,U\}\underline{U}d_3xd_3p - \sum_s\int \frac{1}{T_0}D(U)\underline{U}d_3xd_3p \qquad (7)$$

is an extremum (null) for all the variations of $\underline{U}$.

Generally the deviation U determines the entropy S of the system with respect to the thermodynamical equilibrium U = 0 and the rate of collisional production of entropy. These quantities are related to the first and the third terms of the functional (7). We first connect the third term to the collisional entropy production. The starting point is the basic symmetry of the collision operator, resulting from the hamiltonian nature of the weak interactions between the particles (see Appendix 2), which imposes that

$$\sum_s\int D(U')U''\,d_3xd_3p = \sum_s\int D(U'')U'd_3xd_3p \qquad (8a)$$

whatever U'($\mathbf{x}$,$\mathbf{p}$,s,t) and U"($\mathbf{x}$,$\mathbf{p}$,s,t). This allows to express the third term of the functional (7) as $\dot{S}(U,\underline{U})$ where $\dot{S}$ is the symmetrical bilinear form in U', U" defined at a given time t by

$$\dot{S}(U',U'') = \dot{S}(U'',U') = -\sum_s\int \frac{1}{T_0}D(U')U''\,d_3xd_3p \qquad (8b)$$

The rate of collisional entropy production is equal to $\dot{S}(U,\underline{U})$. Indeed, the entropy S with respect to the thermodynamical equilibrium is given by

$$S = -\left(\sum_s \int F\ln(F)d_3x d_3p - \sum_s \int F_0\ln(F_0)d_3x d_3p\right) \qquad (9)$$

Therefore the rate of collisional entropy production is equal to $-\sum_s \int C(F)(1+\ln(F))d_3x d_3p$, i. e. to $-\sum_s \int D(U)\left(1+\ln(\frac{\nu_0}{(2\pi T_0 m)^{3/2}})+\frac{-H+U}{T_0}\right)d_3x d_3p$. In view of the conservation of particles and energy under collisions: $\int D(U)d_3x d_3p = 0$ and $\sum_s \int D(U)Hd_3x\, d_3p = 0$, it becomes equal to $-\sum_s \int \frac{1}{T_0}D(U)Ud_3x d_3p$, i. e. to $\dot{S}(U,\underline{U})$. For reference we notice that

$$\dot{S}(\mathbf{U},\underline{\mathbf{U}}) = \frac{1}{2T_0^2}\sum_{ss'}\int F(\mathbf{x},\mathbf{p},s,t)F(\mathbf{x},\mathbf{p}',s',t)\,d_3x d_3p d_3p'$$

$$X_{kl}\left(\frac{\partial \mathbf{U}(\mathbf{x},p,s,t)}{\partial p_k} - \frac{\partial \mathbf{U}(x,p',s',t)}{\partial p_k}\right)\left(\frac{\partial \mathbf{U}(x,p,s,t)}{\partial p_l} - \frac{\partial \mathbf{U}(x,p',s',t)}{\partial p_l}\right)$$

$$X_{kl} = X_{lk} = 2\pi e(s)^2 e(s')^2 \ln(\Lambda)\cdot\frac{W\delta_{kl} - W_k W_l}{|W|^3};\quad W = \frac{\partial H(x,p,s,t)}{\partial p} - \frac{\partial H(x,p',s',t)}{\partial p}$$

We will often use below that $\dot{S}(U,\underline{U})$ is the sum of symmetric bilinear forms $X_{kl}\left(\frac{\partial U(x,p,s,t)}{\partial p_k} - \frac{\partial U(x,p',s',t)}{\partial p_k}\right)\left(\frac{\partial \underline{U}(x,p,s,t)}{\partial p_l} - \frac{\partial \underline{U}(x,p',s',t)}{\partial p_l}\right)$ for all possible x and p,s.

The second term of the functional (7) reflecting the motion of the individual particles in the hamiltonian H may be expressed at a given t as $\Sigma(U,\underline{U})$ where

$$\Sigma(U',U'') = \sum_s \int \frac{\nu_0}{(2\pi T_0 m)^{3/2} T_0^2}\exp\left(\frac{-H+U'}{T_0}\right)\{H,U'\}U''\,d_3x d_3p \qquad (10a)$$

It is readily verified by integrations by parts in space x,p that

$$\Sigma(U,\underline{U}) = \sum_s \int \frac{-F}{T_0}\{H,\underline{U}\}d_3x d_3p \qquad (10b)$$

so that $\Sigma(U,\underline{U})$ cancels exactly if $\{H,U\} = 0$ or $\{H,\underline{U}\} = 0$, i. e. if one of the dynamical variables U or $\underline{U}$, is a constant of motion for the hamiltonian H at the considered time t. At first order in U, the functional $\Sigma(U,\underline{U})$ is an antisymmetrical bilinear form in $U,\underline{U}$

$$\Sigma(U,\underline{U}) = \sum_s \int \frac{-F}{T_0^2}\{H,\underline{U}\}U d_3x d_3p = \sum_s \int \frac{-F}{T_0}\{U,\underline{U}\} d_3x d_3p = -\Sigma(\underline{U},U)$$

In view of the equations (8) and (10), the variational principle (7) reads (Samain and Werkoff 1977)

$$\sum_s \int \frac{1}{T_0}\frac{\partial F}{\partial t}\underline{U} d_3x d_3p + \Sigma(U,\underline{U}) + \dot{S}(U,\underline{U}) \text{ extremum in } \underline{U} \qquad (11)$$

As $\Sigma(U,H) = 0$ and $\dot{S}(U,H) = 0$ we obtain by using the principle (11) with a variation of $\underline{U}$ proportional to H

$$\sum_s \int \frac{1}{T_0}\frac{\partial F}{\partial t} H d_3x d_3p = 0 \qquad (12)$$

For the sake of clarity, we will indeed in the rest of the present chapter 2 consider isolated systems of particles confined in an imposed static field rather than field-plasma systems. Namely, rather than letting the field $\mathbf{B}(\mathbf{x},t), \mathbf{E}(\mathbf{x},t)$ to adjust to the electric currents and charges developped by the plasma particles, we will impose a static, purely magnetic configuration $\mathbf{B} = \mathbf{B}_0 = \nabla \times \mathbf{A}_0(x)$, $\mathbf{E} = 0$ by fitting at each time electric currents and charges independent of the plasma. The relaxation that we study then involves a system of particles governed by the static hamiltonian $H = H_0 = \frac{(\mathbf{p} - e_s\mathbf{A}_0(\mathbf{x}))}{2m_s}$, and of course by the interaction hamiltonian. As $\frac{\partial H}{\partial t} = 0$, the derivative $\frac{\partial F}{\partial t}$ in the first term of the functionals (7) or (11) reduces to $F\frac{\partial U}{T_0 \partial t}$. At a given time t, we have

$$\sum_s \int \frac{1}{T_0}\frac{\partial F}{\partial t}\underline{U} d_3x d_3p = -2S(\frac{\partial U}{\partial t},\underline{U}) \qquad (13)$$

where S(U',U") is the symmetrical bilinear form defined by

$$S(U',U'')=\sum_s \int \frac{-F}{2T_0^2} U'U'' d_3x d_3p \qquad (14)$$

We will find in the § 3.1 that the relation (13) is still applicable in the case of isolated field-plasma systems, with a new definition of the symmetrical bilinear form S involving the field perturbation created by the deviation U. In all cases the principles (11) or (7) equivalent to the kinetic equation become

$$-2S\left(\frac{\partial U}{\partial t},\underline{U}\right)+\Sigma(U,\underline{U})+\dot{S}(U,\underline{U}) \qquad \text{or} -4S\left(\frac{\partial U}{\partial t},\underline{U}\right)+2\Sigma(U,\underline{U})+\dot{S}(\underline{U},\underline{U}) \qquad (15)$$

is an extremum for all the variations of $\underline{U}$ around U.

The functionals $\dot{S}(U',U''),\Sigma(U',U''),S(U',U'')$ defined by the equations(8,10,14) depend on the state of the system, determined by the deviation U, at the considered time t. The principles (15) then give the variation $\frac{\partial U}{\partial t}$ exactly. However it is only if S(U',U") is taken at zeroth order in U that the quantity S(U,U) is equal to the entropy S given by the equation (9). Let us stress that this equality is not necessary for the exact principles (15) to be valid. The equation $S(U,U) = S$ at second order in U may be directly established from the equations (9,14) by expressing that the number of particles and the system energy is the same in the state U and in the relaxed thermodynamical state U = 0. One may as well notice that, at second order in U, we have $\frac{d}{dt}(S(U,U))=S(\frac{\partial U}{\partial t},U)+S(U,\frac{\partial U}{\partial t})$. Because of the symmetry $S(U,\underline{U}) = S(\underline{U},U)$ and of the equation (13), we then have $\frac{dS(U,U)}{dt}=2S(\frac{\partial U}{\partial t},U)=-\sum_s \int \frac{1}{T_0}\frac{\partial F}{\partial t} U d_3x d_3p$. On the other hand, by taking into account the particle conservation $\int \frac{\partial F}{\partial t} d_3x d_3p=0$ and the equation (12), we obtain

$$\frac{d}{dt}S=-\sum_s \int \frac{\partial F}{\partial t}\ln(F) d_3x d_3p=-\sum_s \int \frac{\partial F}{\partial t}\frac{-H+U}{T_0} d_3x d_3p=-\sum_s \int \frac{1}{T_0}\frac{\partial F}{\partial t} U d_3x d_3p \qquad (16)$$

so that $\frac{d}{dt}S=\frac{d}{dt}S(U,U)$. That equation during any relaxation process leading to the thermodynamical equilibrium state where $S=0, S(U,U)=0$ implies that $S(U,U)=S$.

2.2 WEAKLY COLLISIONAL, INTEGRABLE REGIMES

At a given time the functional $\Sigma(U,\underline{U})$ cancels if U or $\underline{U}$ is a constant of motion such that $\{H,U \text{ or } \underline{U}\}=0$. If it is not the case, $\Sigma(U,\underline{U})$ is proportional to the variation rate of U or $\underline{U}$ along the trajectories produced by H at time t. These variations take place at a characteristic frequency $\frac{1}{\tau_{kinetic}} \sim \frac{\{H,U \text{ or } \underline{U}\}}{U \text{ or } \underline{U}}$ which, according to the structure of U and $\underline{U}$, may be a Larmor frequency ω_c or a parallel transit frequency $\frac{V_{G//}}{L}$ of the guiding centre or a still slower transverse drift frequency $\frac{V_{G\perp}}{L}$. Roughly we have, N being the number of particles

$$\Sigma(U,\underline{U}) \sim N \frac{U\underline{U}}{T_0^2 \tau_{kinetic}} \text{ if } \{H,U\} \text{ and } \{H,\underline{U}\} \neq 0 ; \Sigma(U,\underline{U})=0 \text{ if } \{H,U\} \text{ or } \{H,\underline{U}\}=0 \quad (17a)$$

The two other functionals $S(\frac{\partial U}{\partial t},\underline{U})$ and $\dot{S}(U,\underline{U})$ which enter in the principle (15) are proportional to the relaxation rate $\frac{1}{\tau_{relaxation}}$ and to a pertinent collisional rate $\frac{1}{\tau_{collision}}$ again depending on the structure of U and $\underline{U}$

$$S(\frac{\partial U}{\partial t},\underline{U}) \sim N \frac{U\underline{U}}{T_0^2 \tau_{relaxation}} \text{ and } \dot{S}(U,\underline{U}) \sim N \frac{U\underline{U}}{T_0^2 \tau_{collision}} \quad (17b)$$

Let us show that an Onsager relaxation automatically occurs in weakly collisional cases where $\frac{1}{\tau_{collision}}$ and $\frac{1}{\tau_{relaxation}}$ are small enough compared to $\frac{1}{\tau_{kinetic}}$, if the motion in the hamiltonian H is integrable. The latter condition means that at each time t there exists angular and action variables for the hamiltonian H, namely, 3 angular variables $\Phi(\mathbf{x},\mathbf{p},s,t)= (\Phi_1,\Phi_2,\Phi_3)$ defined modulo 2π, canonically conjugate to 3 action variables $\mathbf{J}(\mathbf{x},\mathbf{p},s,t) = (J_1,J_2,J_3)$, the hamiltonian H being a function $h(\mathbf{J},s,t)$ of the action variables $\mathbf{J}$. The action variables $\mathbf{J}$ (and the variation rates $\{H,\Phi\}= \frac{\partial h(\mathbf{J},s,t)}{\partial \mathbf{J}}$) are constants of motion. The trajectories in the 6 dimensional space $\mathbf{x},\mathbf{p}$ determined by H at the considered time t are indefinitely drawn on 3 dimensional toruses labelled by the 3 values $\mathbf{J}$ and univoquely parametrized by the 3 angular variables Φ. Any constant of motion $Z(\mathbf{x},\mathbf{p},s,t)$ is a function $z(\mathbf{J},s,t)$. With a static hamiltonian $H = H(\mathbf{x},\mathbf{p},s)$, the dynamical variables Φ and $\mathbf{J}$ are of course also static. In view of the estimations (17), a situation where $\frac{1}{\tau_{kinetic}}$ is large enough compared to $\frac{1}{\tau_{collision}}$ and $\frac{1}{\tau_{relaxation}}$ implies that the variations of $\Sigma(U,\underline{U})$ when one varies $\underline{U}$ largely dominate the variations of

$S(\frac{\partial U}{\partial t},\underline{U})$ and $\dot{S}(U,\underline{U})$, except if $\{H,U\} \neq 0$. The principle (15) then effectively obliges that $\{H,U\} \neq 0$, i. e. obliges that $U \neq u(\mathbf{J},s,t)$, $F \neq f(\mathbf{J},s,t)$. We define $f(\mathbf{J},s,t)$ at each time t as the Φ average value $<F>_{\mathbf{J}}$ of $F(\mathbf{x},\mathbf{p},s,t)$ over the 3D torus in space $\mathbf{x},\mathbf{p}$ specified by $\mathbf{J}$, and use the relation (6) to define $u(\mathbf{J},s,t)$ in terms of $f(\mathbf{J},s,t)$ and $h(\mathbf{J},s,t)$. Our basic trick is to choose the working dynamical variable $\underline{U}(\mathbf{x},\mathbf{p},s,t)$ so that it is an exact function $\underline{u}(\mathbf{J},s,t)$ and therefore $\{H,\underline{U}\}=0$. The functional $\Sigma(U,\underline{U})$ then cancels exactly and may be discarded from the principle (15). If the collisionality and therefore the difference $U' = U - u(\mathbf{J},s,t)$ are weak enough, one may replace in the remaining functionals $S(\frac{\partial U}{\partial t},\underline{U})$ and $\dot{S}(U,\underline{U})$ the dynamical variables $U(\mathbf{x},\mathbf{p},s,t)$ and $\frac{\partial U(\mathbf{x},\mathbf{p},s,t)}{\partial t}$ by the good approximations $u(\mathbf{J},s,t)$ and $\frac{\partial u(\mathbf{J},s,t)}{\partial t}$. The principle (15) then determines at all orders in U the relaxation of the function $u(\mathbf{J},s,t)$ by imposing that at each time

$$-4S\left(\frac{\partial \underline{u}(\mathbf{J},s,t)}{\partial t},\underline{u}(\mathbf{J},s,t)\right)+\dot{S}(\underline{u}(\mathbf{J},s,t),\underline{u}(\mathbf{J},s,t)) \qquad (18)$$

is an extremum for all the variations of $\underline{u}(\mathbf{J},s,t)$ around $u(\mathbf{J},s,t)$. The principle (18) obviously implies an Onsager relaxation for the vector $\Lambda(t)$ representing the function $u(\mathbf{J},s,t)$ of arguments $\mathbf{J},s$ at each time. This is normal since we are in a Van Kampen situation of a system of individually integrable particles relaxing towards a thermodynamical state under the action of a small interaction hamiltonian. Putting

$$u(\mathbf{J},s,t) = \Lambda^a(t)u_a(\mathbf{J},s)\ ;\ \underline{u}(\mathbf{J},s,t)=\underline{\Lambda}^a(t)\,u_a(\mathbf{J},s) \qquad (19a)$$

where the functions $u_a(\mathbf{J},s)$ form a proper basis, the vector $\Lambda = (\Lambda^a)$ satisfies the principle (4), with the symmetric tensors S_{ab} and $\dot{S}_{ab}$ given by

$$S_{ab}=S(u_a(\mathbf{J},s),u_b(\mathbf{J},s))\ ,\ \dot{S}_{ab}=\dot{S}(u_a(\mathbf{J},s),u_b(\mathbf{J},s)) \qquad (19b)$$

Let us notice that the notations of type $u(\mathbf{J},s,t)$ have 2 different meanings according to the context: at the considered time t, either $u(J,s,t)$ is a dynamical variable function of $\mathbf{x},\mathbf{p},s$ via $\mathbf{J}(\mathbf{x},\mathbf{p},s,t)$, as for instance in $\dot{S}(u,u)$ and in the derivatives $\frac{\partial u}{\partial \mathbf{p}}$ that we will use below; or $u(\mathbf{J},s,t)$ is a simple function of arguments $\mathbf{J},s$, as in the equation (19a). We stress that the principle (18) exactly produces at a given time the derivative $\frac{d\Lambda}{dt}$ of the vector Λ representing the function $u(\mathbf{J},s,t)$: in other words it is not restricted to the linear range in Λ. This is simply due to the fact that the bilinear forms $\dot{S}(U',U'')$ and $S(U',U'')$ are defined at each time t by taking into account the state of the system at that time, and will remain

valid with the field-plasma systems considered in the chapter 3. The study of an Onsager relaxation is of course greatly simplified if S(U',U") is taken at zeroth order in Λ, i.e. in U. However the fact that the Onsager relaxation covered by the principle (18) is not restricted to the linear range in Λ is important in principle. It meets the preoccuppations of several authors (Krommes and Hu 1993), even if the principle (18) directly issued from the kinetic equation introduces the non linearities in Λ by a very different way.

We notice also that, for the principle (18) to be applicable, the 3 dynamical variables $\mathbf{J}(\mathbf{x},\mathbf{p},s)$ need not to be really action variables: they may be as well 3 independent static functions of the action variables, i.e. 3 independent static constants of motion, that we will still note $\mathbf{J}(\mathbf{x},\mathbf{p},s)$. The condition of validity of the principle (18) is that $\dot{S}(U,\underline{u}) \# \dot{S}(u,\underline{u})$ or $\dot{S}(U',\underline{u}) \ll \dot{S}(u,\underline{u})$. One may estimate $U' = U - u \# U - \langle U\rangle_{\mathbf{J}}$ from the principle (15): the latter approximately states that $\Sigma(U',\underline{U}') + \dot{S}(u,\underline{U}')$ is an extremum in $\underline{U}'$ submitted to the constraint $\langle \underline{U}'\rangle_{\mathbf{J}}=0$. Using the equations (8,10) it first appears that the structure of U' reflects the variations W of the dynamical variable $D(u(\mathbf{J},s,t))$ over each 3D torus $\mathbf{J}$. Putting $U' \sim \lambda W$ and $\frac{1}{\tau_{kinetic}} \sim \left|\frac{\{H,W\}}{W}\right|$, it then comes that $N\ \frac{\lambda\underline{\lambda}W^2}{T_0^2\tau_{kinetic}} + \underline{\lambda}\dot{S}(u,W)$ is roughly an extremum in $\underline{\lambda}$, i. e. $\lambda \sim \frac{1}{N}\frac{\dot{S}(u,W)}{W^2/T_0^2}\tau_{kinetic}$. The validity condition is $\lambda\dot{S}(W,\underline{u}) \ll \dot{S}(u,\underline{u})$. In view of the Schwarz inequality $|\dot{S}(u,W)| < (\dot{S}(u,u)\dot{S}(W,W))^{1/2}$, it is satisfied if $\frac{1}{\tau_{kinetic}} >> \frac{1}{\tau_{collision}}$ where $\frac{1}{\tau_{collision}} \sim \frac{1}{N}\frac{\dot{S}(W,W)}{W^2/T_0^2}$.

To be useful the vector Λ must involve a number of components Λ^a as reduced as possible. We therefore abandon the definitions (19) representing the full structure of the function u(J,s,t), and try to build up a simplest Λ space. We will build up at each time, within the vectorial space (u) formed by all the functions u(J,s,t), two complementary subspaces (u_I) and (u_{II}) such that, if one decomposes the functions u achieved during the relaxations in the form $u = u_I + u_{II}$, the entropy S(u,u) is close to $S(u_I,u_I)$. The principle (18) then nearly states that $-4\dot{S}(\frac{\partial u_I}{\partial t},\underline{u}_I) + \dot{S}(\underline{u}_I+\underline{u}_{II},\underline{u}_I+\underline{u}_{II})$ is an extremum in $\underline{u}_I$ around u_I within (u_I) and in $\underline{u}_{II}$ around u_{II} within (u_{II}). The minimum value of $\dot{S}(\underline{u}_I+\underline{u}_{II},\underline{u}_I+\underline{u}_{II})$ when one varies $\underline{u}_{II}$ within the subspace (u_{II}) for given $\underline{u}_I$ is a quadratic form $\dot{S}_{reduced}(\underline{u}_I,\underline{u}_I)$. The evolution of the function $u_I(\mathbf{J},s,t)$ is such that at each time

$$-4\dot{S}(\frac{\partial u_I}{\partial t},\underline{u}_I) + \dot{S}_{reduced}(\underline{u}_I,\underline{u}_I) \qquad (20)$$

is an extremum for all the variations of $\underline{u}_I(\mathbf{J},s,t)$ around $u_I(\mathbf{J},s,t)$within the subspace (u_I). The new vector Λ represents the function $u_I(\mathbf{J},s,t)$. Being $(u_{Ia}(\mathbf{J},s))$ a basis for the vectorial subspace (u_I), we replace the definitions (19) by

$$u_I(\mathbf{J},s,t) = \Lambda^a(t)u_{Ia}(\mathbf{J},s) \; ; \; \underline{u}_I(\mathbf{J},s,t) = \underline{\Lambda}^a(t)u_{Ia}(\mathbf{J},s)$$
$$S_{ab} = S(u_{Ia}(\mathbf{J},s),u_{Ib}(\mathbf{J},s)) \; ; \; \dot{S}_{ab} = \dot{S}_{reduced}(u_{Ia}(\mathbf{J},s),u_{Ib}(\mathbf{J},s))$$

A convenient starting point to build up the subspaces (u_I) and (u_{II}) is the working assumption, to be verified a posteriori, that the values of S(u,u) achieved by the system during the relaxation process are mainly determined by the values $\Lambda_a(t)$ of some typical characteristics of the deviation from thermodynamical equilibrium, for instance typical temperature and density variations, typical average velocities, etc. Those values Λ_a will appear below as the covariant components $S_{ab}\Lambda^b$ of the new vector Λ. At a given time, by imposing the set of values Λ_a, one imposes constraints of the following form to the function $u(\mathbf{J},s,t)$

$$L_a(u) = \Lambda_a \qquad (21)$$

where each $L_a(u)$ is a linear form in u. We associate $u_I(J,s,t)$ to the set (Λ_a), namely, u_I is the function u which minimizes the entropy S(u,u) under the constraints (21). Our working assumption means that the functions u_I achieved during the relaxation process are such that S(u,u) and $S(u_I,u_I)$ are approximately equal. The accuracy of that equality is in fact the accuracy of the principle (20) with respect to the principle (18). Of course the minimization leading to u_I has not to be performed with a better accuracy. The basis $u_{Ia}(\mathbf{J},s)$ introduced above is obtained from that minimization: using Lagrange multiplicators, it appears that $u_I = \Lambda^a u_{Ia}$ where each function $u_{Ia}(\mathbf{J},s)$ is determined by the linear form L_a, namely, is such that $S(u_{Ia},u') \neq L_a(u')$ whatever $u'(\mathbf{J},s)$. The quantities Λ^a and Λ_a then appear as the contravariant and covariant components of the vector Λ since $\Lambda_a = L_a(u_I) = \Lambda^b L_a(u_{Ib}) \neq \Lambda^b S(u_{Ia},u_{Ib}) = S_{ab}\Lambda^b$. The vectorial subspace (u_{II}) is formed by all the functions $u(\mathbf{J},s,t)$ such that $L_a(u) = 0$ for all a values. It is "perpendicular" to the subspace (u_I) in the sense that $S(u',u'') \neq 0$ whatever u' within (u_I) and u'' within (u_{II}). We therefore have $S(u_I+u_{II},u_I+u_{II}) \neq S(u_I,u_I) + S(u_{II},u_{II})$ so that the working assumption $S(u,u) \neq S(u_I,u_I)$ may be written $S(u_{II},u_{II}) \ll S(u_I,u_I)$.

Let us apply the principle (20) in a fixed, purely magnetic, axisymmetrical tokamak configuration. All the trajectories, either trapped or passing, are integrable and close to the magnetic surfaces. We may form $\mathbf{J}$ with the 3 exact constants of motion H,μ,Rp_φ where the hamiltonian $H(\mathbf{x},\mathbf{p},s)$ is the kinetic energy $\frac{mV^2}{2}$ (since $\Psi = 0$), the

magnetic moment $\mu(\mathbf{x},\mathbf{p},s) = \frac{mV_\perp{}^2}{2B}(1 + O(\frac{\rho_c}{L}))$ specifies the amplitude of the Larmor motion and the angular momentum Rp_φ in the direction φ around the major axis specifies the magnetic surface occupied by the particle: indeed $Rp_\varphi = ReA_\varphi + RmV_\varphi = e\psi_{pol}(v(\mathbf{x})) + RmV_\varphi = e\psi_{pol}(1 + O(\frac{\rho_c}{L}))$, where $v(\mathbf{x})$ is the volume of the magnetic surface passing through $\mathbf{x}$ and $\psi_{pol}(v)$ is $\frac{1}{2\pi}\times$the magnetic flux embrassed by the major turns drawn on the magnetic surface of volume v. Any trajectory $\mathbf{J} = (H,\mu,M)$ is close to the magnetic surface specified by the poloidal flux $\psi_{pol} = \frac{M}{e}$. That situation allows to a weakly collisional plasma exhibiting $F \# f(\mathbf{J},s,t)$ and $U \# u(\mathbf{J},s,t)$ to be in quasi thermodynamical equilibrium on each magnetic surface and at the same time to exhibit any density and temperature profiles from the magnetic axis to the surrounding wall. However, in the Onsager line that we follow, we assume a weak deviation $U << T_0$ of the plasma from its thermodynamical equilibrium. We assume the plasma delimited and isolated by two hamiltonian particle barriers located on 2 relatively magnetic surfaces, so that it relaxes at constant particle and energy contents towards a thermodynamical equilibrium with finite particle densities $n_0(s)$ and a finite temperature T_0. Since $\Psi_0 = 0$, the densities $n_0(s)$ are the constants $\nu_0(s)$ in the equation (5). A deviation $U << T_0$ means in particular small variations of the density n and temperature T with respect to the values n_0,T_0. In spite of this constraint, the practical gradients ∇n, ∇T may be achieved by bringing the barriers closer.

We make the working assumption that the entropy S(u,u), given by the equation (14), is mainly determined by the average densities n(v,s,t) and temperatures T(v,s,t) on each magnetic surface, labelled by its volume v. That assumption anticipates that the macroscopic velocities of the various particle species or subspecies play a negligible role in S(u,u). The situation will be quite different with the field-plasma systems considered in the chapter 3, where the entropy will involve the macroscopic velocities through the electric currents controlling the magnetic energy. We introduce the small differences

$$\tilde{n}(v,s,t) = n(v,s,t) - n_0(s)\ ,\ \tilde{T}(v,s,t) = T(v,s,t) - T_0$$

We express that the deviation $U = u(J,s,t)$ achieves given profiles $\tilde{n}(v,s,t)$ and $\tilde{T}(v,s,t)$ by calculating from the distribution function (6) the particle and energy contents $n(v,s,t)dv$ and $\frac{3}{2}n(v,s,t)T(v,s,t)dv$ between 2 neighbouring surfaces v,v+dv. We obtain at first order in U

$$\int \frac{\exp(-H/T_0)}{(2\pi m T_0)^{3/2}} \delta(v(\mathbf{x})-v) \frac{u}{T_0} d_3x d_3p = \frac{\tilde{n}(v,s,t)}{n_0(s)}$$

$$\frac{2}{3}\int \frac{\exp(-H/T_0)}{(2\pi m T_0)^{3/2}} \left(\frac{H}{T_0}-\frac{3}{2}\right) \delta(v(\mathbf{x})-v) \frac{u}{T_0} d_3x d_3p = \frac{\tilde{T}(v,s,t)}{T_0} \qquad (22)$$

The equations (22) play the role of the equations (21). A deviation $u_I(\mathbf{J},s,t)$ which approximately minimizes $S(u,u)$ under the constraints (22) is given by

$$\frac{u_I}{T_o} = \frac{\tilde{n}(v_{drift},s,t)}{n_o(s)} + \left(\frac{H}{T_o}-\frac{3}{2}\right)\frac{\tilde{T}(v_{drift},s,t)}{T_o} \qquad (23a)$$

where $v_{drift}(\mathbf{x},\mathbf{p},s)$ is any dynamical variable which is both constant of motion and close to $v(\mathbf{x})$. We choose the simplest $v_{drift}(\mathbf{x},\mathbf{p},s)$, namely that $\psi_{pol}(v_{drift}) = \frac{Rp_\varphi}{e}$. Then

$$v_{drift} = v(\mathbf{x}) + \frac{RmV\varphi}{e\partial\psi_{pol}/\partial v} + O(L\rho_c^2) \qquad (23b)$$

The dynamical variable $u_I(\mathbf{J},s,t)$ produces a derivative $\frac{\partial u_I}{\partial \mathbf{p}}$ at given $\mathbf{x}$ consisting of 2 terms: an isotropic term $\frac{\tilde{T}(v(\mathbf{x}),t)}{T_o}\frac{\partial H}{\partial \mathbf{p}} = \frac{\tilde{T}(v(\mathbf{x}),t)}{T_o}\mathbf{V}$ which reflects the temperature variation $\tilde{T}$ at each position $\mathbf{x}$, and a directional term

$$\left(\frac{\partial u_I}{\partial \mathbf{p}}\right)_{dir} = T_0\left(\frac{\partial \tilde{n}}{n_0 e\,\partial v}\frac{e\partial v_{drift}}{\partial \mathbf{p}} + \frac{\partial \tilde{T}}{T_o e\,\partial v}\left(\left(\frac{H}{T_0}-\frac{3}{2}\right)\frac{e\partial v_{drift}}{\partial \mathbf{p}} + \frac{e(v_{drift}-v(x,t))}{T_0}\mathbf{V}\right)\right) \qquad (24)$$

It will be important below that $\frac{e\partial v_{drift}(\mathbf{x},\mathbf{p},s)}{\partial \mathbf{p}} = \frac{R}{\partial\psi_{pol}/\partial v}R\nabla\varphi$ is independent of $\mathbf{p}$ and s at a given position x. That independence means that at lowest order in $\frac{\rho_c}{L}$ $v_{drift}(\mathbf{x},\mathbf{p},s)$ - $v(\mathbf{x}) = \frac{e\partial v_{drift}(\mathbf{x},\mathbf{p},s)}{\partial \mathbf{p}}\cdot\frac{m\mathbf{V}}{e}$ at lowest order in $\frac{\rho_c}{L}$. Equivalently, it means that, for all the individual trajectories, the deviation with respect to the magnetic surfaces scales as $\frac{mV}{eB} \sim \rho_c$. The derivative $\left(\frac{\partial u_I}{\partial \mathbf{p}}\right)_{dir}$, properly averaged, represents the φ diamagnetic velocities of the various species at various energies, producing via the poloidal field the Lorentz forces balancing the pressure forces $-\nabla(nT)$. At a given $\mathbf{x}$ position, u_I contains an even part in $\mathbf{V}$, proportional to the net values $\tilde{n}$ and $\tilde{T}$, and a odd part $\sim \left(\frac{\partial u_I}{\partial \mathbf{p}}\right)_{dir\varphi} mV_\varphi$. When one calculates the negative entropy $S(u_I,u_I)$ from the equations (14,23), it appears that the dominant terms are produced by the even part of u_I, the odd part producing a term

which is smaller in a ratio $\sim \frac{\rho_c^2}{L^2}$. In fact the contribution of the even part of u_I reflects the structuration in space **x** of the densities and temperatures, while the little contribution of the odd part reflects the fact that the small orderer kinetic energy associated to the diamagnetism is no longer available for thermal motions. Introducing the working profiles $\tilde{\underline{n}}(v,s,t)$ and $\tilde{\underline{T}}(v,s,t)$ to be varied around $\tilde{n}(v,s,t)$ and $\tilde{T}(v,s,t)$, which define

$$\frac{\underline{u}_I}{T_o} = \frac{\tilde{\underline{n}}(v_{drift},s,t)}{n_o(s)} + (\frac{H}{T_o} - \frac{3}{2})\frac{\tilde{\underline{T}}(v_{drift},s,t)}{T_o}$$

one obtains

$$S(u_I,\underline{u}_I) = \frac{-1}{2}\sum_s \int (\frac{\tilde{n}(v,s,t)}{n_o(s)}\frac{\tilde{\underline{n}}(v,s,t)}{n_o(s)} + \frac{3}{2}\frac{\tilde{T}(v,s,t)}{T_o}\frac{\tilde{\underline{T}}(v,s,t)}{T_o})n_0(s)dv \qquad (25)$$

That expression gives the first term in the principle (20)

$$-4S(\frac{\partial u_I}{\partial t},\underline{u}_I) = 2\sum_s \int (\frac{1}{n_o(s)^2}\frac{\partial \tilde{n}(v,s,t)}{\partial t}\underline{n}(\tilde{v},s,t) + \frac{3}{2T_o^2}\frac{\partial \tilde{T}(v,s,t)}{\partial t}\underline{T}(\tilde{v},s,t))n_o(s)dv \qquad (26)$$

The Onsager relaxation of the profiles $\tilde{n}(v,s,t), \tilde{T}(v,s,t)$, equivalent to the function $u_I(\mathbf{J},s,t)$, is obtained by expressing that the functional $-4S(\frac{\partial u_I}{\partial t},\underline{u}_I) + \dot{S}_{reduced}(\underline{u}_I,\underline{u}_I)$ is an extremum for all the variations of $\tilde{\underline{n}}(v,s,t), \tilde{\underline{T}}(v,s,t$ around $\tilde{n}(v,s,t), \tilde{T}(v,s,t)$. The problem of computing that relaxation has thus been brought back to a minimization problem: the calculation of the minimum value $\dot{S}_{reduced}(\underline{u}_I,\underline{u}_I)$ of the collisional entropy production rate $\dot{S}(\underline{u}_I+\underline{u}_{II},\underline{u}_I+\underline{u}_{II})$ when one varies $\underline{u}_{II}$ within the subspace (u_{II}). The latter is of course now defined by the eqs (22) made homogeneous by cancelling the RHS. Minimizing the entropy production rate is in fact a well known technique for determining the structure of the distribution functions in weakly collisional confined plasmas (Robinson and Bernstein 1962, Rosenbluth et al 1972). We notice that $\underline{u}_I$ enters into $\dot{S}(\underline{u}_I+\underline{u}_{II},\underline{u}_I+\underline{u}_{II})$ and therefore into $\dot{S}_{reduced}(\underline{u}_I,\underline{u}_I)$ through differences of 2 derivatives $\frac{\partial \underline{u}_I}{\partial \mathbf{p}}$ taken at the same **x** with 2 different **p**,s. The differences of derivatives of type $\frac{\tilde{T}(v(\mathbf{x}),t)}{T_o}\mathbf{V}$ expressing the local temperature variations lead to collisional energy exchanges from a species to another. To simplify we assume the same temperature for all species, $\tilde{T}(v,s,t) = \tilde{T}(v,t)$, so that such exchanges do not occur. Then only the derivatives $\frac{\partial \underline{u}_I}{\partial \mathbf{p}}$ of the directional type

(24) can be involved in $\dot{S}_{reduced}(\underline{u}_I,\underline{u}_I)$. It first results that $\underline{u}_I$ enters into $\dot{S}_{reduced}(\underline{u}_I,\underline{u}_I)$ through derivatives $\frac{\partial\tilde{\underline{n}}(v,s,t)}{n_0(s)\partial v}$ and $\frac{\partial\tilde{\underline{T}}(v,t)}{T_0\partial v}$ taken on the same magnetic surface v. The functional $\dot{S}_{reduced}(\underline{u}_I,\underline{u}_I)$ has therefore the form

$$\dot{S}_{reduced}(\underline{u}_I,\underline{u}_I)=\int\Big(\sum_{ss'}\alpha_{ss'}(v)\frac{\partial\tilde{\underline{n}}(v,s,t)}{n_0(s)\partial v}\frac{\partial\tilde{\underline{n}}(v,s',t)}{n_0(s')\partial v}$$
$$+2\sum_s\beta_s(v)\frac{\partial\tilde{\underline{n}}(v,s,t)}{n_0(s)\partial v}\frac{\partial\tilde{\underline{T}}(v,t)}{T_0\partial v}+\gamma(v)\frac{\partial\tilde{\underline{T}}(v,t)}{T_0\partial v}\frac{\partial\tilde{\underline{T}}(v,t)}{T_0\partial v}\Big)dv \qquad (27)$$

with of course $\alpha_{ss'}=\alpha_{s's}$. If we now take into account that $\frac{e\partial v_{drift}(\mathbf{x},\mathbf{p},s)}{\partial\mathbf{p}}$ in the equation (24) depends only on the position $\mathbf{x}$, it appears that the profile $\tilde{n}(v,s,t)$ enters into the differences of 2 $\frac{\partial\underline{u}_I}{\partial\mathbf{p}}$ taken at the same x with 2 different $\mathbf{p}$,s through differences of two $\frac{\partial\tilde{\underline{n}}(v,s,t)}{n_0(s)e(s)\partial v}$ taken at the same v with two different s. Therefore the expression (27) must be invariant if one changes $\frac{\partial\tilde{\underline{n}}(v,s,t)}{n_0(s)\partial v}$ into $\frac{\partial\tilde{\underline{n}}(v,s,t)}{n_0(s)\partial v}$ + e(s) × a constant. This imposes that

$$\sum_s e(s)\alpha_{ss'}=0\ ;\ \sum_s e(s)\beta_s=0 \qquad (28)$$

Using the equations (26,27), we express that the functional $-4S(\frac{\partial u_I}{\partial t},\underline{u}_I)+\dot{S}_{reduced}(\underline{u}_I,\underline{u}_I)$ is an extremum for all the variations of $\tilde{\underline{n}}(v,s,t),\tilde{\underline{T}}(v,s,t)$ around $\tilde{n}(v,s,t),\tilde{T}(v,s,t)$ and thus obtain the relaxation equations

$$\frac{\partial\tilde{n}(v,s,t)}{\partial t}=\frac{\partial}{\partial v}\Big(\sum_s\alpha_{ss'}(v)\frac{\partial\tilde{n}(v,s',t)}{n_0(s')\partial v}+\beta_s(v)\frac{\partial\tilde{T}(v,t)}{T_0\partial v}\Big)$$
$$\frac{3}{2}\Big(\sum_s n_0(s)\Big)\frac{\partial\tilde{T}(v,t)}{T_0\partial t}=\frac{\partial}{\partial v}\Big(\sum_s\beta_s(v)\frac{\partial\tilde{n}(v,s,t)}{n_0(s)\partial v}+\gamma(v)\frac{\partial\tilde{T}(v,t)}{T_0\partial v}\Big) \qquad (29)$$

where we could replace $n_0(s)$ by $n(v,s,t)$, $\frac{\partial\tilde{n}(v,s,t)}{\partial t}$ by $\frac{\partial n(v,s,t)}{\partial t}$ and $\frac{\partial\tilde{n}(v,s,t)}{\partial v}$ by $\frac{\partial n(v,s,t)}{\partial v}$, and similarily T_0 by $T(v,s,t)$, $\frac{\partial\tilde{T}(v,s,t)}{\partial t}$ by $\frac{\partial T(v,s,t)}{\partial t}$ and $\frac{\partial\tilde{T}(v,s,t)}{\partial v}$ by $\frac{\partial T(v,s,t)}{\partial v}$. The Onsager symmetries consist of the symmetry $\alpha_{ss'}=\alpha_{s's}$ and of the presence of the same coefficient β_s in the two equations (29). With respect to the principle (15), they result from the weak collisionality and from the existence of the constant of motion

$v_{drift}(x,p,s) \neq v(x)$, meaning that all the individual particle trajectories are integrable and close to the magnetic surfaces. The equations (28), which impose $\sum_s e(s)\frac{\partial \tilde{n}(v,s,t)}{\partial t} = 0$, i.e. the ambipolarity of the particle fluxes, do not represent Onsager symmetries. They result from the independence of $\frac{e\partial v_{drift}(x,p,s)}{\partial p}$ with respect to p,s at given $\dot{x}$, meaning that all the integrable trajectories deviate from the magnetic surfaces by a distance scaling as ρ_c.

We still have to justify our working hypothesis $S(u,u) \neq S(u_I,u_I)$ or equivalently $S(u_{II},u_{II}) << S(u_I,u_I)$. The basic element which determines u_{II} complementing u_I given by the equation (23) is that the φ diamagnetic velocities $(\frac{\partial u_I}{\partial \mathbf{p}})_{dir}$ given by the equation (24) are relatively large, namely represent $\frac{B}{B_{pol}}\frac{\nabla nT}{neB}$ or $\frac{R}{r}\times$ the Pfirsch Schluter velocities. In order to decrease the collisional frictions reflected by $\dot{S}(u_I + u_{II}, u_I + u_{II})$, the function $u_{II}(\mathbf{J},s,t)$ then opposes as much as possible to the odd component $\sim (\frac{\partial u_I}{\partial \mathbf{p}})_{dir\varphi}\, mV_\varphi$ of u_I. In the trapped domain, there is no constant of motion with such an odd structure and in fact $u_{II} = 0$. On the contrary, in each of the passing domains $V_{//} > 0$ and < 0, an effective opposition to the odd component of u_I is possible by an $u_{II}(\mathbf{J},s,t)$ related to the constant of motion $\frac{V_{//}}{|V_{//}|}(H - \mu B_{max}(v_{drift}))^{1/2}$, $B_{max}(v)$ being the maximum B value on the magnetic surface v, which behaves as $V_{G//}$ in the bulk of these domains. Finally u_{II} reduces the derivative $\frac{\partial u_I + u_{II}}{\partial \mathbf{p}}$ to the level of the Pfirsch Schluter velocities in the bulk of the passing domain, by taking values $\sim (\frac{-\partial u_I}{\partial \mathbf{p}})_{dir\varphi}\, mV_\varphi$. Such an u_{II} has a small influence in the entropy $S(u_I+u_{II}, u_I+u_{II})$: the quantity $S(u_{II},u_{II})$ appears to be smaller than $S(u_I,u_I)$ in a ratio $\sim \frac{\rho_c^2}{L^2}$. Incidentally the derivative $\frac{\partial u_I + u_{II}}{\partial \mathbf{p}} \sim (\frac{\partial u_I}{\partial p})_{dir} \sim \frac{B}{B_{pol}}\frac{\nabla p}{neB}$ which subsists whithin and in the neighbouring of the trapped domains, averaged at each point x in p space, i.e. multiplied by $\sim(\frac{r}{R})^{1/2}$, represents the macroscopic velocities along φ of the various species, i.e. the bootstrap velocities.

2.3 COLLISIONAL OR NON INTEGRABLE SITUATIONS

We now consider toroidally confined plasmas without imposing a weak collisionality. We assume that the flux lines generate nested magnetic surfaces of volume $v(\mathbf{x})$; more generally the individual passing particles have integrable trajectories close to the magnetic surfaces, the deviation being then of course $\sim \rho_c$. We cannot extend that assumption to the trapped particles in rippled tokamaks or in stellarators, where the drift

velocity $v_{G\perp}$ largely displaces some class of trapped particles across the magnetic surfaces. In fact we will hereafter consider 2 possibilities:

1/ The configuration is axisymmetric, or exhibits a constant B on each magnetic surface so that all the particles are passing. All the particles have integrable trajectories close to the magnetic surfaces, the deviation scaling as ρ_c. There exists, besides H and μ, a constant of motion $v_{drift}(\mathbf{x},\mathbf{p},s)$ # $v(\mathbf{x})$ such that $\frac{e\partial v_{drift}(\mathbf{x},\mathbf{p},s)}{\partial p}$ at a given $\mathbf{x}$ is independent of $\mathbf{p}$,s. In those situations, it will appear that the Onsager relaxation (29) and the equation (28) implying the ambipolarity hold true at any collisionality.

2/ The configuration is general, but we assume a simplified behaviour of the trapped particles: some class of trapped particles have integrable individual trajectories close to the magnetic surfaces, the deviation being much smaller than the radial scale $\sim$ r of the configuration, but not necessarily $\sim \rho_c$; the particles in the other trapped domains experience large displacements $\sim$ r across the magnetic surfaces at a velocity $\sim v_{G\perp}$. It will appear that the Onsager relaxation (29) is only insured if the latter particles are detrapped by collisions long before they have completed a radial excursion comparable to the radial scale r, i.e. if the detrapping rate $\frac{1}{\tau_{detrapping}}$ is such that

$$\frac{1}{\tau_{detrapping}} >> \frac{V_{G\perp}}{r} \qquad (30)$$

The condition $\frac{\rho_c}{L} << 1$ implies the possibility of the Kruskal separation (Kruskal 1962) of a fast cyclotron phase $\varphi_c(\mathbf{x},\mathbf{p},s)$ from 5 independent slow dynamical variables, that we may form with $H(\mathbf{x},\mathbf{p},s)$, $\mu(\mathbf{x},\mathbf{p},s)$ and the three coordinates $\mathbf{x}_G(\mathbf{x},\mathbf{p},s) = \mathbf{x} + m\mathbf{V}_\perp \times \frac{\mathbf{B}}{eB^2} + O(\rho_c^2)$ of the guiding centre. Any dynamical variable $Z(\mathbf{x},\mathbf{p},s,t)$ is a function $Z(\varphi_c,H,\mu,\mathbf{x}_G,\varepsilon,t)$ which is 2π periodic in φ_c; the sign ε of $V_{G//}$ included in the argument will play an important role below. At all orders in $\frac{\rho_c}{L}$, μ is a constant of motion, i.e. $\{H,\mu\} = 0$, and on the other hand the cyclotron frequency $\omega_c = \{H,\varphi_c\}$, the guiding centre velocity $V_G = \{H,x_G\}$, the jacobian $\Im = \frac{D(\mathbf{x},\mathbf{p})}{D(\varphi_c, H, \mu, \mathbf{x}_G)}$ are functions of $H,\mu,\mathbf{x}_G,\varepsilon,s$ independent of φ_c. The condition $\omega_c >> \frac{1}{\tau_{collision}}$ and $\frac{1}{\tau_{relaxation}}$ together with the principle (15) imply that U and F are nearly independent of φ_c: U # $u(H,\mu,\mathbf{x}_G,\varepsilon,s,t)$ and F # $f(H,\mu,\mathbf{x}_G,\varepsilon,s,t)$. We precisely define the function

$f(H,\mu,\mathbf{x}_G,\varepsilon,s,t)$ as the φ_c average $\int_0^{2\pi} F\frac{d\varphi_c}{2\pi}$ of F and the function $u(H,\mu,\mathbf{x}_G,\varepsilon,s,t)$ by applying the relation (6) between H,u,and f. We choose the working dynamical variable $\underline{U}(\mathbf{x},\mathbf{p},s,t)$ so that it is exactly a function $\underline{u}(H,\mu,\mathbf{x}_G,\varepsilon,s,t)$ independent of φ_c. It then appears that the kinetic functional $\Sigma(U,\underline{U})$ given by the equations (10) is equal to $\Sigma(u,\underline{u})$ at all orders in $\frac{\rho_c}{L}$, and no longer involves the cyclotron motion but only the guiding centre motion

$$\Sigma(U,\underline{U})=\Sigma(u,\underline{u})=\sum_s\int \frac{F}{T_0^2}(\mathbf{V}_G\cdot\frac{\partial u}{\partial \mathbf{x}_G})\,\underline{u}\,d_3x d_3p=\sum_s\int \frac{-F}{T_0}(\mathbf{V}_G\cdot\frac{\partial \underline{u}}{\partial \mathbf{x}_G})\,d_3x d_3p \qquad (31)$$

where Fd_3xd_3p may be replaced by $f\,\Im\,2\pi\,dH\,d\mu\,d_3x_G$ independent of φ_c. With a small error of order $\frac{1}{\omega_c\tau_{relaxation}}$ and $\frac{1}{\omega_c\tau_{collision}}$, we replace $S(\frac{\partial U}{\partial t},\underline{U})$ by $S(\frac{\partial u}{\partial t},\underline{u})$ and $\dot{S}(U,\underline{U})$ by $\dot{S}(u,\underline{u})$. The principle (15) then determines an autonomous relaxation of the function $u(H,\mu,\mathbf{x}_G,\varepsilon,s,t)$ by imposing that at each time

$$-2S(\frac{\partial u}{\partial t},\underline{u})+\Sigma(u,\underline{u})+\dot{S}(u,\underline{u}) \quad \text{or} \quad -4S(\frac{\partial u}{\partial t},\underline{u})+2\Sigma(u,\underline{u})+\dot{S}(\underline{u},\underline{u}) \qquad (32)$$

is an extremum for all the variations of $\underline{u}(H,\mu,\mathbf{x}_G,\varepsilon,s,t)$ around $u(H,\mu,x_G,\varepsilon,s,t)$. The principle (32) is equivalent to a gyrokinetic equation (Hazeltine and Meiss 1992) valid at first order in $\frac{1}{\omega_c\tau_{relaxation}}$, $\frac{1}{\omega_c\tau_{collision}}$ and at all orders in $\frac{\rho_c}{L}$.

Again, we make the working assumption to be verified a posteriori that the entropy S(u,u) given by the equation (14) achieved during the relaxation process is approximately determined by the profiles $\tilde{n}(v,s,t)=n(v,s,t)-n_0(s)$ and $\tilde{T}(v,s,t)=T(v,s,t)-T_0$. The equations (22) still express that the deviation u, now a function $u(H,\mu,\mathbf{x}_G,\varepsilon,s,t)$), achieves such profiles. At each time t, we associate to the profiles $\tilde{n}(v,s,t),\tilde{T}(v,s,t)$ the function $u_I(H,\mu,\mathbf{x}_G,\varepsilon,s,t)$ which approximately minimizes S(u,u) under the constraints (22). Our working hypothesis means that $S(u,u)\ \#\ S(u_I,u_I)$. We find that u_I is still given by the equation (23a), but of course only if the constant of motion $v_{drift}(\mathbf{x},\mathbf{p},s)\ \#\ v(x)$ exists everywhere in space $\mathbf{x},\mathbf{p}$. In fact we now write

$$\frac{u_I}{T_o}=\frac{\tilde{n}(w,s,t)}{n_o(s)}+(\frac{H}{T_o}-\frac{3}{2})\frac{\tilde{T}(w,s,t)}{T_o} \qquad (33a)$$

where w is any dynamical variable which is both independent of φ_c and close to $v(\mathbf{x})$. The equations (25,26) are then applicable. We have 2 possible situations

1/ In the axisymmetric or $\nabla_{//}B = 0$ configurations, we may take

$$w = v_{drift}(\mathbf{x},\mathbf{p},s) \tag{33b}$$

For an axisymmetric configuration, the constant of motion v_{drift} is given by the equation (23b). In a $\nabla_{//}B = 0$ configuration, we have, $\frac{1}{\boldsymbol{R}}$ being the curvature of the flux lines

$$v_{drift} = v(\mathbf{x}_G) + \frac{a(\mathbf{x}_G)}{e} m V_{G//} \;;\; a(\mathbf{x}) = \int^{\mathbf{x}} \left((\frac{1}{\boldsymbol{R}} \times \frac{\mathbf{B}}{B^2}) \cdot \nabla v(\mathbf{x}) \right) dx_{//} \tag{33c}$$

For both configurations, $\frac{e\partial v_{drift}(\mathbf{x},\mathbf{p},s)}{\partial \mathbf{p}}$ depends only on $\mathbf{x}$ and

$$v_{drift} - v(\mathbf{x}) = \frac{e\partial v_{drift}(\mathbf{x},\mathbf{p},s)}{\partial \mathbf{p}} \cdot \frac{m\mathbf{V}}{e} + O(L\rho_c^2)$$

2/ In all cases we may take

$$w = v(\mathbf{x}_G) = v(\mathbf{x}) + (\nabla v(\mathbf{x}) \times \frac{-\mathbf{B}}{eB^2}) \cdot m\mathbf{V}_\perp + O(L\rho_c^2) \tag{33d}$$

Let generally u_{even} and u_{odd} be the even and odd components in ε of a function $u(H,\mu,x_G,\varepsilon,s,t)$. It will be important below that in all cases

$$\frac{u_{Ieven}}{T_o} = \frac{\tilde{n}(v(\mathbf{x}_G),s,t)}{n_o(s)} + (\frac{H}{T_o} - \frac{3}{2}) \frac{\tilde{T}((v(\mathbf{x}_G),s,t)}{T_o} \;;\; u_{Iodd} << u_{Ieven} \tag{33e}$$

Within the vectorial space (u) formed by the functions $u(H,\mu,\mathbf{x}_G,\varepsilon,s,t)$, the functions u_I associated to the various profiles $\tilde{n}(v,s,t), \tilde{T}(v,s,t)$ via the equations (33) form a vectorial subspace (u_I). Within (u_I), we define $\underline{u}_I$ to be varied around u_I by changing $\tilde{n}(v,s,t), \tilde{T}(v,s,t)$ in the equation (33a) into $\underline{\tilde{n}}(v,s,t), \underline{\tilde{T}}(v,s,t)$

$$\frac{\underline{u}_I}{T_o} = \frac{\underline{\tilde{n}}(w,s,t)}{n_o(s)} + (\frac{H}{T_o} - \frac{3}{2}) \frac{\underline{\tilde{T}}(w,s,t)}{T_o}$$

The functionals $S(u_I,\underline{u}_I)$ and $S(\frac{\partial u_I}{\partial t},\underline{u}_I)$ are then still given by the equations (25,26). We define also the vectorial subspace (u_{II}) formed by the functions $u(H,\mu,\mathbf{x}_G,\varepsilon,s,t)$ which satisfy the equations (22) made homogeneous by cancelling the RHS. We decompose u and $\underline{u}$ in the form $u_I + u_{II}$ and $\underline{u}_I + \underline{u}_{II}$, where the functions u_{II} and $\underline{u}_{II}$ belong to the subspace (u_{II}). Under our working assumption $S(u,u) \# S(u_I,u_I)$, equivalent to $|S(u_{II},u_{II})| << |S(u_I,u_I)|$, the principle (32) states that at each time

$$-2S\left(\frac{\partial u_I}{\partial t},\underline{u}_I\right) + \Sigma\left(u_I+u_{II},\underline{u}_I+\underline{u}_{II}\right) + \dot{S}\left(u_I+u_{II},\underline{u}_I+\underline{u}_{II}\right) \qquad (34)$$

is an extremum for all the variations of $\underline{u}_I$ and of $\underline{u}_{II}$. Because the antisymmetrical form Σ superimposes to the symmetrical form $\dot{S}$, we do not recover the easy way of the weakly collisional, integrable situations, where the extremalization with respect to $\underline{u}_{II}(J,s,t)$ was immediatly leading to the principle (20) implying an Onsager relaxation for the function u_I and thereby for the profiles $\tilde{n}(v,s,t),\tilde{T}(v,s,t)$. The principle (34) gives a priori no guarantee for such a relaxation: the sum $\Sigma + \dot{S}$ in the expression (34), once it has been extremalized with respect to $\underline{u}_{II}(H,\mu,\mathbf{x}_G,\varepsilon,s,t)$, becomes a bilinear form $\dot{S}_{reduced}(u_I,\underline{u}_I)$ in u_I and $\underline{u}_I$, but, because of the presence of Σ, there is no obvious reason for that bilinear form to be symmetrical. We will show that it is however the case under the condition (30). The principle (34) will then become effectively equivalent the principle (20) for u_I or $\tilde{n},\tilde{T}$.

One may verify by using the equations (31) and (33a,33b,33d) that $\Sigma(u_I,\underline{u}_I)=0$. The bilinear form $\dot{S}_{reduced}(u_I,\underline{u}_I)$ is then the value of the bilinear form in u_I,u_{II} and $\underline{u}_I,\underline{u}_{II}$

$$\dot{S}(u_I,\underline{u}_I)+(\Sigma+\dot{S})(u_{II},\underline{u}_{II})+(\Sigma+\dot{S})(u_I,\underline{u}_{II})+(-\Sigma+\dot{S})(\underline{u}_I,u_{II}) \qquad (35)$$

when it has been made extremum with respect to $\underline{u}_{II}$ within (u_{II}) (Nguyen 1992). That extremalization means that $(\Sigma+\dot{S})(u_{II},\underline{u}_{II}) + (\Sigma+\dot{S})(u_I,\underline{u}_{II})$ cancels whatever $\underline{u}_{II}$, which imposes a linear relation $u_{II} = Tu_I$ allowing to derive u_{II} from u_I. It remains $\dot{S}_{reduced}(u_I,\underline{u}_I) = \dot{S}(u_I,\underline{u}_I)+(-\Sigma+\dot{S})(\underline{u}_I,Tu_I)$. Generally, it may be shown that such an extremalization in $\underline{u}_{II}$ of a bilinear form $A(u_I,\underline{u}_I)+B(u_{II},\underline{u}_{II})+C(u_I,\underline{u}_{II})+D(\underline{u}_I,u_{II})$, if the latter is symmetrical in u_I,u_{II} and $\underline{u}_I,\underline{u}_{II}$, leads to a bilinear form in u_I and $\underline{u}_I$ which is also symmetrical. On the other hand, the result of the extremalization process is obviously unchanged if one changes $\underline{u}_{II}$ into $O\underline{u}_{II}$ where O is a linear operator transforming the subspace (u_{II}) into (u_{II}) bijectively. Therefore the extremalization in $\underline{u}_{II}$ of a bilinear form $A(u_I,\underline{u}_I)+B(u_{II},\underline{u}_{II})+C(u_I,\underline{u}_{II})+D(\underline{u}_I,u_{II})$ leads to a symmetrical bilinear form in u_I and $\underline{u}_I$ if $A(u_I,\underline{u}_I)+B(u_{II},O\underline{u}_{II})+C(u_I,O\underline{u}_{II})+D(\underline{u}_I,u_{II})$ is invariant when one exchanges u_I,u_{II} and

$\underline{u}_I, \underline{u}_{II}$. Applying that result to the bilinear form (35), it appears that a sufficient, but not at all necessary, condition for having $\dot{S}_{reduced}(u_I, \underline{u}_I) = \dot{S}_{reduced}(\underline{u}_I, u_I)$ is that there exists an operator O which insures

$$\begin{aligned} &\dot{S}(u_I, \underline{u}_I) + (\Sigma + \dot{S})(u_{II}, O\underline{u}_{II}) + (\Sigma + \dot{S})(u_I, O\underline{u}_{II}) + (-\Sigma + \dot{S})(\underline{u}_I, u_{II}) \\ &\# \dot{S}(\underline{u}_I, u_I) + (\Sigma + \dot{S})(\underline{u}_{II}, Ou_{II}) + (\Sigma + \dot{S})(\underline{u}_I, Ou_{II}) + (-\Sigma + \dot{S})(u_I, \underline{u}_{II}) \end{aligned} \qquad (36)$$

for the various components of u_I, u_{II} and $\underline{u}_I, \underline{u}_{II}$. Our strategy to demonstrate the symmetry $\dot{S}_{reduced}(u_I, \underline{u}_I) = \dot{S}_{reduced}(\underline{u}_I, u_I)$ is based on the parity operator P in $\varepsilon = \frac{V_{G//}}{|V_{G//}|}$, such that

$$Pu(H, \mu, \mathbf{x}_G, \varepsilon, s, t) = u(H, \mu, \mathbf{x}_G, -\varepsilon, s, t)$$

We will indeed show that the equation (36) holds true with O = -P under the condition (30). For that we have to show that the various components of u_I, u_{II} and $\underline{u}_{II}$ verify

$$\Sigma(u_I, Pu_{II}) \# \Sigma(u_I, u_{II}) \text{ and } |\dot{S}(u_I, Pu_{II}) + \dot{S}(u_I, u_{II})| \ll \dot{S}(u_I, u_I) \qquad (37)$$

and

$$\Sigma(u_{II}, P\underline{u}_{II}) + \dot{S}(u_{II}, P\underline{u}_{II}) \# \Sigma(\underline{u}_{II}, Pu_{II}) + \dot{S}(\underline{u}_{II}, Pu_{II}) \qquad (38)$$

In view of the equation (31), we write $\Sigma = \Sigma_{//} + \Sigma_{\perp}$, where $\Sigma_{//}$ and $\Sigma_{\perp}$ represent respectively the parallel and the much slower transverse motion of the guiding centre

$$\Sigma_{//}(u, \underline{u}) = \sum_s \int \frac{f}{T_0^2} V_{G//} \frac{\partial u}{\partial x_{G//}} \underline{u} d_3x d_3p \; ; \; \Sigma_{\perp}(u, \underline{u}) = \sum_s \int \frac{f}{T_0^2} \mathbf{V}_{G\perp} \cdot \frac{\partial u}{\partial \mathbf{x}_{G\perp}} \underline{u} d_3x d_3p \quad ;$$

$$\Sigma_{\perp}(u, \underline{u}) = \sum_s \int \frac{f}{T_0^2} V_{G\perp} \cdot \frac{\partial u}{\partial x_{G\perp}} \underline{u} d_3x d_3p$$

We will use the fact that

$$\Sigma_{//}(u, Pu') = -\Sigma_{//}(Pu, u') = +\Sigma_{//}(u', Pu) \text{ , while } \Sigma_{\perp}(u, Pu') \neq \Sigma_{\perp}(u', Pu) \qquad (39a)$$

On the other hand, we will use the symmetry of the coulombian collisions with respect to an inversion of the parallel velocities, which is expressed by the equation : $\dot{S}(u, u') = \dot{S}(Pu, Pu')$ or, by taking into account that PP is the identity

$$\dot{S}(u, Pu') = \dot{S}(Pu, u') = \dot{S}(u', Pu) \qquad (39b)$$

equivalent to state that $\dot{S}(u, u')$ cancels if u and u' have opposite parities in ε.

The first relation (37) is readily verified in view of the relations (33e). As $\dot{S}(u,u') = 0$ if u and u' have opposite parities in ε, the relation $|\dot{S}(u_I,Pu_{II})+\dot{S}(u_I,u_{II})| << \dot{S}(u_I,u_I)$ is satisfied if we have $|\dot{S}(u_{Ieven},u_{IIeven})| << \dot{S}(u_{Ieven},u_{Ieven})$. We then notice that $\dot{S}(u,u')$ couples functions $u(H,\mu,x_G,s,t)$ and $u'(H,\mu,x_G,s,t)$ through products of the same type, namely, either of the type $\frac{\partial u}{\partial x_G}\frac{\partial u'}{\partial x_G}$ or of the type $\frac{\partial u}{\partial H \text{ or } \mu}\frac{\partial u'}{\partial H \text{ or } \mu}$. By using again the structure (33e) it then appears that the relation $|\dot{S}(u_{Ieven},u_{IIeven})| << \dot{S}(u_{Ieven},u_{Ieven})$ is satisfied if a similar relation holds through the entropy (14), namely if $|S(|u_{Ieven}|,|u_{IIeven}|)| << |S(u_{Ieven},u_{Ieven})|$. The latter relation is a consequence of our basic working assuption $|S(u_{II},u_{II})| << |S(u_I,u_I)|$.

In view of the relations (39), the relation (38) is verified if $\dot{S}(u_{II},\underline{u}_{II})$ or $\Sigma_{//}(u_{II},\underline{u}_{II})$ largely dominates $\Sigma_{\perp}(u_{II},\underline{u}_{II})$. As the ratio $\Sigma_{\perp}/\Sigma_{//}$ reflects the ratio $\frac{V_{G\perp}}{V_{G//}} \sim \frac{\rho_c}{L}$, the Onsager character of the relaxation is formally insured at lowest order in $\frac{\rho_c}{L}$, independently of the collisionality (Sugama and Horton 1996). However problems arise when the functions u_{II}, $\underline{u}_{II}$ exhibit very low derivative $\frac{\partial}{\partial x_{G//}}$. The key difficulty occurs, at low collisionality, when there exists trapped particles performing displacements $\sim$ r across the magnetic surfaces at the velocity $v_{G\perp}$. Such large radial displacements combined with the radial temperature and density gradients produce an enhanced perturbation u_{II} localized within the considered trapped domain. For a collision rate smaller than the bounce frequencies, that enhanced u_{II} is independent of the bounce phases. This means that $\Sigma_{//}(u_{II},\underline{u}_{II})$ cancels so that the relation (38) now demands that $\dot{S}(u_{II},\underline{u}_{II})$ largely dominates $\Sigma_{\perp}(u_{II},\underline{u}_{II})$ .The considered u_{II} is not a constant of motion if we exclude the extremely low detrapping rates $\frac{1}{\tau_{detrapping}} << \frac{V_{G\perp}}{r}$ allowing to the particles to complete an excursion $\sim$ r without being perturbed by collisions. This means that $\Sigma_{\perp}(u_{II},\underline{u}_{II})$ cannot cancel: per particle, we have in fact $\Sigma_{\perp}(u_{II},\underline{u}_{II}) \sim \frac{V_{G\perp}}{r}\frac{u_{II}\underline{u}_{II}}{T_0^2}$. Then for $\dot{S}(u_{II},\underline{u}_{II}) \sim \frac{u_{II}\underline{u}_{II}}{T_0^2\,\tau_{detrapping}}$ to largely dominate $\Sigma_{\perp}(u_{II},\underline{u}_{II})$, the condition (30) must be fulfilled.

With a bilinear form $\dot{S}_{reduced}(u_I,\underline{u}_I)$ in $u_I,\underline{u}_I$ now symmetrical, the principle (34) becomes equivalent to the principle (20). The first entropy term is given by the equation (26). The quadratic form $\dot{S}_{reduced}(\underline{u}_I,\underline{u}_I)$ in $\underline{u}_I$ is the value of the expression (35) with $u_I=\underline{u}_I$, namely $\dot{S}(\underline{u}_I,\underline{u}_I)+(\Sigma+\dot{S})(u_{II},-P\underline{u}_{II})+(\Sigma+\dot{S})(\underline{u}_I,-P\underline{u}_{II})+(-\Sigma+\dot{S})(\underline{u}_I,u_{II})$, made extremum with respect to $\underline{u}_{II}$ within the subspace (u_{II}). As that expression is symmetrical

in $u_{II}, \underline{u}_{II}$, the value of $\dot{S}_{reduced}(\underline{u}_I, \underline{u}_I)$ is the minimum value with repect to $\underline{u}_{II}$ of $\dot{S}(\underline{u}_I, \underline{u}_I) + (\Sigma+\dot{S})(\underline{u}_{II}, -P\underline{u}_{II}) + (\Sigma+\dot{S})(\underline{u}_I, -P\underline{u}_{II}) + (-\Sigma+\dot{S})(\underline{u}_I, \underline{u}_{II})$. Since $-Pu = -u_{even} + u_{odd}$, $\Sigma(u,u') = -\Sigma(u',u)$ and $\dot{S}(u,u') = 0$ if u and u' have opposite parities, we find that $\dot{S}_{reduced}(\underline{u}_I, \underline{u}_I)$ is the minimum in $\underline{u}_{II}$ within (u_{II}) of the expression

$$\dot{S}(\underline{u}_I, \underline{u}_I) + \dot{S}(\underline{u}_{IIodd}, \underline{u}_{IIodd}) - \dot{S}(\underline{u}_{IIeven}, \underline{u}_{IIeven}) + 2\Sigma(\underline{u}_{IIeven}, \underline{u}_{IIodd}) + C(\underline{u}_I, \underline{u}_{II}) \qquad (40a)$$

where $C(\underline{u}_I, \underline{u}_{II}) = 2\dot{S}(\underline{u}_{Iodd}, \underline{u}_{IIodd}) - 2\Sigma(\underline{u}_I, \underline{u}_{IIeven})$. We recover our two basic possibilities:

1/ We consider an axisymmetric or $\nabla_{//}B = 0$ configuration and $\underline{u}_I$ is expressed in terms of the constant of motion v_{drift} via equationsof type (33a,33b).Then $\Sigma(\underline{u}_I, \underline{u}_{IIeven}) = 0$ and

$$C(\underline{u}_I, \underline{u}_{II}) = 2\dot{S}(\underline{u}_{Iodd}, \underline{u}_{IIodd}) = 2\dot{S}(\underline{u}_I, \underline{u}_{IIodd}) \qquad (40b)$$

The fact that $\underline{u}_I$ enters into $\dot{S}_{reduced}(\underline{u}_I, \underline{u}_I)$ via a bilinear form of type $\dot{S}(\underline{u}_I, u')$ implies that it enters by the differences of 2 derivatives $(\frac{\partial \underline{u}_I}{\partial \mathbf{p}})_{dir}$ of the type (24) , if we still assume the same temperature for all species. As furthermore $\frac{e\partial v_{drift}(\mathbf{x},\mathbf{p},s)}{\partial \mathbf{p}}$ depends only on $\mathbf{x}$, we have all the ingredients to recover the equations (27,28,29).

2/ We use for $\underline{u}_I$ the expresion of type (33a,33c) in terms of $v(\mathbf{x}_G)$, which is applicable in all cases. Then $\underline{u}_I$ is even in ε and it comes

$$C(\underline{u}_I, \underline{u}_{II}) = -2\Sigma(\underline{u}_I, \underline{u}_{IIeven}) = -2\Sigma_{\perp}(\underline{u}_I, \underline{u}_{IIeven}) \qquad (40c)$$

The profiles $\tilde{\underline{n}}(v,s,t)$ and $\tilde{\underline{T}}(v,t)$ enter into $\dot{S}_{reduced}(\underline{u}_I, \underline{u}_I)$ by derivatives $\frac{\partial \tilde{\underline{n}}}{n_0 \partial v}$ and $\frac{\partial \tilde{\underline{T}}}{T_0 \partial v}$ and we still recover the equations (27,29). However the equations (28) are no longer insured. The strongly collisional Pfirsch-Schlüter regimes are an exception from that point of view. Indeed the minimization of the expression (40a,40c) with a strong collisionality imposes that $\underline{u}_{IIeven}$ cancels $\dot{S}(\underline{u}_{IIeven}, \underline{u}_{IIeven})$, which means that $\underline{u}_{IIeven}$ has the form $a(\mathbf{x}_G,s) + b(\mathbf{x}_G)H$. At each $\mathbf{x}$ point, a and b make extremum $2\Sigma_{//}(\underline{u}_{IIeven}, \underline{u}_{IIodd}) - 2\Sigma_{\perp}(\underline{u}_{Ieven}, \underline{u}_{IIeven})$. This imposes that $\underline{u}_{IIodd}$ produces at each point the Pfirsch Schluter parallel particle and energy fluxes due to the gradients $\underline{u}_{IIodd}$. The value of $\dot{S}_{reduced}(\underline{u}_I, \underline{u}_I)$ is finally $\dot{S}(\underline{u}_I, \underline{u}_I)$ plus the minimum value of $\dot{S}(\underline{u}_{IIodd}, \underline{u}_{IIodd})$ with $\underline{u}_{IIodd}$ submitted to that constraint. The Pfirsch Schluter parallel particle and energy fluxes and

therefore that minimum are invariant if one changes $\frac{\partial\tilde{\tilde{\underline{n}}}(v,s,t)}{n_0(s)\partial v}$ into $\frac{\partial\tilde{\tilde{\underline{n}}}(v,s,t)}{n_0(s)\partial v} + e(s)\times$constant. This is true also of $\dot{S}(\underline{u}_I,\underline{u}_I)$ since $\frac{e\partial v(\mathbf{x}_G)}{\partial \mathbf{p}}$ depends only on $\mathbf{x}$. The equations (28) result from these two invariances.

A striking point is that to derive the Onsager behaviour of the relaxation from the kinetic equation in collisional or non integrable situations, we have been obliged to use the symmetries of the individual motions and of the collisions with respect to an inversion of velocities along the magnetic lines, so that the equations (39) hold true. Typically our demonstration demands that the interaction hamiltonian is invariant when one changes the sign of the parallel velocities. We have seen on the contrary in the § 2.2 that in weakly collisional and integrable situations the Onsager behaviour occurs whatever the weak interaction as long as it is hamiltonian. One must stress here that the relations (37,38) derived from the relations (39) are sufficient, but not at all necessary to insure the symmetry $\dot{S}_{reduced}(u_I,\underline{u}_I)=\dot{S}_{reduced}(\underline{u}_I,u_I)$ implying an Onsager relaxation. It remains possible that the latter may be established via another line whatever the interaction hamiltonian. The fact that in all cases an Onsager relaxation demands that the plasma layers on the successive magnetic surfaces are independent enough suggests the following line. At the starting point, one would decompose the plasma into layers of width $D << r$ delimited by a set of successive magnetic surfaces and fully isolated from each other by thin barriers of width $d << D$ localized around each of these surfaces. In a magnetically confined toroidal plasma, such barriers do not represent a huge perturbation: they may be obtained by suppressing the interacions between the particles and also the variations of the magnetic intensity B within the thin intervals d, assumed $>> \rho_c$. The presence of the barriers allows stationnary states of the plasma, where the temperature and the densities are constant within each interval D but are allowed to vary from an interval to the next. The suppression of the barriers, i.e. the reintroduction of the interactions and B variations, would then be considered as an hamiltonian perturbation responsible for a diffusion of the plasma in the space of these stationnary states towards its thermodynamical equilibrium. It is conceivable that the considered hamiltonian perturbation is weak enough so that it finally induces transition probabilities verifying the detailed balance equations leading to an Onsager relaxation of the Van Kampen type.

3. FIELD-PLASMA SYSTEMS

3.1 VARIATIONAL PRINCIPLES

We now assume that the confining field $\mathbf{B} = \nabla\times\mathbf{A}$, $\mathbf{E} = -\nabla\Psi - \frac{\partial \mathbf{A}}{\partial t}$ is coupled to the plasma by the Maxwell equations and study the slow relaxation of the system formed by that field and the plasma. For the system to be isolated, we surround it by an hamiltonian particle barrier which is also perfectly conducting, so that it forbids any transfer of energy through the Poynting vector $\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$. Over the barrier the tangential component $\mathbf{A}_t$ of the vector potential $\mathbf{A}$ is then frozen and the potential Ψ is assigned a null value. In a toroidal plasma, the 2 barriers are of course installed on 2 frozen magnetic surfaces flanking the plasma. Under those boundary conditions, the field $\mathbf{A},\Psi$ in the plasma is determined by the densities of electric current $\mathbf{I}(\mathbf{x},t) = \sum_s \int F\mathbf{V} d_3p$ and charge $\rho(\mathbf{x},t) = \sum_s \int F d_3p$ developped by the distribution of the plasma particles. We use the gauge $\mathrm{div}\mathbf{A} = 0$. As the field is slowly evolving, we may assume the "electrostatic" field $-\nabla\Psi$ much larger than the "inductive" field $\frac{-\partial \mathbf{A}}{\partial t}$. The Maxwell equations then reduce to $\nabla\times\nabla\times\mathbf{A} = \mu_0(\mathbf{I} + \varepsilon_0\frac{\partial \nabla\Psi}{\partial t})$ and to the Poisson equation $\varepsilon_0\Delta\Psi = -\rho$ and are equivalent to state that, at each time, the field $\mathbf{A},\Psi$ under the above boundary and gauge constraints makes extremum the lagrangian

$$\int \left(\frac{\varepsilon_0}{2}(\nabla\Psi)^2 - \frac{1}{2\mu_0}(\nabla\times\mathbf{A})^2\right) d_3x + \int (\mathbf{I}\cdot\mathbf{A} - \rho\Psi) d_3x \qquad (41)$$

$\mathbf{I}(\mathbf{x},t)$ and $\rho(\mathbf{x},t)$ being kept constant when $\mathbf{A},\Psi$ is varied. Since $H = \frac{(\mathbf{p}-e\mathbf{A})^2}{2m} + e\Psi$, a variation $\delta\mathbf{A},\delta\Psi$ induces a variation $\delta H = \frac{(\mathbf{p} - e\mathbf{A})}{m}\cdot(-e\delta\mathbf{A}) + e\delta\Psi = -e\mathbf{V}.\delta\mathbf{A} + e\delta\Psi$ so that the variation $\int (\mathbf{I}\cdot\delta\mathbf{A} - \rho\delta\Psi) d_3x$ may be replaced by $-\sum_s \int F\delta H d_3x d_3p$. In fact, the dynamical variable $H(\mathbf{x},\mathbf{p},s,t)$ at a given time t is a functional of the field $\mathbf{A}(\mathbf{x},t)$, $\Psi(\mathbf{x},t)$ at that t (since $H = \frac{(\mathbf{p}-e\mathbf{A})^2}{2m} + e\Psi$), a situation that we will express by the notation $H(\mathbf{x},\mathbf{p},s,t) = H(\mathbf{x},\mathbf{p},s|\mathbf{A},\Psi)$. The principle (41) is equivalent to state that the field $\mathbf{A},\Psi$ makes extremum

$$\int (\frac{\varepsilon_0}{2}(\nabla\Psi)^2 - \frac{1}{2\mu_0}(\nabla\times\mathbf{A})^2)d_3x$$
$$+ \sum_s \int \frac{\nu_0(s)T_0}{(2\pi T_0 m(s))^{3/2}} \exp\left(\frac{-H(\mathbf{x},\mathbf{p},s|\mathbf{A},\Psi) + U(\mathbf{x},\mathbf{p},s,t)}{T_0}\right) d_3x d_3p \qquad (42)$$

$U(\mathbf{x},\mathbf{p},s,t)$ being kept constant when $\mathbf{A},\Psi$ is varied.

When the system reaches its thermodynamical equilibrium, the field $\mathbf{A},\Psi$ takes the value $\mathbf{A}_0(\mathbf{x}),\Psi_0(\mathbf{x})$ satisfying the static boundary conditions but corresponding to the electric currents and charges $\mathbf{I}_0(\mathbf{x}),\rho_0(\mathbf{x})$ produced by the maxwellian distribution (5). Obviously $\mathbf{I}_0=0$ so that the magnetic field $\mathbf{B}_0=\nabla\times\mathbf{A}_0$ is the vacuum field achieving the fixed magnetic fluxes $\oint \mathbf{A}_t\cdot d\mathbf{x}$ across the various contours drawn over the field-plasma barrier. By positioning close enough the 2 toric barriers flanking a toroidal plasma, the practical magnetic configuration may be reproduced. The relaxed electric potential Ψ_0, assumed to cancel on both barriers, is determined by the Poisson equation $-\varepsilon_0\Delta\Psi_0=\rho_0$ with the charge density $\rho_0=\sum_s ev_0\exp(\frac{-e\Psi_0}{T_0})$ resulting from the distribution (5). Assuming that $\sum_s ev_0=0$, the solution is $\Psi_0 = 0$, implying constant relaxed densities $n_0(\mathbf{x},s) = \nu_0(s)$ and $\rho_0 = 0$.

At a given time, the deviation U entirely determines through the principle (42) the field A,Ψ and the hamiltonian H, which are therefore functionals of U. We express that situation by writing

$$H(\mathbf{x},\mathbf{p},s,t) = H(\mathbf{x},\mathbf{p},s|U)$$
$$\tilde{\mathbf{A}}(\mathbf{x},t)=\mathbf{A}(\mathbf{x},t)-\mathbf{A}_0(\mathbf{x})=\tilde{\mathbf{A}}(\mathbf{x}|U)\ ;\ \tilde{\Psi}(\mathbf{x},t)=\Psi(\mathbf{x},t)-\Psi_0(\mathbf{x})=\Psi(\mathbf{x},t)=\tilde{\Psi}(\mathbf{x}|U) \qquad (43a)$$

The functionals $\tilde{\mathbf{A}}(\mathbf{x}|U),\tilde{\Psi}(\mathbf{x}|U),H(\mathbf{x},\mathbf{p},s|U)$ depend only on the constants $\nu_0(s)$ and T_0 in the equation (6) and on the boundary conditions imposed to the field $\mathbf{A},\Psi$. A differential variation dU induces variations $d\mathbf{A}=\tilde{\mathbf{A}}(\mathbf{x}|U+dU)-\tilde{\mathbf{A}}(\mathbf{x}|U)$, $d\Psi=\tilde{\Psi}(\mathbf{x}|U+dU)-\tilde{\Psi}(\mathbf{x}|U)$ and also a variation $dH = H(\mathbf{x},\mathbf{p},s \mid U+dU) - H(\mathbf{x},\mathbf{p},s \mid U) = -e\mathbf{V}\cdot d\mathbf{A} + ed\Psi$. There exists at each time t a well defined linear operator G acting in the space of dynamical variables functions of $\mathbf{x},\mathbf{p},s$ at that t , such that , whatever the dynamical variable dU

$$dH = G(dU) \qquad (43b)$$

The kinetic equation is still equivalent to the principles (7) or (11). However, the set (13,14) is no longer valid in the present case where the hamiltonian $H(\mathbf{x},\mathbf{p},s,t)$ depends

on t, since $\frac{\partial F(\mathbf{x},\mathbf{p},s,t)}{\partial t}$ is now given by $\frac{\partial F}{\partial t} = F(\frac{\partial U}{T_0 \partial t} - \frac{\partial H}{T_0 \partial t})$ instead of $F\frac{\partial U}{T_0 \partial t}$. We have in fact $\frac{\partial H}{\partial t} = G(\frac{\partial U}{\partial t})$ so that $\frac{\partial F}{\partial t} = \frac{F}{T_0}(\frac{\partial U}{\partial t} - G(\frac{\partial U}{\partial t}))$. The equation (13) becomes again valid if we replace the definition (14) of the bilinear form S by the new definition

$$S(U',U'') = \sum_s \int \frac{-F}{2T_0^2} U'U'' d_3x d_3p + \sum_s \int \frac{F}{2T_0^2} G(U')U'' d_3x d_3p \qquad (44)$$

The principles (15) then give exactly at each time t the variation $\frac{\partial U}{\partial t}$. It is crucial at this point that the deviation U imposes a each time the state of field-plasma system, via the principle (42) for the field $\mathbf{A}(\mathbf{x},t), \Psi(\mathbf{x},t)$ and the hamiltonian $H(\mathbf{x},\mathbf{p},s,t)$, and then via the equation (6) for the distribution function $F(\mathbf{x},\mathbf{p},s,t)$. On the other hand, for principles of type (4) to be eventually recovered, it is essential that S(U',U") = S(U",U'). It is the case because of the principle (42). Indeed, the latter implies that in the space of dynamical variables at a given time, a variation dU induces variations dA, $d\Psi$ and dH = G(dU) such that the differential element $\sum_s \int \frac{\nu_0}{(2\pi T_0 m)^{3/2}} \exp(\frac{-H+U}{T_0}) dH d_3x d_3p$ is integrable, since it is equal to $d\left(\int (\frac{\varepsilon_0}{2}(\nabla\Psi)^2 - \frac{1}{2\mu_0}(\nabla \times \mathbf{A})^2) d_3x \right)$. That integrability means that 2 different variations dU', dH' = G(dU') on one hand and dU", dH" = G(dU") on the other must produce equal values of $\sum_s \int \frac{\nu_0}{(2\pi T_0 m)^{3/2}} \exp(\frac{-H+U}{T_0}) \frac{-dH' + dU'}{T_0} dH'' d_3x d_3p$ and $\sum_s \int \frac{\nu_0}{(2\pi T_0 m)^{3/2}} \exp(\frac{-H+U}{T_0}) \frac{-dH'' + dU''}{T_0} dH' d_3x d_3p$. The symmetry S(U',U") = S(U",U') results. It is worth stressing that it results from the lagrangian structure of the Maxwell equations, expressed by the principles (41,42), and not simply from the conservation in time of the energy of the field-plasma system. An important consequence of the symmetry S(U',U") = S(U",U') is to allow to identify S(U,U) at second order in U with the entropy *S* of the system with respect to the thermodynamical equilibrium U = 0, by using the same argument as at the end of the § 2.1, namely, the fact that $\frac{dS(U,U)}{dt} = 2S(\frac{\partial U}{\partial t}, U)$ and the equations (13,16).

Again an Onsager relaxation is immediately recovered in the weakly collisional regimes where $\frac{1}{\tau_{collision}}$ and $\frac{1}{\tau_{relaxation}}$ are small compared to $\frac{1}{\tau_{kinetic}}$, if at each time t the motion in the hamiltonian H(x,p,s,t) is integrable, i.e. if at each t there exists 3 angular variables $\Phi(\mathbf{x},\mathbf{p},s,t)$ and 3 action variables $\mathbf{J}(\mathbf{x},\mathbf{p},s,t)$, with $H(\mathbf{x},\mathbf{p},s,t) = h(\mathbf{J},s,t)$. The arguments used in the § 2.2 (namely, the low collisionality implies U # u(**J**,t,s),

F # f(**J**,t,s) and the choice $\underline{U}=\underline{u}(\mathbf{J},t,s)$ cancels $\Sigma(U,\underline{U})$) still allow to pass from the principle (15) to the principle (18). At this point the fact that the dynamical variables $\mathbf{J}(\mathbf{x},\mathbf{p},s,t)$ are no longer static but depend slowly on time could make an important difference with respect to the static situation of the § 2.2. Indeed $\frac{\partial U(\mathbf{x},\mathbf{p},s,t)}{\partial t}$ in the principle (15) should lead in the principle (18) to $\frac{\partial u(\mathbf{J},s,t)}{\partial t} + \frac{\partial u(\mathbf{J},s,t)}{\partial \mathbf{J}} \cdot \frac{\partial \mathbf{J}(\mathbf{x},\mathbf{p},s,t)}{\partial t}$ and not simply to \f(∂u(**J**,s,t);∂t). However, we may safely use the principle (18) with $\frac{\partial u(\mathbf{J},s,t)}{\partial t}$ as it stands. This is important since the vector $\Lambda(t)$ representing at each time t the function u(**J**,s,t) of argument **J**,s then satisfies the principle (18), on the basis of the equations (19). To establish that point, one first notices that at a given t the 3 dynamical variables $\mathbf{J}(\mathbf{x},\mathbf{p},s,t)$ are fully determined by the field $\mathbf{A}(\mathbf{x},t),\Psi(\mathbf{x},t)$ at that time t, or in other words: $\mathbf{J}(\mathbf{x},\mathbf{p},s,t) = \mathbf{J}(\mathbf{x},\mathbf{p},s|\mathbf{A},\Psi)$. Similarily the function h(**J**,s,t) is a functional h(**J**,s|**A**,Ψ). Of course those notations imply that **A**,Ψ belongs to the set **E** of the fields creating integrable trajectories. The key element is then the form of the variation $d\mathbf{J}(\mathbf{x},\mathbf{p},s) = \mathbf{J}(\mathbf{x},\mathbf{p},s|\mathbf{A}+d\mathbf{A},\Psi+d\Psi) - \mathbf{J}(\mathbf{x},\mathbf{p},s|\mathbf{A},\Psi)$ corresponding to a variation $d\mathbf{A}(\mathbf{x}),d\Psi(\mathbf{x})$ of the field **A**,Ψ within **E**: namely, the dynamical variable d**J** is a sum $\sum_{\mathbf{n}} \mathbf{X}_{\mathbf{n}}(J,s)\exp(i\,\mathbf{n}\cdot\Phi)$ over the triplets **n** of integers (n_1,n_2,n_3), where the term $\mathbf{X}_{\mathbf{n}=(0,0,0)}$ cancels (Rebut and Samain 1969, Samain 1970). This particular form of d**J**, together with the fact that $d_3x d_3p = d_3\Phi d_3J$, makes that $\frac{\partial u}{\partial \mathbf{J}} \cdot \frac{\partial \mathbf{J}}{\partial t}$ disappears when one passes from the principle (15) to the principle (18). It also allows to replace in the principle (42) the functional $\exp(\frac{-H(\mathbf{x},\mathbf{p},s|\mathbf{A},\Psi)+U(\mathbf{x},\mathbf{p},s,t)}{T_0}) d_3x d_3p$ by $\exp(\frac{-h(\mathbf{J},s|\mathbf{A},\Psi)+u(\mathbf{J},s,t)}{T_0})(2\pi)^3 d_3J$. That principle then guarantees a direct link from the function u(**J**,s,t) of argument **J**,s to the field $\mathbf{A}(\mathbf{x},t)$, $\Psi(\mathbf{x},t)$, and thereby to the structure in **x**,**p** of $\mathbf{J}(\mathbf{x},\mathbf{p},s,t)$. The knowledge of that structure allows in turn to calculate the coefficients S_{ab} and $\dot{S}_{ab}$ given by the equations (19b). Finally the Onsager scheme for the vector $\Lambda(t)$ representing the function u(**J**,s,t) relies on the principle (18) equivalent to the principle (4) through the equations (19), but as well on the principle (42) which allows to know what sort of bilinear forms $S_{ab}\Lambda^a\Lambda^b$ and $\dot{S}_{ab}\Lambda^a\Lambda^b$ must be employed.

We now derive convenient expressions of S(U',U") at zeroth order in U, i.e. of S(U,$\underline{U}$) at first order in U. We remark that if $U = u(\mathbf{Z}(\mathbf{x},\mathbf{p},s,t),s,t)$ and if **Z** at given **x**,**p** has a relative variation ~ U during the relaxation, $\frac{\partial U}{\partial t} = \frac{\partial u}{\partial t}(1+O(U))$. This will allow below to identify $S(\frac{\partial U}{\partial t},\underline{U})$ with $S(\frac{\partial u}{\partial t},\underline{u})$ for U =u(H,μ,v_{drift} or $\mathbf{x}_G$,s,t)) .At first order in U, the current and charge densities **I**(**x**,t), ρ(**x**,t) at time t are linear functionals **I**(**x**|U), ρ(**x**|U) given by the equation (6) (where $\Psi = \tilde{\Psi}$, $\sum_s e v_0 = 0$)

$$\mathbf{I}(\mathbf{x},t)=\mathbf{I}(\mathbf{x}|U)=\sum_s\int e\,F\,\mathbf{V}\frac{U}{T_0}d_3p \tag{45a}$$

$$\rho(\mathbf{x},t)=\rho(\mathbf{x}|U)=\sum_s\int e\,F\left(\frac{U}{T_0}-\frac{e\tilde{\Psi}(\mathbf{x}|U)}{T_0}\right)d_3p \tag{45b}$$

The functionals $\tilde{\mathbf{A}}(\mathbf{x}|U)$ and $\tilde{\Psi}(\mathbf{x}|U)$ also become linear in U, and $G(U)=-e\mathbf{V}\cdot\tilde{\mathbf{A}}(\mathbf{x}|U)+e\tilde{\Psi}(\mathbf{x}|U)$, so that the expression (44) becomes

$$S(U,\underline{U})=-\sum_s\int\frac{1}{2T_0^2}FU\underline{U}d_3xd_3p+\sum_s\int\frac{F}{2T_0^2}e^2\tilde{\Psi}(\mathbf{x}|U)\tilde{\Psi}(\mathbf{x}|\underline{U})d_3xd_3p$$
$$-\int\frac{1}{2T_0}\left(\tilde{\mathbf{A}}(\mathbf{x}|U)\cdot\mathbf{I}(\mathbf{x}|\underline{U})-\tilde{\Psi}(\mathbf{x}|U)\rho(\mathbf{x}|\underline{U})\right)d_3x \tag{46}$$

In principle the electric potential $\tilde{\psi}$ is determined from the charge density ρ via the Poisson equation. If the spatial scales in the plasma structure are much larger than the Debye length, one may a priori impose $\rho = 0$, $\mathrm{div}\mathbf{I} = 0$. This implies that $\rho(\mathbf{x}|U) = 0$, and, in view of the equation (45b), the functional $\tilde{\Psi}(\mathbf{x}|U)$ is determined in terms of U by the equation

$$\tilde{\Psi}(\mathbf{x}|U)=\frac{1}{\sum_s\int Fe^2d_3p}\sum_s\int FeUd_3p=\frac{1}{\sum_s n_0e^2}\sum_s\int FeUd_3p\;;(n_0(s)=\nu_0(s)) \tag{47}$$

Putting $\rho(\mathbf{x}|U) = 0$ in the expression (46), where $\tilde{\Psi}(\mathbf{x}|U)$ is now given by the equation (47), we obtain

$$S(U,\underline{U})=-\sum_s\int\frac{F}{2T_0^2}(U-e\tilde{\Psi}(\mathbf{x}|U))(\underline{U}-e\tilde{\Psi}(\mathbf{x}|\underline{U}))d_3xd_3p$$
$$-\frac{1}{2T_0}\int\tilde{\mathbf{A}}(\mathbf{x}|U)\cdot\mathbf{I}(\mathbf{x}|\underline{U})d_3x \tag{48}$$

When one builds up a deviation $U(\mathbf{x},\mathbf{p},s,t)$ in practise, one starts from a set of values of densities $\tilde{n}$, temperatures, macroscopic velocities, etc, and one fits U to these values through the equation (6). The equation (47) leads to use the values of the electric potential $\tilde{\psi}$ as a similar ingredient for building U. The potential $\tilde{\psi}$ and its working version $\tilde{\Psi}$ then enters into the functionals of the principles (15,18,32), which must be extremalized with respect to $\tilde{\Psi}$ on the same grounds as with respect to $\tilde{n}$, etc. That extremalization with

respect to $\tilde{\Psi}$ around $\tilde{\psi}$ will express the neutrality constraint $\rho = 0$ within the evolution of the system, and therefore will express that $\mathrm{div}\mathbf{I} = 0$. Let us notice that, with $\mathrm{div}\mathbf{I} = 0$, the Maxwell equations now reduce to $\nabla\times\nabla\times\mathbf{A} = \mu_0\mathbf{I}$: for a given U, the current density I(x|U), provided by the equation (45a), creates the field $\tilde{\mathbf{A}}(\mathbf{x}|U)$ via the Ampère law $\nabla\times\nabla\times\tilde{\mathbf{A}} = \mu_0\mathbf{I}$ and the boundary condition $\tilde{\mathbf{A}}_t = 0$. The symmetry $S(U,\underline{U}) = S(\underline{U},U)$ based on the expression (48) then implies the "reciprocity theorem" (Landau and Lifshitz 1960)

$$\int (\tilde{\mathbf{A}}(\mathbf{x}|U)\cdot\mathbf{I}(\mathbf{x}|\underline{U}))\,d_3x = \int (\tilde{\mathbf{A}}(\mathbf{x}|\underline{U})\cdot\mathbf{I}(\mathbf{x}|U))\,d_3x = \int \frac{1}{\mu_0}(\nabla\times\tilde{\mathbf{A}}(\mathbf{x}|U)\cdot\nabla\times\tilde{\mathbf{A}}(\mathbf{x}|\underline{U}))d_3x$$

In a toroidal configuration the field $\mathbf{A}(\mathbf{x},t)$ imposes at each time the form of the magnetic surfaces, i.e. the volume $v(\mathbf{x},t)$ of the magnetic surface passing by $\mathbf{x}$. Via the integrals $\oint \mathbf{A}\cdot d\mathbf{x}$, it then determines the poloidal flux $2\pi\psi_{pol}(v,t)$ embrassed by the major turns drawn on the magnetic surface of volume v, and the toroidal flux $2\pi\psi_{tor}(v,t)$ embrassed by the minor turns. At a given t, the deviation U, since it entirely determines the field $\mathbf{A}(\mathbf{x},t)$, determines as well the functions $\psi_{pol}(v,t)$ and $\psi_{tor}(v,t)$ of v. At the thermodynamical equilibrium, the volume function $v(\mathbf{x},t)$ has taken the relaxed form $v_0(\mathbf{x})$ and the flux functions $\psi_{pol}(v,t)$ $\psi_{tor}(v,t))$ the forms $\psi_{0pol}(v)$, $\psi_{0tor}(v)$. We introduce the functionals in U

$$\tilde{\psi}_{pol}(v|U) = \tilde{\psi}_{pol}(v,t) = \psi_{pol}(v,t) - \psi_{0pol}(v)\,;\ \tilde{\psi}_{tor}(v|U) = \tilde{\psi}_{tor}(v,t) = \psi_{tor}(v,t) - \psi_{0tor}(v)$$

which are linear at first order in U. We will transform the second term in the expression (48) by taking into account that we know a priori that

$$\mathrm{div}\mathbf{I} = 0\ ;\ \mathbf{I}\times\mathbf{B}\ \#\ \nabla p\ ;\ p\ \#\ p(v(\mathbf{x},t),t) \qquad (49)$$

p being the plasma pressure. The equations (49) imply that the current density $\mathbf{I}(\mathbf{x},t)$ produces well defined amperages $2\pi\iota_{pol}(v,t)$ and $2\pi\iota_{tor}(v,t)$ across the minor and the major turns. We define

$$I_{pol}(v,t) = \frac{\partial\iota_{pol}(v,t)}{\partial v}\ ;\ I_{tor}(v,t) = \frac{\partial\iota_{tor}(v,t)}{\partial v}$$

The mechanical equilibrium $\mathbf{I}\times\mathbf{B} = \nabla p$ imposes that (Mercier 1974, p 33)

$$I_{tor}(v,t)\frac{\partial\psi_{pol}(v,t)}{\partial v} - I_{pol}(v,t)\frac{\partial\psi_{tor}(v,t)}{\partial v} = \frac{\partial p(v,t)}{(2\pi)^2\partial v} \qquad (50a)$$

We base the topology of our minor and major turns, which then determines the functions $\psi_{tor}(v,t)$, $I_{tor}(v,t)$ and $\psi_{pol}(v,t)$, $I_{pol}(v,t)$, on Hamada angular coordinates $\theta_{Ha}(\mathbf{x},t)$, $\varphi_{Ha}(\mathbf{x},t)$ defined modulo 2π, parametrizing each magnetic surface v. We define the minor turns Γ_{min} on a surface v by varying θ_{Ha},φ_{Ha} by $2\pi,0$ and the major turns Γ_{maj} by varying θ_{Ha},φ_{Ha} by $0,2\pi$. The use of the Hamada coordinates allows to write (Mercier 1974, p 37)

$$\mathbf{I}(\mathbf{x},t) = I_{tor}(v,t)\nabla v\times\nabla\theta_{Ha} + I_{pol}(v,t)\nabla v\times\nabla\varphi_{Ha}$$

$$I_{tor}(v,t) = \frac{1}{(2\pi)^2}\mathbf{I}(\mathbf{x},t)\cdot\nabla\varphi_{Ha}(x,t)\ ;\ I_{pol}(v,t) = \frac{-1}{(2\pi)^2}\mathbf{I}(\mathbf{x},t)\cdot\nabla\theta_{Ha}(x,t) \quad (50b)$$

where $v,\theta_{Ha},\varphi_{Ha}$ stand for $v(\mathbf{x},t)$, etc. By taking into account the equation (45a) we then obtain

$$I_{pol}(v,t)=I_{pol}(v|U)=\sum_s\int eF\frac{U}{T_0}\frac{\partial H}{\partial \mathbf{p}}\cdot\frac{-\partial\theta_{Ha}(x,t)}{(2\pi)^2\partial\mathbf{x}}d_3p=\sum_s\int eF\frac{\partial U}{\partial \mathbf{p}}\cdot\frac{-\partial\theta_{Ha}}{(2\pi)^2\partial\mathbf{x}}d_3p$$

$$I_{tor}(v,t)=I_{tor}(v|U)=\sum_s\int eF\frac{U}{T_0}\frac{\partial H}{\partial \mathbf{p}}\cdot\frac{\partial\varphi_{Ha}(\mathbf{x},t)}{(2\pi)^2\partial\mathbf{x}}d_3p=\sum_s\int eF\frac{\partial U}{\partial \mathbf{p}}\cdot\frac{\partial\varphi_{Ha}}{(2\pi)^2\partial\mathbf{x}}d_3p \quad (51a)$$

The equation (50a) links I_{tor} and I_{pol} to the profile of the pressure $p(v,t)=\sum_s n(v,s,t)T(v,s,t)$, expressed in terms of U via the equation (6)

$$I_{tor}(v|U)\frac{\partial\psi_{pol}}{\partial v}-I_{pol}(v|U)\frac{\partial\psi_{tor}}{\partial v}=\frac{\partial\tilde{p}(v|U)}{(2\pi)^2\partial v}$$

$$\tilde{p}(v,t)=p(v,t)-p_0(v,t)=\sum_s n_0T_0\left(\frac{\tilde{n}}{n_0}+\frac{\tilde{T}}{T_0}\right)=\sum_s\int\frac{2}{3}FHUd_3p=\tilde{p}(v|U) \quad (51b)$$

Let us stress that, for a given U, the functionals $I_{tor}(v|U)$ and $I_{pol}(v|U)$, defined by the equations (51), determines $\mathbf{I}$ via the eqs (50b) , then $\tilde{\mathbf{A}}(\mathbf{x}|U)$ via the Ampère law $\nabla\times\nabla\times\tilde{\mathbf{A}}=\mu_0\mathbf{I}$ and the boundary condition $\tilde{\mathbf{A}}_t=0$, and thus finally the functionals $\tilde{\psi}_{tor}(v|U),\tilde{\psi}_{pol}(v|U)$. A key point (see Appendix 3) is that the last term in the expression (48) may be written

$$-\frac{1}{2T_0}\int\tilde{\mathbf{A}}(\mathbf{x}|U)\cdot\mathbf{I}(\mathbf{x}|\underline{U})d_3x=-\int\frac{(2\pi)^2}{2T_0}\left(\tilde{\psi}_{pol}(v|U)I_{tor}(v|\underline{U})-\tilde{\psi}_{tor}(v|U)I_{pol}(v|\underline{U})\right)dv \quad (52)$$

so that

$$S(U,\underline{U})=-\sum_s\int\frac{F}{2T_0^2}(U-e\tilde{\Psi}(\mathbf{x}|U))(\underline{U}-e\tilde{\Psi}(\mathbf{x}|\underline{U}))d_3xd_3p$$

$$-\int \frac{(2\pi)^2}{2T_0}\left(\tilde{\psi}_{pol}(v|U)\,I_{tor}(v|\underline{U}) - \tilde{\psi}_{tor}(v|U)\,I_{pol}(v|\underline{U})\right)dv \qquad (53)$$

3.2 NEOCLASSICAL TRANSPORT

With respect to the imposed field situations considered in the § 2.2 and 2.3, the main difference is that the entropy S(U,U), now given by the equation (53), is influenced by the average velocties of each species through the electric current densitiy I which controls the magnetic energy of the field plasma system. Our working assumption that S(U,U) is simply determined by the average density and temperature profiles $n(v,t,s)=n_0(s)+\tilde{n}(v,s,t)$ and $T(v,t,s)=T_0+\tilde{T}(v,s,t)$ is no longer realistic. We must add the poloidal and toroidal current profiles $I_{pol}(v,t)$ and $I_{tor}(v,t)$, and of course also the average electric potential $\Psi(v,t)=\tilde{\Psi}(v,t)$ over each magnetic surface v. In fact we have to add only one of the profiles I_{pol}, I_{tor}: because of the equations (51b), we may consider the profile I_{pol}, for instance, as determined by the profiles $\tilde{n},\tilde{T}$, and I_{tor}. Accordingly, we will take as working assumption to be verified a posteriori that S(U,U) is determined by the profiles $\tilde{n}(v,s,t),\tilde{T}(v,s,,t),\tilde{\Psi}(v,t)$ and $I_{tor}(v,t)$. The equations expressing that the deviation U achieves at a given time t the profiles $\tilde{n}(v,t,s)$ and $\tilde{T}(v,t,s)$ are driven from the equations (6) and are quite similar to the equations (22). Those expressing that U achieves the profiles $\tilde{\Psi}(v,t)$ and $I_{tor}(v,t)$ are driven from the equations (47) and (51). The whole set has the form

$$\int \frac{\exp(-H/T_0)}{(2\pi m_s T_0)^{3/2}}\delta(v(\mathbf{x},t)-v)\frac{U}{T_0}d_3x d_3p = \frac{\tilde{n}(v,s,t)}{n_0(s)} + \frac{e(s)\tilde{\Psi}(v,t)}{T_0}$$

$$\frac{2}{3}\int \frac{\exp(-H/T_0)}{(2\pi m T_0)^{3/2}}\left(\frac{H}{T_0}-\frac{3}{2}\right)\delta(v(\mathbf{x},t)-v)\,\frac{U}{T_0}d_3x d_3p = \frac{\tilde{T}(v,s,t)}{T_0}$$

$$\sum_s \int \frac{n_0\exp(-H/T_0)}{(2\pi m T_0)^{3/2}}\,\delta(v(\mathbf{x},t)-v)\frac{eU}{T_0}d_3x d_3p = \left(\sum_s n_0 e^2\right)\frac{\tilde{\Psi}(v,t)}{T_0}$$

$$\sum_s \int n_0 \frac{\exp(-H/T_0)}{(2\pi m T_0)^{3/2}}\,e\left(\frac{\partial U}{\partial \mathbf{p}}\cdot\frac{\partial \varphi_{Ha}(\mathbf{x},t)}{(2\pi)^2\partial \mathbf{x}}\right)d_3p = I_{tor}(v,t)) \qquad (54)$$

In the last integral, $\mathbf{x}$ is a position (arbitrarily chosen) on the magnetic surface of volume v, of equation $v(\mathbf{x},t) = v$.

Let us first consider an axisymmetric tokamak in a weakly collisional regime. At each time the deviation U is nearly a function u(**J**,s,t) of 3 action variables **J**(**x**,**p**,s,t) for the hamiltonian H at the considered t, which then satisfies the principle (18). With S(U,U) taken at first order in U, we may in fact express U as u(**J**,s,t) with **J** meaning the 3 constants of motion $H = \frac{mV^2}{2} + e\Psi(\mathbf{x},t)$, μ and the constant of motion $v_{drift}(\mathbf{x},\mathbf{p},s,t)$ # $v(\mathbf{x},t)$ given by the equation (23b). We will pass of course from the principle (18) to the principle (20). The first step is to associate to the profiles $\tilde{n}(v,s,t), \tilde{T}(v,s,t), \tilde{\Psi}(v,t)$ and $I_{tor}(v,t)$ a deviation $u_I(\mathbf{J},s,t)$ which approximately minimizes S(U,U) given by the equation (53) under the constraints (54). The second term of S(U,U) is directly imposed by the given profiles $\tilde{n}, \tilde{T}$, and I_{tor} which determine I_{pol} via the equations (51b) and then $\tilde{\psi}_{tor}, \tilde{\psi}_{pol}$ from I_{tor}, I_{pol} via the Ampère law. Therefore u_I has only to minimize the first term $S' = \sum_s \int \frac{-F}{2T_0^2}(U - e\Psi)^2 d_3x d_3p$. Accordingly u_I contains a term of type (23a), namely $u_I' = \frac{\tilde{n}(v_{drift},s,t)}{n_0} + \frac{e(s)\tilde{\Psi}(v_{drift},t)}{T_0} + (\frac{H}{T_0} - \frac{3}{2})\frac{\tilde{T}(v_{drift},s,t)}{T_0}$. As long as the ordered kinetic energy associated to average velocities of the various species is small, the odd part of u_I' has a small influence on S'. To obtain the actual u_I, we add to u_I' a further odd constant of motion, which again does not influence S', but allows to satisfy the last constraint (54) involving I_{tor}. That term is built up from $\varepsilon(H - \mu B_{max}(v_{drift},t))$, where $B_{max}(v,t)$ is the maximum value of the magnetic field B over the magnetic surface of volume v and ε is either 0 in the trapped domain $H - \mu B_{max} < 0$ or $\frac{V_{//}}{|V_{//}|}$ in the passing domain $H - \mu B_{max} > 0$. We are in fact led to take

$$\frac{u_I}{T_0} = \frac{\tilde{n}(v_{drift},s,t)}{n_0} + \frac{e(s)\tilde{\Psi}(v_{drift},t)}{T_0} + (\frac{H}{T_0} - \frac{3}{2})\frac{\tilde{T}(v_{drift},s,t)}{T_0}$$
$$+ e(s)\,\varepsilon\,\frac{H - \mu B_{max}(v_{drift},t)}{T_0}\Big(a(v_{drift},t)\,I_{tor}(v_{drift},t) +$$
$$\sum_s n_0(s)\,e(s)\Big(b(v_{drift},s,t)\Big(\frac{\partial\tilde{n}(v_{drift},s,t)}{e(s)n_0(s)\partial v} + \frac{\partial\tilde{\Psi}(v_{drift},t)}{T_0\partial v}\Big) + c(v_{drift},s,t)\,\frac{\partial\tilde{T}(v_{drift},s,t)}{e(s)T_0\partial v}\Big)\Big) \quad (55)$$

giving rise to a derivative $\frac{\partial u_I}{\partial \mathbf{p}}$ at a given **x** which, besides the isotropic term $\frac{\tilde{T}(v(x,t),s,t)}{T_o}V$, exhibits the following directional term

$$(\frac{\partial u_I}{\partial \mathbf{p}})_{dir} = T_0\Big(\Big(\frac{\partial\tilde{n}}{n_0 e\,\partial v} + \frac{\partial\tilde{\Psi}}{T_0\partial v}\Big)\frac{e\partial v_{drift}}{\partial \mathbf{p}} + \frac{\partial\tilde{T}}{T_o e\,\partial v}\Big(\Big(\frac{H}{T_0} - \frac{3}{2}\Big)\frac{e\partial v_{drift}}{\partial \mathbf{p}} + \frac{e(v_{drift} - v(\mathbf{x},t))}{T_0}\mathbf{V}\Big)\Big)$$

$$+\Big(a\,I_{tor}+\Big(\sum_s b\,n_0 e\Big(\frac{\partial\tilde{n}}{en_0\partial v}+\frac{\partial\tilde{\Psi}}{T_0\partial v}\Big)\Big)+\Big(\sum_s c\,n_0 e\,\frac{\partial\tilde{T}}{eT_0\partial v}\Big)\Big)e\,\varepsilon\,(\mathbf{V}-\frac{B_{max}}{B}\mathbf{V}_\perp) \qquad (56)$$

The functions a(v,t), b(v,s,t), c(v,s,t) are found by substituting the expression (56) in the constraint (54) involving I. An important point is that, if $\frac{e\partial v_{drift}(\mathbf{x},\mathbf{p},s,t)}{\partial\mathbf{p}}$ is independent of $\mathbf{p}$,s, the function b(v,s,t) appears to be independent of s. It is indeedthe case in tokamaks, where $a=\pi^{1/2}B_{max}\Big(\sum_s n_0 e^2\,\frac{\partial\psi_{tor}}{\partial v}V_{thermal}\Big)^{-1}$ (with $v_{thermal}=\big(\frac{2T_0}{m}\big)^{1/2}$) and b = c $= -\pi^{1/2}B_{max}T_0\Big(\sum_s n_0 e^2\frac{\partial\psi_{tor}}{\partial v}\frac{\partial\psi_{pol}}{\partial v}(2\pi)^2V_{thermal}\Big)^{-1}$. We build up $\underline{u}_I(\mathbf{J},s,t)$ by replacing the profiles $\tilde{n}(v,s,t),\tilde{T}(v,s,t),\tilde{\Psi}(v,t),I_{tor}(v,t)$ in the expression (55) by the working profiles $\tilde{\underline{n}}(v,s,t),\tilde{\underline{T}}(v,s,t),\tilde{\underline{\Psi}}(v,t),\underline{I}_{tor}(v,t)$, to be varied around $\tilde{n}(v,s,t),\tilde{T}(v,s,t),\tilde{\Psi}(v,t),I_{tor}(v,t)$. After insertion of u_I and $\underline{u}_I$ in the expression (53) and elimination of I_{pol} by using the equations (51b), one finally obtains the first term of the principle (20) in the form

$$\begin{aligned}-4S\Big(\frac{\partial u_I}{\partial t},\underline{u}_I\Big)=2\sum_s\int\Big(&\frac{1}{(n_0(s))^2}\frac{\partial\tilde{n}(v,s,t)}{\partial t}\tilde{\underline{n}}(v,s,t)+\frac{3}{2T_0^2}\frac{\partial\tilde{T}(v,s,t)}{\partial t}\tilde{\underline{T}}(v,s,t)\Big)n_0(s)dv\\&+2\int\frac{(2\pi)^2}{T_0}\Big(\frac{\partial\tilde{\underline{\psi}}_{pol}(v,t)}{\partial t}-\frac{\partial\psi_{pol}/\partial v}{\partial\psi_{tor}/\partial v}\frac{\partial\tilde{\underline{\psi}}_{tor}(v,t)}{\partial t}\Big)\underline{I}_{tor}(v,t)\,dv\\&+2\int\frac{1}{\partial\psi_{tor}/\partial v}\frac{\partial\tilde{\underline{\psi}}_{tor}(v,t)}{\partial t}\sum_s n_0(s)\frac{\partial}{\partial v}\Big(\frac{\tilde{\underline{n}}(v,s,t)}{n_0(s)}+\frac{\tilde{\underline{T}}(v,s,t)}{T_0}\Big)dv\end{aligned} \qquad (57)$$

The second step to apply the principle (20) is to minimize $\dot{S}(\underline{u}_I+\underline{u}_{II},\underline{u}_I+\underline{u}_{II})$ for given u_I when one varies $\underline{u}_{II}(\mathbf{J},s,t)$ submitted to the constraints (54) made homogeneous by cancelling the right hand side. We still obtain the structure of the corresponding minimum $\dot{S}_{reduced}(\underline{u}_I,\underline{u}_I)$ by using the fact that $\underline{u}_I(\mathbf{J},s,t)$ enters into $\dot{S}(\underline{u}_I+\underline{u}_{II},\underline{u}_I+\underline{u}_{II})$ and therefore into $\dot{S}_{reduced}(\underline{u}_I,\underline{u}_I)$ through differences of 2 $\frac{\partial\underline{u}_I}{\partial\mathbf{p}}$ for 2 different $\mathbf{p}$,s at the same $\mathbf{x}$. In view of the expression (56) one first finds, assuming $\tilde{T}(v,s,t)=\tilde{T}(v,t)$, that $\dot{S}_{reduced}(\underline{u}_I,\underline{u}_I)$ is a sum for all possible v of quadratic forms in $\underline{X}_s=\frac{\partial\tilde{\underline{n}}(v,s,t)}{n_0(s)\,\partial v}+\frac{e(s)\partial\tilde{\underline{\Psi}}(v,t)}{T_0\partial v},\underline{Y}=\frac{\partial\tilde{\underline{T}}(v,t)}{T_0\partial v}$ and $\underline{I}_{tor}(v,t)$

$$\begin{aligned}\dot{S}_{reduced}(\underline{u}_I,\underline{u}_I)=\int\Big(&\sum_{ss'}\alpha_{ss'}(v,t)\underline{X}_s\underline{X}_{s'}+\gamma(v,t)\underline{Y}\,\underline{Y}+\eta(v,t)\,\underline{I}_{tor}\,\underline{I}_{tor}\\&+2\sum_s\beta_s(v,t)\underline{X}_s\underline{Y}+2\sum_s\kappa_s(v,t)\underline{X}_s\underline{I}_{tor}+2\lambda(v,t)\underline{Y}\,\underline{I}_{tor}\Big)\,dv\end{aligned} \qquad (58)$$

where of course $\alpha_{ss'} = \alpha_{s's}$. In axisymmetric tokamaks, $\frac{e\partial v_{drift}(\mathbf{x},\mathbf{p},s,t)}{\partial \mathbf{p}}$ is independent of p,s, which implies that b(v,s,t) is independent of s. Taking also into account that $\sum_s n_0(s)e(s)=0$, it then appears that the differences of 2 $\frac{\partial \underline{u}_I}{\partial \mathbf{p}}$ for 2 different $\mathbf{p}$,s at the same $\mathbf{x}$ are invariant if one changes $\frac{\partial \tilde{\underline{n}}(v,s,t)}{n_0(s)\partial v} + \frac{e(s)\partial \tilde{\underline{\Psi}}(v,t)}{T_0\partial v}$ into $\frac{\partial \tilde{\underline{n}}(v,s,t)}{n_0(s)\partial v} + \frac{e(s)\partial \tilde{\underline{\Psi}}(v,t)}{T_0\partial v} + e(s)$ × a constant. This means that $\dot{S}_{reduced}(\underline{u}_I, \underline{u}_I)$ is invariant when one changes $\underline{X}_s$ into $\underline{X}_s + e(s)$ × a constant, that is to say

$$\sum_s e(s)\alpha_{ss'}=0 ; \sum_s e(s)\beta_s=0 ; \sum_s e(s)\kappa_s=0 \tag{59}$$

We then apply the principle (20) by extremalizing $-4S(\frac{\partial u_I}{\partial t}, \underline{u}_I) + \dot{S}_{reduced}(\underline{u}_I, \underline{u}_I)$ as it appears from the equations (57,58) with respect to the working profiles $\tilde{\underline{n}}(v,s,t), \tilde{\underline{T}}(v,s,t), \underline{\Psi}(v,t), \underline{I}_{tor}(v,t)$ around the actual profiles $\tilde{n}, \tilde{T}, \tilde{\Psi}$ and I_{tor}. It comes

$$\frac{\partial \tilde{n}(v,s,t)}{\partial t} = \frac{\partial}{\partial v}(-\Gamma_s + n_0(s)\frac{\partial \tilde{\psi}_{tor}(v,t)/\partial t}{\partial \psi_{tor}(v,t)/\partial v})$$

$$-\Gamma_s = \Big(\sum_{s'} \alpha_{ss'}\big(\frac{\partial \tilde{n}(v,s',t)}{n_0(s')\partial v} + \frac{e(s')\partial \tilde{\Psi}(v,t)}{T_0\partial v}\big)\Big) + \beta_s\frac{\partial \tilde{T}(v,t)}{T_0\partial v} + \kappa_s I_{tor}(v,t) \tag{60a}$$

$$\frac{3}{2}\Big(\sum_s n_0(s)\Big)\frac{\partial \tilde{T}(v,t)}{T_0\partial t} = \frac{\partial}{\partial v}\Big(\Big(\sum_s \beta_s\big(\frac{\partial \tilde{n}(v,s,t)}{n_0(s)\partial v} + \frac{e(s)\partial \tilde{\Psi}(v,s,t)}{T_0\partial v}\big)\Big)$$

$$+\gamma\frac{\partial \tilde{T}(v,t)}{T_0\partial v} + \lambda I_{tor}(v,t) + \Big(\sum_s n_0(s)\Big)\frac{\partial \tilde{\psi}_{tor}(v,t)/\partial t}{\partial \psi_{tor}(v,t)/\partial v}\Big) \tag{60b}$$

$$0 = \frac{\partial}{\partial v}\Big(\sum_s e(s)\Gamma_s\Big) \tag{60c}$$

$$\frac{(2\pi)^2}{T_0}\frac{E}{\partial \psi_{tor}(v,t)/\partial v} = \Big(\sum_s \kappa_s\big(\frac{\partial \tilde{n}(v,s,t)}{n_0(s)\partial v} + \frac{e(s)\partial \tilde{\Psi}(v,s,t)}{T_0\partial v}\big)\Big) + \lambda\frac{\partial \tilde{T}(v,t)}{T_0\partial v} + \eta I_{tor}(v,t)$$

$$E = -\frac{\partial \tilde{\psi}_{pol}(v,t)}{\partial t}\frac{\partial \psi_{tor}(v,t)}{\partial v} + \frac{\partial \tilde{\psi}_{tor}(v,t)}{\partial t}\frac{\partial \psi_{pol}(v,t)}{\partial v} \tag{60d}$$

Since the ratio $\frac{-\partial\psi_{tor}(v,t)/\partial v}{\partial\psi_{pol}(v,t)/\partial v}$ is the safety factor $\frac{\nabla_{//}\varphi_{Ha}}{\nabla_{//}\theta_{Ha}}$, the quantity $\frac{E}{\partial\psi_{tor}(v,t)/\partial v}$ is readily identified with $\frac{1}{2\pi}\times$ the inductive voltage $\int E_{//}dx_{//}$ along the flux lines per major turn. The equations (60) express the relaxation of the profiles $\tilde{n}(v,s,t), \tilde{T}(v,t), \tilde{\Psi}(v,t)$ and $I_{tor}(v,t)$ if one uses the Ampère law to relate $\frac{\partial\tilde{\psi}_{pol}(v,t)}{\partial t}, \frac{\partial\tilde{\psi}_{tor}(v,t)}{\partial t}$ to $\frac{\partial I_{pol}(v,t)}{\partial t}, \frac{\partial I_{tor}(v,t)}{\partial t}$ and then, via the equations (51b), to $\frac{\partial\tilde{n}(v,s,t)}{\partial t}, \frac{\partial\tilde{T}(v,t)}{\partial t}, \frac{\partial I_{tor}(v,t)}{\partial t}$. The considered situation is general in the sense that the shape of each magnetic surface of volume v and the flux functions $\psi_{pol}(v,t), \psi_{tor}(v,t)$ may evolve in the course of time. However, since we have used an expression of S(U,$\underline{U}$) at first order in U, some effects are absent in the eqs (60): for instance the energy transfer from the field to the particles, namely $\int \mathbf{E}\cdot\mathbf{I}\,d_3x = \int (-\frac{\partial\tilde{\psi}_{pol}}{\partial t} I_{tor} + \frac{\partial\tilde{\psi}_{tor}}{\partial t} I_{pol})(2\pi)^2 dv$, being of order U^2, does not appear in $\frac{\partial T}{\partial t}$; it may be introduced by using the general expression (44) of the bilinear form S(U',U"). The minor turns Γ_{minor} along which the Hamada coordinates $\theta_{Ha}, \varphi_{Ha}$ vary by $(2\pi,0)$, rather than the major turns Γ_{major} along which $\theta_{Ha}, \varphi_{Ha}$ vary by $(0,2\pi)$, have been chosen to play a preferential role in the equations (60), whose sructure must be invariant if one changes of Hamada coordinates, passing from θ_{Ha} and φ_{Ha} to $\theta'_{Ha} = l\,\theta_{Ha} + m\,\varphi_{Ha}$ and $\varphi'_{Ha} = p\,\theta_{Ha} + q\varphi_{Ha}$ where the integers l, m, p, q are such that $lq - mp = 1$ (see Appendix 3). The integers p and q determine the new minor turns $\Gamma'_{minor} = q\,\Gamma_{minor} - p\,\Gamma_{major}$ along which $\theta'_{Ha}, \varphi'_{Ha}$ vary by $(2\pi,0)$, embrassing the new "toroidal" flux $\psi'_{tor} = q\,\psi_{tor} - p\,\psi_{pol}$ and the new "toroidal" current $I'_{tor} = q\,I_{tor} - p\,I_{pol}$. It appears that the quantity E given by the equation (60c) is invariant. The new coefficients $\alpha'_{ss'}$, β'_s, etc are readily derived from the invariance of the entropy production rate (58) expressed in terms of $\underline{X}_s, \underline{Y}, \underline{I}_{tor}$ on one side and of $\underline{X}_s, \underline{Y}, \underline{I}'_{tor}$ on the other, by taking into account the relation $\underline{I}'_{tor} = q\,\underline{I}_{tor} - p\underline{I}_{tor}\frac{\partial\psi_{pol}/\partial v}{\partial\psi_{tor}/\partial v} - p\frac{\partial\tilde{p}/\partial v}{(2\pi)^2\partial\psi_{tor}/\partial v}$ derived from the equations (51b). It appears that for a given safety factor $\frac{-\partial\psi_{tor}/\partial v}{\partial\psi_{pol}/\partial v}$ the new equations (60) are determined by the integers p and q which determine the new minor turns Γ'_{minor}. The structure of the equations (60) is therefore fully determined by the topology of the flux lines and that of the minor turns.

The above analysis may be extended to the collisional or non integrable situations. Of course the dynamical variables U and $\underline{U}$ are now functions $u(H,\mu,\mathbf{x}_G,\varepsilon,s,t)$ and $\underline{u}(H,\mu,\mathbf{x}_G,\varepsilon,s,t)$ and one must start from the principle (34) rather than

from the principle (20). The analysis of the § 2.3 allowing to pass from the principle (34) to the principle (20) is still applicable. The function u_I is now given by the equation (55) in the axisymmetric configurations or the $\nabla_{//}B = 0$ configurations for which the constant of motion v_{drift} exists. In all cases one may use the equation (55) by replacing v_{drift} by $v(\mathbf{x}_G)$. The functional $-4\dot{S}(\frac{\partial u_I}{\partial t}, \underline{u}_I)$ is given by the equation (57) and $\dot{S}_{reduced}(\underline{u}_I, \underline{u}_I)$ is the minimum of the expression (40a) with respect to $\underline{u}_{II}$, now submitted to the constraints (53) made homogeneous by cancelling the RHS. Finally, the equations (58,59,60) apply in the axisymmetric or $\nabla_{//}B = 0$ configurations. In the general case, where one cannot state that all the trajectories are integrable and close to the magnetic surfaces by a distance $\sim \rho_c$, the equation (59) does not apply. The ambipolarity of the particle fluxes, i.e. the equation (60c), is then introduced by the extremalization with respect to $\underline{\Psi}(v,t)$ when applying the principle (20). Sugama and Horton have shown that the ambipolarity allows to build up an Onsager matrix which does not involve the electric potential (Sugama and Horton 1996). This is quite clear in a variational presentation of the Onsager relaxation: the working potential $\underline{\Psi}(v,t)$, since it does not appear in the expression (57) of $-4\dot{S}(\frac{\partial u_I}{\partial t}, \underline{u}_I)$, must minimize the expression (58) of $\dot{S}_{reduced}(\underline{u}_I, \underline{u}_I)$, which then becomes a quadratic form in $\underline{n}(v,s,t), \underline{T}(v,t), \underline{I}_{tor}(v,t)$ only.

In an axisymmetric configuration, where the equations (59) hold true, $\underline{\Psi}(v,t)$ is involved neither in $-4\dot{S}(\frac{\partial u_I}{\partial t}, \underline{u}_I)$ nor in $\dot{S}_{reduced}(\underline{u}_I, \underline{u}_I)$. When one applies the principle (20), the extremalization with respect to $\underline{\tilde{\Psi}}(v,t)$ gives no information . The ambipolarity (60c) results from the equation (59) directly. The electric potential $\tilde{\Psi}(v,t)$ and the associated rotation of the plasma along φ around the major axis are let free by the principle (20). However this is only true as long as the inertia and the viscosity effects associated to that rotation are not taken into account. A strong potential gradient $\frac{\partial \tilde{\Psi}(v,t)}{\partial v}$ induces via the equation (56) a derivative $(\frac{\partial u_I}{\partial \mathbf{p}})_{dir\,rot} = T_0 \frac{\partial \tilde{\Psi}(v_{drift},t)}{T_0 \partial v} \frac{e \partial v_{drift}}{\partial \mathbf{p}} = R\Omega(v_{drift},t)\, R\nabla\varphi$ where $\Omega(v,t) = \frac{\partial \tilde{\Psi}(v,t)}{\partial v}\left(\frac{\partial \psi_{pol}(v,t)}{\partial v}\right)^{-1}$, which represents the rotational velocity at a given $\mathbf{x}$ of the various subspecies $\mathbf{p}$,s. of the plasma. That gradient enters into $\dot{S}_{reduced}(\underline{u}_I, \underline{u}_I)$ by the differences of 2 $(\frac{\partial \underline{u}_I}{\partial \mathbf{p}})_{dir\,rot}$ taken at the same $\mathbf{x}$ for 2 different $\mathbf{p}$,s , i. e. by the differences of $\frac{\partial \Omega(v,t)}{\partial v}(v_{drift} - v(\mathbf{x},t)) = \frac{\partial \Omega(v,t)}{\partial v} \frac{Rm V_\varphi}{e \partial \psi_{pol}/\partial v}$. For reasons of parity in $\mathbf{V}$, there is no coupling between such differences and the differences of 2 $(\frac{\partial \underline{u}_I}{\partial \mathbf{p}})_{dir}$ due to

$\frac{\partial \tilde{\underline{n}}}{\partial v}$, $\frac{\partial \tilde{\underline{I}}(v,t)}{\partial v}$, $\underline{I}_{tor}(v,t)$. Therefore the rotation enters into $S_{reduced}(\underline{u}_I,\underline{u}_I)$ by a decoupled quadratic form $\int a_{viscosity}(v,t)\frac{\partial \underline{\Omega}(v,t)}{\partial v}\frac{\partial \underline{\Omega}(v,t)}{\partial v}dv$. On the other hand the term $(\frac{\partial u_I}{\partial \mathbf{p}})_{dir\ rot}\cdot m\mathbf{V} = R\Omega m V_\varphi$ of u_I contributes to $S(u_I,u_I)$, given by the equation (53), by a quantity $-\sum_s\int \frac{n_0 m}{2T_o}\Omega\Omega R^2 d_3x$, equal to $\frac{-1}{T_0}\times$ the rotational kinetic energy. It then contributes to $-4S(\frac{\partial u_I}{\partial t},\underline{u}_I)$ by $2\sum_s\int \frac{n_0 m}{T_o}\frac{\partial \underline{\Omega}(v,t)}{\partial t}\underline{\Omega}(v,t)\,R^2 d_3x = 2\int a_{inertia}(v,t)\frac{\partial \underline{\Omega}(v,t)}{\partial t}\underline{\Omega}(v,t)dv$. When one applies the principle (20), the extremalization with respect to $\tilde{\underline{\Psi}}(v,t)$ around $\tilde{\Psi}(v,t)$, or equivalently to $\underline{\Omega}(v,t)$ around $\Omega(v,t)$, provides the relaxation equation $a_{inertia}\frac{\partial \Omega}{\partial t} = \frac{\partial}{\partial v}(a_{viscosity}\frac{\partial \Omega}{\partial v})$. This expresses the the balance of the inertia and viscosity forces along φ, but as well the ambipolarity of the particle fluxes: the radial polarization current due to the variation in time of the radial electric field is compensated by the radial current due to the viscosity forces.

3.3 TURBULENT SITUATIONS

We will now try to extend the above results when our isolated toroidal field-plasma system involves a saturated turbulence, whose level relaxes towards 0 together with the general deviation from the thermodynamical equilibrium. That saturated turbulence is the consequence of the initial situation, either directly or through an enhancement by instability. We will assume hereafter weakly collisional regimes. Let us begin by a situation where the consistent field $\mathbf{A}(\mathbf{x},t)$ $\Psi(\mathbf{x},t)$ remains quasi static, although it involves a complex component perturbing the configuration. It is convenient to take as a guiding example a perturbation creating magnetic islands on a set of resonant magnetic surfaces. A first possibility is that the island chains are well separated so that, between the island chains and within each island, a magnetic surface passes through each point $\mathbf{x}$, by embrassing the volume $v(x,t)$ and the fluxes $\psi_{tor}(v,t)$, $\psi_{pol}(v,t)$. Of course the magnetic surfaces exhibit a different topology between the island chains, where they are nested around the main magnetic axis, and within each island where they surround a secondary magnetic axis. The situation remains similar to those studied above: one may expect an Onsager relaxation for the vector $A(t)$ representing the profiles $\tilde{n}$, $\tilde{T}$, $\tilde{\Psi}$ and I_{tor} on the successive magnetic surfaces between and within the islands, if the plasma layers around these surfaces are independent enough. In our weakly collisional regime, such an independence of the layers demands that the particle motion is integrable and close to the magnetic surfaces. This is typically the case if we neglect $\nabla_{//}B$ so that all the particles are passing. We may then assume the existence of 3 angular variables $\Phi(\mathbf{x},\mathbf{p},s,t)$ and 3

action variables $\mathbf{J}(\mathbf{x},\mathbf{p},s,t)$ for the Hamiltonian $H = h(\mathbf{J},s,t)$: for passing particles and a large safety factor we have $J_1 = \frac{-m}{e}\mu$, J_2 # $e\psi_{tor}(v(\mathbf{x}_G,t),t)$, J_3 # $e\psi_{pol}(v(\mathbf{x}_G,t),t) + RmV_{G//}$. Of course, the action variables $\mathbf{J}(\mathbf{x},\mathbf{p},s,t)$ changes of structure with respect to $\mathbf{x},\mathbf{p}$ when one passes in the phase space $\mathbf{x},\mathbf{p}$ from a type of trajectory topology to another, i.e., in our guiding example, when one passes from an island to a space between 2 island chains. The weak collisionality imposes that, in each domain DJ corresponding to a type of trajectory topology, the deviation $U(\mathbf{x},\mathbf{p},s,t)$ is nearly a function $u(\mathbf{J},s,t)$. The function $u(\mathbf{J},s,t)$, continuous in space $\mathbf{x},\mathbf{p}$, changes of structure with respect to $\mathbf{J}$ when one passes from a domain DJ to another. This does not prevent that function to satisfy the principle (18) implying an Onsager relaxation on the basis of the equations (19). One may indeed show that the principle (18) may be applied at a given time by freezing the various domains DJ. Within or near small islands, one must really use the 3 action variables $\mathbf{J}$ to express U as $u(\mathbf{J},s,t)$: if one expresses U as $u(\mathbf{J},t)$ with $\mathbf{J}$ meaning the 3 constants of motion H, μ and v_{drift} widely used above, it is not guaranteed that $S(\frac{\partial U(\mathbf{x},\mathbf{p},s,t)}{\partial t}, \underline{U}(\mathbf{x},\mathbf{p},s,t))$ is safely replaced by $S(\frac{\partial u(\mathbf{J},s,t)}{\partial t}, \underline{U}(\mathbf{J},s,t))$ when one goes from the principles (15) to the principle (18). In each domain DJ the function $u(\mathbf{J},s,t)$ then reflects the profiles $\tilde{n}, \tilde{T}, \tilde{\Psi}$ and I_{tor}, expressed versus ψ_{tor} rather than versus v (since J_2 # $e\psi_{tor}(\mathbf{x}_G,t)$). With that small change, the principle (20) is still applicable on the basis of equations of type (55,57,58).

Another regime occurs when the resonant surfaces are close to each other so that the magnetic islands overlap. The particle trajectories are no longer integrable. We will consider the actual hamiltonian H as a sum $H_{int}(\mathbf{x},\mathbf{p},s,t) + H'(\mathbf{x},\mathbf{p},s,t)$, where H_{int} is integrable and H' is a small resonant term destroying the integrability. In the simplest approach, H_{int} is the hamiltonian of the unperturbed configuration in the absence of the turbulent field creating the islands and H' is the hamiltonian perturbation associated to that field. In the presence of the hamiltonian H_{int} only, the trajectories are described by 3 angular variables $\Phi(\mathbf{x},\mathbf{p},s,t)$ and 3 action variables $\mathbf{J}(\mathbf{x},\mathbf{p},s,t))$ such that $H_{int} = h(\mathbf{J},s,t)$. The quasi static perturbation H' is expressed in terms of the dynamical variables Φ, $\mathbf{J}$ as

$$H' = \sum_{\mathbf{n}} h_{\mathbf{n}}(\mathbf{J},s,t)\exp(i\,\mathbf{n}\cdot\Phi) \qquad (61)$$

where $\mathbf{n}$ is a set of 3 integers n_1,n_2,n_3. For a small H' and a weak collisionality, the dynamical variables H, F, U are approximately functions $h(\mathbf{J},s,t)$, $f(\mathbf{J},s,t)$, $u(\mathbf{J},s,t)$ representing the average of H, F, U over Φ at given $\mathbf{J}$. However, the change from the principles (15) to the principle (18) is now impossible since the dynamical variables $\mathbf{J}$ are not constants of motion for the actual hamiltonian H: the basic relation $\Sigma(U,\underline{u}(\mathbf{J},s,t)) = 0$

does not hold. Nevertheless F and U tend to be constant along the trajectories in the presence of the actual hamiltonian H. Because of the non integrability of these trajectories and of their resulting stochasticity, such a large scale of U along the trajectories imposes fine scales to U in the transverse directions. The adjustment is such that the fine transverse scales are at the limit of being felt by the particles in view of the collisional diffusion they experience during their transit over the large longitudinal scale (Rechester and Rosenbluth 1978). From the point of view of the principle (15), we first notice that the structure of U and of $\underline{U}$ around their averaged $u(\mathbf{J},s,t)$ and $\underline{u}(\mathbf{J},s,t)$ over Φ at given $\mathbf{J}$ disappears by integration in the entropy functionals $S(U,\underline{U})$, $S(\frac{\partial U}{\partial t},\underline{U})$, that we may write as well $S(u,\underline{u})$, $S(\frac{\partial u}{\partial t},\underline{u})$. On the other hand we notice that the large longitudinal scale of U, $\underline{U}$ tends to decrease the functional $\Sigma(U,\underline{U})$ while the fine transverse scales enhance $\dot{S}(U,\underline{U})$, since they correspond to large gradients in $\mathbf{p}$ space. By this mechanism, $\dot{S}(U,\underline{U})$ is reaching the level of $\Sigma(U,\underline{U})$ in spite of the assumed weak collisionality. The principle (15) leads to extremalize the bilinear form $\Sigma(U,\underline{U})+\dot{S}(U,\underline{U})$ in U and $\underline{U}$ under the constraint that U and $\underline{U}$ exhibit the averages $u(\mathbf{J},s,t)$ and $\underline{u}(\mathbf{J},s,t)$ over Φ at given $\mathbf{J}$. This changes $\Sigma(U,\underline{U})+\dot{S}(U,\underline{U})$ into a finite bilinear form $\dot{S}_{QL}(u,\underline{u})$ in u and $\underline{u}$. For a weak enough H', satisfying however the Chirikov criterium (the magnetic islands overlap in our guiding example), the quasilinear theory allows to calculate that $\dot{S}_{QL}(u,\underline{u})$ via a formal development at second order in H', as it allows to establish the diffusion equation of the particles in space J. One finds

$$\dot{S}_{QL}\left(\underline{u}(\mathbf{J},s,t),\underline{u}(\mathbf{J},s,t)\right) = \dot{S}(\underline{u},\underline{u}) + \sum_s \int \frac{f}{2T_0^2} \sum_{\mathbf{n}} (\mathbf{n}\cdot\frac{\partial \underline{u}}{\partial \mathbf{J}})(\mathbf{n}\cdot\frac{\partial \underline{u}}{\partial \mathbf{J}}) |h_{\mathbf{n}}|^2 2\pi\,\delta(\mathbf{n}\cdot\frac{\partial h}{\partial \mathbf{J}})(2\pi)^3 d_3J \qquad (62)$$

The bilinear form $\dot{S}_{QL}(u,\underline{u})$ is symmetrical, i.e. $\dot{S}_{QL}(u,\underline{u})=\dot{S}_{QL}(\underline{u},u)$, by virtue of the basic $D_{kl} = D_{lk}$. The principle (18) becomes applicable by replacing $\dot{S}(\underline{u},\underline{u})$ by $\dot{S}_{QL}(\underline{u},\underline{u})$.

Let us apply that principle (18) in a tokamak, by neglecting the collisional term $\dot{S}$ in $\dot{S}_{QL}$. We assume that the turbulent field has its wave numbers mainly perpendicular to the flux lines, so that the friction forces that it exerts on the species or subspecies are also perpendicular. This means that the quasilinear entropy production rate $\dot{S}_{QL}(\underline{u},\underline{u})$ is mainly influenced by the transverse diamagnetic velocities of the various species and subspecies. The action variable $\frac{J_3(\mathbf{x},\mathbf{p},s,t)}{e}$ is close to $\psi_{pol}(v(\mathbf{x},t))$, the poloidal flux embrassed by the unperturbed magnetic surface of volume $v(\mathbf{x},t)$. For given J_3, the action variables J_1 and J_2 determine the transverse and parallel motions. We define v_{drift}

as the function of J_3 close to $v(\mathbf{x},t)$ such that $\psi_{pol}(v_{drift}) = \frac{J_3}{e(s)}$. It appears that the values of $S\left(\frac{\partial u}{\partial t}, \underline{u}\right)$ and of $\dot{S}_{QL}(\underline{u},\underline{u})$ are correctly produced by taking

$$\frac{u(\mathbf{J},s,t)}{T_0} = \frac{\tilde{n}(v_{drift},s,t)}{n_0} + \frac{e(s)\tilde{\Psi}(v_{drift},t)}{T_0} + (\frac{H}{T_0} - \frac{3}{2})\frac{\tilde{T}(v_{drift},s,t)}{T_0} \quad ; \quad \psi_{pol}(v_{drift}) = \frac{J_3}{e(s)}$$

$$\text{implying}: \mathbf{n}\cdot\frac{\partial \underline{u}}{T_0\partial \mathbf{J}} = \left(\frac{\partial \underline{\tilde{n}}}{n_0\partial v} + \frac{e\,\partial \underline{\tilde{\Psi}}}{T_0\partial v} + (\frac{h}{T_0} - \frac{3}{2})\frac{\partial \underline{\tilde{T}}}{T_0\partial v}\right)\frac{n_3}{e\,\partial\psi_{pol}/\partial v} + \frac{\tilde{T}}{T_0^2}(\mathbf{n}\cdot\frac{\partial h}{\partial \mathbf{J}}) \qquad (63)$$

The principle (18) in its quasilinear version then states that the functional

$$2\sum_s \int \left(\frac{1}{(n_o(s))^2}\frac{\partial \tilde{n}(v,s,t)}{\partial t}\underline{\tilde{n}}(v,s,t) + \frac{3}{2T_o^2}\frac{\partial \tilde{T}(v,s,t)}{\partial t}\underline{\tilde{T}}(v,s,t)\right)n_o(s)dv$$

$$+\sum_s \int \frac{f}{2T_0^2}\sum_{\mathbf{n}}\left(\frac{\partial \underline{\tilde{n}}}{n_0\partial v} + \frac{e\,\partial \underline{\tilde{\Psi}}}{T_0\partial v} + (\frac{h(\mathbf{J},s,t)}{T_0} - \frac{3}{2})\frac{\partial \underline{\tilde{T}}}{T_0\partial v}\right)\frac{n_3 T_0}{e\,\partial\psi_{pol}/\partial v} + \frac{\tilde{T}}{T_0}(\mathbf{n}\cdot\frac{\partial h(\mathbf{J},s,t)}{\partial \mathbf{J}})\Big)^2$$

$$|h_{\mathbf{n}}|^2 2\pi\delta(\mathbf{n}\cdot\frac{\partial h(\mathbf{J},s,t)}{\partial \mathbf{J}})(2\pi)^3 dJ_1\,dJ_2\,|e\partial\psi_{pol}(v,t)/\partial v|dv \qquad (64)$$

with J_3 identified with $e(s)\psi_{pol}(v,t)$, is an extremum for all the variations of the working profiles $\underline{\tilde{n}}, \underline{\tilde{\Psi}}, \underline{\tilde{T}}$ around the actual profiles $\tilde{n}, \tilde{\Psi}, \tilde{T}$. The term $\frac{\tilde{T}}{T_0}(\mathbf{n}\cdot\frac{\partial h}{\partial \mathbf{J}})$ disappears from the functional (64) because it is multiplied by $\delta(\mathbf{n}\cdot\frac{\partial h}{\partial \mathbf{J}})$. Accordingly, the extremalization of the latter with respect to $\underline{\tilde{n}}, \underline{\tilde{T}}$ around $\tilde{n}, \tilde{T}$, leads to Onsager relations between $\frac{\partial \tilde{n}}{\partial t}$ and $\frac{\partial \tilde{T}}{\partial t}$ on one hand and $\frac{\partial \tilde{n}}{n_0\partial v} + \frac{e\partial\tilde{\Psi}}{T_0\partial v}$ and $\frac{\partial \tilde{T}}{T_0\partial v}$ on the other hand involving particle and energy fluxes but no heating power by the turbulent field. This is normal since the latter is a quasi static, non oscillating field. The extremalization with respect to $\underline{\tilde{\Psi}}$ around Ψ insures the ambipolarity of the fluxes. It may be used to eliminate $\underline{\tilde{\Psi}}$ from the functional (64), which then involves only the density and temperature profiles.

Let us now assume that the turbulent perturbing field is no longer quasi static but exhibits oscillations in time, meaning in our guiding example a rotation of the islands. We assume the oscillations of the system quasi periodic: the consistent field $\mathbf{A}, \Psi$ involving the turbulence and the dynamical variables H,F,U linked by the equations (6) are periodic functions of period 2π of a set of q phases $\alpha = (\alpha_1, \ldots, \alpha_q)$

$$\mathbf{A},\Psi = [A,\Psi](\mathbf{x},\alpha,s,t) \text{ and } H,F,U = [H,F,U](\mathbf{x},\mathbf{p},\alpha,s,t) \qquad (65)$$

In addition, the phases α have at each time a variation rate $\frac{d\alpha}{dt} = \omega(t) = (\omega_1(t),\ldots,\omega_q(t))$, assumed of course much larger than the relaxation rate, and uncommensurable so that the useful cell of extension $(2\pi)^q$ in space α is ergodically covered at each t. The explicit dependence on time t in the functions (65) and in $\omega(t)$ reflects the slow relaxation of the isolated field-plasma system towards its thermodynamical equilibrium, namely the relaxation of $[\mathbf{A},\Psi](\mathbf{x},\alpha,s,t)$ towards $[A_0,\Psi_0](\mathbf{x},s)$, of $[H,F,U](\mathbf{x},\mathbf{p},\alpha,s,t)$ towards $[H_0,F_0,U_0](\mathbf{x},\mathbf{p},s)$ and of the frequencies $\omega(t)$ towards a null value. A convenient way to generalize the results obtained above with a quasi static turbulence is to consider the dependence of $[\mathbf{A},\Psi](\mathbf{x},\alpha,s,t)$ and $[H,F,U](\mathbf{x},\mathbf{p},\alpha,s,t)$ on the phases α on the same grounds as the dependence on $\mathbf{x}$ or $\mathbf{x},\mathbf{p}$. This means that we install our field $\mathbf{A},\Psi$ and our particles in an extended space $\mathbf{x},\alpha$ of dimension 3+q. To preserve the hamiltonian nature of the particle motion, we use the well known trick of introducing an artificial set $\mathbf{j} = (j_1,\ldots j_q)$ canonically conjugate to $\alpha = (\alpha_1,\ldots,\alpha_q)$. The particles are thus installed in an extended phase space of dimension 2(3+q), namely $\mathbf{x},\alpha$ conjugate to $\mathbf{p},\mathbf{j}$. We choose as hamiltonian in that extended phase space

$$H(\mathbf{x},\mathbf{p},\alpha,\mathbf{j},s,t) = H(\mathbf{x},\mathbf{p},\alpha,s,t) + \omega(t)\cdot\mathbf{j} \qquad (66a)$$

This extended hamiltonian indeed produces variation rates $\frac{d\alpha}{dt} = \frac{\partial H}{\partial \mathbf{j}}$ of the phases α which coincide with the actual variation rates ω. Also the variation rate $\frac{dW}{dt} = \frac{\partial W}{\partial t} + \{H,W\}$ of any dynamical variable $W(\mathbf{x},\mathbf{p},\alpha,s,t)$ which is "natural" - i.e. independent of $\mathbf{j}$ and 2π periodic in $\alpha_1,\ldots,\alpha_q$ - coincides with the actual $\frac{dW}{dt} = \frac{\partial W}{\partial t} + \frac{\partial W}{\partial \alpha}\cdot\omega + \{H,W\}$. It is convenient to impose hamiltonian barriers preventing the particles to escape from a large box in space $\mathbf{j}$. The distribution $F(\mathbf{x},\mathbf{p},\alpha,\mathbf{j},s,t)$ of the particles in the extended phase space $\mathbf{x},\mathbf{p},\alpha,\mathbf{j}$ verifies the kinetic equation $\frac{\partial F}{\partial t} + \{H,\ F\} = C(F)$, with C being the usual collision operator differential in $\mathbf{p}$. The natural distribution $F(\mathbf{x},\mathbf{p},\alpha,s,t)$ verifies that kinetic equation. In fact it may be shown that F coincides with F (except in an irrelevant zone near the boundaries of the large allowed box in space $\mathbf{j}$; also F and F differ by a mutiplicative constant, omitted in what follows). The relaxation of the natural, oscillating system and that of the quasi static extended system are therefore equivalent. The entropy of the extended system at a time t

is $-\sum_s \int F \ln(F)\, d_3x d_3p d_q\alpha d_q j = -\sum_s \int F \ln(F) d_3x d_3p \frac{d_q\alpha}{(2\pi)^q}$, equal to the natural entropy $-\sum_s \int F \ln(F) d_3x d_3p$ since the useful cell of extension $(2\pi)^q$ in space α is ergodically covered at each time t. We are led to define the deviation of the extended system from the thermodynamical equilibrium by the dynamical variable U linked to H and F by the equation (6). As $F = F$ is independent of $\mathbf{j}$, we obtain

$$U(\mathbf{x},\mathbf{p},\alpha,\mathbf{j},s,t) = U(\mathbf{x},\mathbf{p},\alpha,s,t) + \omega(t)\cdot\mathbf{j} \qquad (66b)$$

This means that the extended deviation $U(\mathbf{x},\mathbf{p},\alpha,\mathbf{j},s,t)$ of our system from the thermodynamical equilibrium is specified by the natural deviation $U(\mathbf{x},\mathbf{p},\alpha,s,t)$ and by the frequencies $\omega(t)$. At this point it is essential to the consistency of the scheme that at a given time t the extended deviation $U(\mathbf{x},\mathbf{p},\alpha,\mathbf{j},s,t)$ fully determines the state of the extended field plasma system. Indeed an extended version of the principle (42) allows to calculate $\mathbf{A}(\mathbf{x},\alpha,t), \Psi(\mathbf{x},\alpha,t)$ together with $H(\mathbf{x},\mathbf{p},\alpha,s,t)$ at a given time t if one knows $U(\mathbf{x},\mathbf{p},\alpha,s,t)$ and $\omega(t)$ at that t. The equation (6) then determines $F(\mathbf{x},\mathbf{p},\alpha,s,t)$. Introducing the working deviation $\underline{U} = \underline{U}(\mathbf{x},\mathbf{p},\alpha,s,t) + \underline{\omega}(t)\cdot\mathbf{j}$, where $\underline{U}$ and $\underline{\omega}$ are to be varied around U and ω, one readily establishes extended versions $S, \dot{S}$ and Σ in $U, \underline{U}$ of S, $\dot{S}$ and Σ in $U, \underline{U}$. The relaxation of U, i. e. of $U(\mathbf{x},\mathbf{p},\alpha,s,t)$ and $\omega(t)$, is then obtained from the extended version of the principles (15). We find in fact that

$$S(U,\underline{U}) = \int S(U,\underline{U}) \frac{d_q\alpha}{(2\pi)^q} \text{ and } \dot{S}(U,\underline{U}) = \int \dot{S}(U,\underline{U}) \frac{d_q\alpha}{(2\pi)^q} \qquad (67)$$

At second order in U the quadratic form $S(U,U)$ is the entropy of our isolated field plasma system with respect to the thermodynamical equilibrium. Of course the antisymmetrical functional $\Sigma(U,\underline{U})$ cancels exactly if U or $\underline{U}$ commutes with H.

Again a first possibility is that the extended hamiltonian $H(\mathbf{x},\mathbf{p},\alpha,\mathbf{j},s,t)$ frozen at a given time t produces integrable trajectories in the extended phase space $\mathbf{x},\mathbf{p},\alpha,\mathbf{j}$, described by 3+q angular variables $\Phi(\mathbf{x},\mathbf{p},\alpha,\mathbf{j},s,t)$ canonically conjugate with 3+q action variables $J(\mathbf{x},\mathbf{p},\alpha,\mathbf{j},s,t)$ such that $H = h(J, s,t)$. It appears that, if it exists, the set Φ, J has a well defined structure with respect to the phases α and the artificial dynamical variables $\mathbf{j}$: it is readily shown that there exists at each time time 3 natural variables Φ of angular type, 3 natural variables $\mathbf{J}$ of action type and q natural dynamical variables $\mathbf{K}$, such that

Φ consists of $\Phi = (\Phi_k(\mathbf{x},\mathbf{p},\alpha,s,t))$ and of $\alpha = (\alpha_r)$

J consists of $\mathbf{J} = (J_k(\mathbf{x},\mathbf{p},\alpha,s,t))$ and of $\mathbf{j} + \mathbf{K} = (j_r + K_r(\mathbf{x},\mathbf{p},\alpha,s,t))$

$k = 1,2,3$; , $r = 1, \ldots, q$

In view of the equation (66a), $H(\mathbf{x},\mathbf{p},\alpha,\mathbf{j},s,t) = h(J,s,t)$ must then have the form $h(\mathbf{J},s,t) + \omega(t)\cdot(\mathbf{j} + \mathbf{K})$, the normal hamiltonian $H(\mathbf{x},\mathbf{p},\alpha,s,t)$ being then identified with $h(\mathbf{J},s,t) + \omega\cdot\mathbf{K}$. We now introduce the point that, in our very weakly collisional regime, the extended deviation $U(\mathbf{x},\mathbf{p},\alpha,\mathbf{j},s,t)$ must be nearly a function $u(J,s,t)$. The structure of $\mathbf{J}$ with respect to $\mathbf{j}$ and the equation (66b) then imposes the following forms of that function and of the natural deviation $U(\mathbf{x},\mathbf{p},\alpha,s,t)$

$$u(J,s,t) = u(\mathbf{J},s,t) + \omega(t)\cdot(\mathbf{j} + \mathbf{K}) \; ; \; U(\mathbf{x},\mathbf{p},\alpha,s,t) = u(\mathbf{J},s,t) + \omega(t)\cdot\mathbf{K}(\mathbf{x},\mathbf{p},\alpha,s,t)$$

while $F = F(\mathbf{x},\mathbf{p},\alpha,s,t)$ # $f(\mathbf{J},s,t)$. The extended deviation $U = u(J,s,t)$ is now characterized by the function $u(\mathbf{J},s,t)$ and the frequencies $\omega(t)$. One may change from the extended principles (15) involving U and $\underline{U}$ to the extended principle (18) involving the function $u(J,s,t) = u(\mathbf{J},s,t) + \omega(t)\cdot(\mathbf{j}+\mathbf{K})$ and its working version $\underline{u}(J,s,t) = \underline{u}(\mathbf{J},s,t) + \underline{\omega}(t)\cdot\mathbf{j}$. Taking into account the eqs (67), the extended principle (18) states that

$$-4\int S\left(\frac{\partial u(\mathbf{J},s,t)}{\partial t} + \frac{\partial \omega(t)}{\partial t}\cdot\mathbf{K}\,,\, \underline{u}(\mathbf{J},s,t) + \underline{\omega}(t)\cdot\mathbf{K}\right)\frac{d_q\alpha}{(2\pi)^q}$$

$$+\int \dot{S}\left(\underline{u}(\mathbf{J},s,t) + \underline{\omega}(t)\cdot\mathbf{K}\,,\, \underline{u}(\mathbf{J},s,t) + \underline{\omega}(t)\cdot\mathbf{K}\right)\frac{d_q\alpha}{(2\pi)^q} \qquad (68)$$

is at each time t an extremum for all the variations of $\underline{u}(\mathbf{J},s,t)$ and of $\underline{\omega}(t)$ around $u(\mathbf{J},s,t)$ and $\omega(t)$. This implies of course an Onsager relaxation for the vector Λ representing at each time t the function $u(\mathbf{J},s,t)$ of argument J,s and the frequencies $\omega(t)$. In a magnetic configuration containing magnetic islands where a magnetic surface passes through each point $\mathbf{x}$ between and within the islands, by embrassing the volume $v(\mathbf{x},\alpha,t)$ and the fluxes $\psi_{tor}(v,t)$, $\psi_{pol}(v,t)$, the 3 action variables $\mathbf{J}(\mathbf{x},\mathbf{p},\alpha,s,t)$ are roughly equal to the 3 action variables for the hamiltonian $H(\mathbf{x},\mathbf{p},\alpha,s,t)$ with frozen α and t (namely, for passing particles and a large safety factor: $J_1 = \frac{-m}{e}\mu$, J_2 # $e\psi_{tor}(v(\mathbf{x}_G,\alpha,t),t)$, J_3 # $e\psi_{pol}(v(\mathbf{x}_G,\alpha,t),t) + RmV_{G//})$. The function $u(\mathbf{J},t)$ is roughly given by a formula of type (55) in terms of the various profiles $\tilde{n}, \tilde{T}, \tilde{\Psi}$ and I_{tor} versus ψ_{tor}. On the other hand, the q dynamical variables $\mathbf{K}(\mathbf{x},\mathbf{p},\alpha,s,t)$ reflect the quasi MHD displacement of the magnetic surfaces, taking place at the velocity $\mathbf{V}_{MHD}(\mathbf{x},\alpha,t)$ perpendicular to $\mathbf{B}$: the

derivative $\frac{\partial(\omega \bullet \mathbf{K})}{\partial \mathbf{p}}$ is roughly identified to $\mathbf{V}_{MHD}(\mathbf{x},\alpha,t)$. Writing $\mathbf{V}_{MHD} = (\omega \bullet \frac{\partial}{\partial \alpha})\mathbf{D}_{MHD}(\mathbf{x},\alpha,t)$, the derivative $\frac{\partial(\underline{\omega} \cdot \mathbf{K})}{\partial \mathbf{p}}$ is then identified to $(\underline{\omega} \cdot \frac{\partial}{\partial \alpha})\ \mathbf{D}_{MHD}$. The dynamical variables $\frac{\partial \omega}{\partial t} \bullet \mathbf{K}$ and $\underline{\omega} \cdot \mathbf{K}$ in the entropy term S of the functional (68) introduce inertia effects due to the displacement of the magnetic surfaces. On the other hand, $\underline{\omega} \cdot \mathbf{K}$ in the entropy production rate term $\dot{S}$ introduces the friction and viscosity effects due to that displacement. Generally the inertia effects are negligible so that the frequencies $\underline{\omega}$ around ω minimize that entropy production rate. Such a minimization is analogous to the minimization with respect to the electric potential $\underline{\Psi}$.

Another possibility is that the extended hamiltonian $H = H_{int} + H'$ where $H_{int} = H_{int}(\mathbf{x},\mathbf{p},\alpha,s,t) + \omega \bullet \mathbf{j}$ produces integrable trajectories, while $H'(\mathbf{x},\mathbf{p},\alpha,s,t)$ is a small resonant perturbation destroying the integrability, to which the quasilinear theory may be applied. In the simplest approach H_{int} is the hamiltonian of the unperturbed configuration in the absence of the turbulent field and H' is the perturbation due to the latter. The 3+q angular variables Φ and the 3+q canonically conjugate action variables $\boldsymbol{J}$ for $H_{int} = h(\boldsymbol{J},s,t)$ are then formed by the 3 angular and action variables $\Phi(\mathbf{x},\mathbf{p},s,t) = (\Phi_1, \Phi_2, \Phi_3)$ and $\mathbf{J}(\mathbf{x},\mathbf{p},s,t) = (J_1, J_2, J_3)$ for the unperturbed hamiltonian $H_{int}(\mathbf{x},\mathbf{p},s,t) = h(\mathbf{J},s,t)$, together with the q phases $\alpha = (\alpha_1,\ldots, \alpha_q)$ and the q artificial momenta $\mathbf{j} = (j_1,\ldots, j_q)$

$$\boldsymbol{\Phi} = (\Phi(\mathbf{x},\mathbf{p},s,t) \text{ and } \alpha)\ ;\ \boldsymbol{J} = (\mathbf{J}(\mathbf{x},\mathbf{p},s,t) \text{ and } \mathbf{j})\ ;\ h(\boldsymbol{J},s,t) = h(\mathbf{J},s,t) + \omega \bullet \mathbf{j}$$

The perturbation H' takes the following form, which extends the expression (61)

$$H' = \sum_{\mathbf{n},\nu} h_{\mathbf{n},\nu}(\mathbf{J},s,t) \exp(i(\mathbf{n} \cdot \Phi + \nu \cdot \alpha)) \qquad (69)$$

$\mathbf{n}$ and ν meaning 3 integers (n_1,n_2,n_3) and q integers $(\nu_1,\ldots,\nu_q)$, respectively. The extended particle distribution function $F = F(\mathbf{x},\mathbf{p},\alpha,s,t)$ and the extended deviation $U = U(\mathbf{x},\mathbf{p},\alpha,s,t) + \omega(t) \bullet \mathbf{j}$ are approximate functions of $\boldsymbol{J}$. This means that $F = F\ \#\ f(\mathbf{J},s,t)$ and $U\ \#\ u(\boldsymbol{J},s,t) = u(\mathbf{J},s,t) + \omega(t) \bullet \mathbf{j}$. One extends the formula (62). by replacing $\mathbf{n}$ by $\boldsymbol{n} = (\mathbf{n} \text{ and } \nu)$, $\mathbf{n} \cdot \frac{\partial}{\partial \mathbf{J}}$ by $\boldsymbol{n} \cdot \frac{\partial}{\partial \boldsymbol{J}} = \mathbf{n} \cdot \frac{\partial}{\partial \mathbf{J}} + \nu \cdot \frac{\partial}{\partial \mathbf{j}}$ and of course $h(\mathbf{J},s,t)$ by $h(\mathbf{J},s,t) + \omega(t) \bullet \mathbf{j}$ and $\underline{u}(\mathbf{J},s,t)$ by $\underline{u}(\mathbf{J},s,t) + \underline{\omega}(t) \cdot \mathbf{j}$. It comes

$$\dot{S}_{QL}\left(\underline{u}(\mathbf{J},s,t),\underline{\omega}\ ;\ \underline{u}(\mathbf{J},s,t),\underline{\omega}\right) = \dot{S}(\underline{u},\underline{u}) +$$

$$\sum_s \int \frac{f}{2T_0^2} \sum_{\mathbf{n}\nu} (\mathbf{n}\cdot\frac{\partial \underline{u}}{\partial \mathbf{J}} + \nu\cdot\underline{\omega})(\mathbf{n}\cdot\frac{\partial \underline{u}}{\partial \mathbf{J}} + \nu\cdot\underline{\omega}) |h_{\mathbf{n}\nu}|^2 2\pi\delta(\mathbf{n}\cdot\frac{\partial h}{\partial \mathbf{J}} + \nu\cdot\omega)(2\pi)^3 d_3J \quad (70)$$

Still applying the equations (63), one extends the principle (64), which becomes a principle of type (4) for the vector $\Lambda(t)$ representing the profiles $\tilde{n}(v,s,t)$, $\tilde{\Psi}(v,t),\tilde{T}(v,s,t)$ and the frequencies $\omega(t)$: at each time t the functional

$$2\sum_s \int \left(\frac{1}{(n_o(s))^2}\frac{\partial \tilde{n}}{\partial t}\tilde{\underline{n}} + \frac{3}{2T_o^2}\frac{\partial \tilde{T}}{\partial t}\tilde{\underline{T}}\right) n_o(s)dv$$

$$+\sum_s \int \frac{f}{2T_0^2} \sum_{\mathbf{n}\nu} \left(\frac{\partial \tilde{\underline{n}}}{n_0 \partial v} + \frac{e\,\partial \tilde{\underline{\Psi}}}{T_0 \partial v} + \left(\frac{h(\mathbf{J},s,t)}{T_0} - \frac{3}{2}\right)\frac{\partial \tilde{\underline{T}}}{T_0 \partial v}\right)\frac{n_3 T_0}{e\,\partial \psi_{pol}/\partial v} + \frac{\tilde{\underline{T}}}{T_0}\left(\mathbf{n}\cdot\frac{\partial h(\mathbf{J},s,t)}{\partial \mathbf{J}}\right) + \nu\cdot\underline{\omega}\Bigg)^2$$

$$|h_{\mathbf{n}\nu}|^2 2\pi\delta(\mathbf{n}\cdot\frac{\partial h(\mathbf{J},s,t)}{\partial \mathbf{J}} + \nu\cdot\omega)(2\pi)^3 dJ_1\, dJ_2\, |e\partial\psi_{pol}(v,t)/\partial v|\, dv \quad (71)$$

with J_3 identified with $e(s)\psi_{pol}(v,t)$, is an extremum for all the variations of the profiles $\tilde{\underline{n}}(v,s,t),\tilde{\underline{\Psi}}(v,s,t),\tilde{\underline{T}}(v,s,t)$ around $\tilde{n}(v,s,t)$, $\tilde{\Psi}(v,t),\tilde{T}(v,s,t)$ and of $\underline{\omega}(t)$ around $\omega(t)$. The extremalization with respect to the profiles $\tilde{\underline{n}},\tilde{\underline{T}}$ leads to an Onsager matrix of the type proposed by Shaing (Shaing 1988), relating the particle and energy fluxes to differences between diamagnetic velocities and phase velocities. The term $\frac{\tilde{\underline{T}}}{T_0}(\mathbf{n}\cdot\frac{\partial h}{\partial \mathbf{J}})$ does not disappear from the functional (71): it introduces in the expression of $\frac{\partial \tilde{T}(v,s,t)}{\partial t}$ for each species s the heating power by the turbulent field, or in other words the energy transfers between the various species via the turbulence. There is of course no reason in the present view for those transfers to be Onsager fluxes, as proposed by Sugama and Horton in another context (Sugama and Horton 1995). The minimization of the second term of the functional (71) with respect to each working frequency $\underline{\omega}_1,...,\underline{\omega}_q$ around the actual frequency ω_1, ..., ω_q gives a set of equations determining the latter. These equations express the energetic balance of the turbulent field. One may also use the minimization with respect to $\underline{\omega}$ to transform the second term of the functional (71) into a quadratic form in the profiles $\tilde{\underline{n}},\tilde{\underline{\Psi}},\tilde{\underline{T}}$ only, the functional (71) thus becoming independent of $\underline{\omega}$. A first possibility is that the component of the turbulent field $(\mathbf{A},\Psi)_1$ which is influenced by the frequency ω_1 is localized very near a well defined magnetic surface $v = v_1$, and so on with ω_2, etc. This means that the coefficients $f(\mathbf{J},s,t)|h_{\mathbf{n}\nu}(\mathbf{J},s,t)|^2$ with J_3 identified to $e\psi_{pol}(v,t)$) cancel except for specific volumes v, namely for $v = v_1$ if ν_1

$\neq 0$, etc. In that case the fluxes, friction forces, heating powers on a magnetic surface v are determined by the profiles $\tilde{n},\tilde{\Psi},\tilde{T}$ on the same surface. In fact that situation is not likely to occur: the principle (71) after elimination of $\underline{\omega}$ then relates the fluxes, friction forces, heating powers on a surface v to the profiles at some distance of that surface.

We stress again that the consistency of the above scheme demands that the knowledge at a given time t of the function $u(\mathbf{J},s,t)$ of argument $\mathbf{J}$,s and of the frequencies $\omega(t)$ allows to calculate the field $\mathbf{A}(\mathbf{x},\alpha,t),\Psi(\mathbf{x},\alpha,t)$ (involving of course the turbulent component) at that t. Such a calculation must take into account the non linear resonances of the turbulent field with the particles, which play a leading role in the saturation of the turbulence. These non linear resonances should also be taken into account in the quasilinear entropy production (70). However it is not the case if our basic partition of H into H_{int} + H' makes use of the unperturbed hamiltonian as integrable hamiltonian H_{int}, the perturbation H' then representing the naked effect of the turbulent field: the fact that the functional (70) is simply quadratic in H' then implies that the non linear resonances involving beatings between the various components of the turbulent field are excluded. To take into account those non linear resonances, and at the same time be justified in using the quasilinear entropy production rate (70) and its transparent Onsager symmetries, one must build up a special integrable hamiltonian H_{int} which properly incorporates a clothing of the particles by the non resonant components of the turbulent field. The construction of that hamiltonian H_{int} and of the associated dynamical variables Φ, J, H', h, $h_{\mathbf{n}\nu}$ is possible if one gives only the function $u(\mathbf{J},s,t)$ and the frequencies $\omega(t)$ (Garbet et al. 1993). Interestingly, it is only justified if the width of the spectrum of the resonant frequencies $\Omega_{\mathbf{n}\nu} = \mathbf{n}.\frac{\partial h(\mathbf{J},s,t)}{\partial \mathbf{J}} + \nu.\omega$ for all the useful $\mathbf{n},\nu$ such that $f(\mathbf{J},s,t)|h_{\mathbf{n}\nu}(\mathbf{J},s,t)|^2 \neq 0$, is muvch larger than the Kolmogorov broadening of the resonances $\gamma_K \sim |D_{kl}\frac{\partial \Omega_{\mathbf{n}\nu}}{\partial J_k}\frac{\partial \Omega_{\mathbf{n}\nu}}{\partial J_l}|^{1/3}$, the D_{kl}s being the particle difusion coefficients in space $\mathbf{J}$. It is plausible that this condition gives the upper limit of the turbulence level which induces an Onsager relaxation.

4. CONCLUSION

Our main conclusion is that the relaxation towards its thermodynamical equilibrium of an isolated toroidal plasma tends to conform to the Onsager scheme (1-4) based on the entropy metric. That scheme is conveniently expressed by the variational principle (4) which carries the physical idea that the vector $\Lambda(t)$ representing the deviation

of the system from the thermodynamical equilibrium relaxes in the form of modes minimizing, at constant entropy, the collisional entropy production rate. The appropriate vector $\Lambda(t)$ at the starting point is the deviation $U(\mathbf{x},\mathbf{p})$ of the particle distribution in phase space $\mathbf{x},\mathbf{p}$ from maxwellian. Our basic tool is the variational principle (15) involving U, equivalent to the kinetic equation. That principle anticipates the principle (4) implying an Onsager evolution, in the sense that it contains two symmetrical functionals S and $\dot{S}$ representing respectively the entropy and the entropy production rate. The symmetry of the functional $\dot{S}$ reflects the symmetry of the Fokker-Planck collision operator in the kinetic equation, i.e. the hamiltonian character of the interactions. For a field-plasma system the symmetry of the functional S reflects the Lagrangian character of the Maxwell equations. However, the principle (15) differs from the principle (4) by an antisymmetrical functional Σ reflecting the individual trajectory effects.

When the particle trajectories are integrable and close to the magnetic surfaces (e.g. in axisymmetric tokamaks), and in weakly collisional regimes, it is straighforward to pass from the principle (15) to the principle (4). Indeed the antisymmetrical functional Σ is eliminated since the dynamical variable $U(\mathbf{x},\mathbf{p})$ is nearly a function $u(\mathbf{J},t)$ of 3 action variables $\mathbf{J}$, constants of motion. The principle (4) automatically applies to the vector $\Lambda(t)$ representing the function $u(\mathbf{J},t)$. That type of Onsager relaxation results from the hamiltonian nature of the system and from the weak collisionality. It could have been anticipated from Van Kampen arguments without using the kinetic equation. One may then pass to a principle (4) for a reduced vector Λ representing the profiles of density, temperature, electric potential, electric current density on the successive magnetic surfaces, which determine the entropy S of the field-plasma system. The entropy production rate functional $\dot{S}_{reduced}$ for that reduced vector Λ is obtained by minimizing the general entropy production rate functional $\dot{S}$ under the constraint that the considered profiles of density, etc are achieved. It covers all the aspects of the neoclassical evolution.

In the case of a collisional regime or of non integrable trajectories, the deviation U is no longer a constant of motion, but simply a function of the slow Kruskal dynamical variables. The antisymmetrical functional Σ is not automatically eliminated. However, one may still arrive to the principle (4) for the vector Λ representing the profiles of density, etc which determine the entropy of the field-plasma system provided that the trapped particles which are not confined are detrapped long before they depart significantly from a magnetic surface. Starting from the general principle (15), the extremalization of $\Sigma + \dot{S}$, when the profiles of density, temperature, electric potential, electric current density are imposed, leads to a functional $\dot{S}_{reduced}$ which is shown to be symmetrical. This proves that a principle (4) applies to the vector Λ representing the profiles of density, etc. The entropy production rate functional $\dot{S}_{reduced}$ for that vector Λ

may be again obtained by a well posed minimization calculus (the minimization of the expressions (40)), and again covers all the aspects of the neoclassical evolution. The toroidal geometry is involved by the topologies of the flux lines and of a single closed contour drawn on the magnetic surfaces. If one neglects the viscosity effects, the ambipolarity of the particle fluxes across the magnetic surface is insured independently of the electrical potential if all the particle trajectories are intregrable and deviate from the magnetic surfaces by a disatance scaling as ρ_c. In the other cases the ambipolarity is expresesed by the minimization of the entropy production rate functional with respect to the eletric potential profile. The derivation from the kinetic equation of the Onsager behaviour of the relaxation is not so general as in the weakly collisional, integrable situations: it is only valid if the hamiltonian interaction is symmetrical with respect to an inversion of the velocities along the magnetic lines. However that restriction could be not physical. From that point of view another approach in the van Kampen framework, where the system would be considered as a set of weakly interacting plasma layers around the successive magnetic surfaces, could be interesting.

Finally, we propound the possibility of Onsager relaxations in turbulent, weakly collisional situations, on the basis that the turbulent field is a component of the isolated field-plasma system. We first consider the case of a quasi static turbulent field. The extension of an Onsager relaxation poses no new problem if the trajectories remain integrable and close to the magnetic surfaces. If the trajectories are no longer integrable but the quasilinear theory is applicable, we recover Onsager relaxations reflecting the symmetries of the quasilinear diffusion. If the turbulent field exhibits finite frequencies ω the latter are considered as an aspect of the deviation of the field-plasma system from its thermodynamical equilibrium. One is led to consider the oscillating phases α (such that $\frac{d\alpha}{dt} = \mathbf{w}$) as new coordinates forming with the 3 usual coordinates $\mathbf{x}$ an extended space, in which the field and in fact the field-plasma system are installed in a quasi static state. The variational techniques applicable to the neoclassical relaxations of collisional may then be extended to the considered turbulent case. This extension leads to Onsager relaxations conform to the scheme (1-4), the vector Λ now representing the frequencies ω together with the usual profiles of density temperature, electric potential, electric current.

APPENDIX 1: NOTATIONS

The notations concern functions and functionals defined at a given time t during the relaxation of the isolated field-plasma system. The substrict 0 means that a function is

taken at the thermodynamical equilibrium. The tilde indicates the difference of a function with respect to its version at the thermodynamical equilibrium. The underlining indicates a working version of a function, to be introduced in a functional and varied around the physical version of the function in the course of a variational calculus.

Functions

$\mathbf{A}(\mathbf{x},t)$, $\Psi(\mathbf{x},t)$ = vector and scalar potentials of the electromagnetic field (see beginning § 2.1); $\mathbf{I}(\mathbf{x},t)$, $\rho(\mathbf{x},t)$ = current and charge densities; $v(\mathbf{x},t)$ = volume of the magnetic surface passing through $\mathbf{x}$; $\theta_{Ha}(\mathbf{x},t)$, $\varphi_{Ha}(\mathbf{x},t)$ = Hamada coordinates (see appendix 3); $\psi_{tor}(v,t)$, $\psi_{pol}(v,t) = \frac{1}{2\pi} \times$ the magnetic fluxes across the minor, major turns on the magnetic surface of volume v; $\iota_{tor}(v,t)$, $\iota_{pol}(v,t) = \frac{1}{2\pi} \times$ the corresponding amperages; $I_{tor}(v,t) = \frac{\partial \iota_{tor}(v,t)}{\partial v}$, $I_{pol}(v,t) = \frac{\partial \iota_{pol}(v,t)}{\partial v}$.

$H(\mathbf{x},\mathbf{p},s,t)$ = hamiltonian of a particle in the field $\mathbf{A},\Psi$; we call constant of motion any dynamical variable $Z(\mathbf{x},\mathbf{p},s,t)$ such that $\{H,Z\} = 0$; $\mathbf{V}(\mathbf{x},\mathbf{p},s,t) = \frac{\partial H}{\partial \mathbf{p}}$ = velocity of a particle in state $\mathbf{x},\mathbf{p}$; $v_0(s)$,T_0 = basic constants in the equation (6); $F(\mathbf{x},\mathbf{p},s,t)$ = particle distribution function in phase space $\mathbf{x},\mathbf{p}$; $U(\mathbf{x},\mathbf{p},s,t)$ = deviation of the field-plasma system with respect to the thermodynamical equilibrium (see equation (6)).

$\mu(\mathbf{x},\mathbf{p},s,t)$ = magnetic moment; $\mathbf{x}_G(\mathbf{x},\mathbf{p},s,t)$ = guiding centre position; $\mathbf{V}_G(\mathbf{x},\mathbf{p},s,t)$ = guiding centre velocity; $\varepsilon = \frac{V_{G//}}{|V_{G//}|}$ (see § 2.3); $\Phi(\mathbf{x},\mathbf{p},s,t) = (\Phi_1,\Phi_2,\Phi_3)$, $\mathbf{J}(\mathbf{x},\mathbf{p},s,t) = (J_1,J_2,J_3)$ = angular and action variables for the hamiltonian $H(\mathbf{x},\mathbf{p},s,t)$ frozen at time t then equal to $h(\mathbf{J},s,t)$ (see § 2.2). $\mathbf{J}(\mathbf{x},\mathbf{p},s,t)$ may designate also the 3 constants of motion H, μ and $v_{drift}(\mathbf{x},\mathbf{p},s,t)$ = constant of motion close to $v(\mathbf{x},t)$

$u(\mathbf{J},s,t)$, = form of the deviation $U(\mathbf{x},\mathbf{p},s,t)$ in a weakly collisional, integrable situation; $u(\mathbf{J},s,t)$ is decomposed as $u_I(\mathbf{J},s,t) + u_{II}(\mathbf{J},s,t)$ so that $u_I(\mathbf{J},s,t)$ determines the entropy of the system. Similarily, $u(H,\mu,\mathbf{x}_G,\varepsilon,s,t)$ = form of U in the general magnetized case.

$n(v,s,t)$, $T(v,s,t)$, $\Psi(v,s,t)$ = average density, temperature, electric potential on the magnetic surface of volume v.

Functionals

$\dot{S}(U',U'')$ (see equation 8), $S(U',U'')$ (see equations 14,44,53), $\Sigma(U',U'')$ (see equations 10) associate a scalar to any 2 dynamical variables $U'(\mathbf{x},\mathbf{p},s,t)$,$U''(\mathbf{x},\mathbf{p},s,t)$; $\dot{S}(U,U)$ is the collisional entropy production rate; $S(U,U)$ is (at second order in U) the entropy with respect to the thermodynamical equilibrium ; $\Sigma(U',U'') = 0$ if U' or U'' are a constant of motion.

At a given time t, the fields $\mathbf{A}(\mathbf{x},t),\Psi(\mathbf{x},t)$ and $\mathbf{I}(\mathbf{x},t),\rho(\mathbf{x},t)$, the hamiltonian $H(\mathbf{x},\mathbf{p},s,t)$, the functions $\psi_{tor}(v,t)$, $\psi_{pol}(v,t)$, $I_{tor}(v,t)$, $I_{pol}(v,t)$ are fully determined by the deviation $U(\mathbf{x},\mathbf{p},s,t)$ and are thus functionals of U. This is expressed by the notations: $\mathbf{A}(\mathbf{x},t) = \mathbf{A}(\mathbf{x}|U)$, etc, $H(\mathbf{x},\mathbf{p},s,t) = H(\mathbf{x},\mathbf{p},s|U)$, $\psi_{tor}(v,t) = \psi_{tor}(v|U)$, etc. Similarily, at a given time t, $H(\mathbf{x},\mathbf{p},s,t)$ is fully determined by $\mathbf{A}(\mathbf{x},t),\Psi(\mathbf{x},t)$ and may be written $H(\mathbf{x},\mathbf{p},s|\mathbf{A},\Psi)$.

APPENDIX 2: VAN KAMPEN AND QUASILINEAR DIFFUSION

Van Kampen starts from the master evolution equation for the probability P_N (t) of finding the system in state N, namely: $\frac{\partial P_N}{\partial t} = \sum_M W_{M\to N} P_M - P_N$, or $\frac{\partial P_N}{\partial t} = \sum_M W_{N\to M}(P_M - P_N)$ by taking into account that $W_{N\to M} = W_{M\to N}$ and that $\sum_M W_{M\to N} = \sum_M W_{N\to M} = 1$. Because of the symmetry $W_{N\to M} = W_{M\to N}$, one has, whatever the auxiliary set (Q_N)

$$\sum_N \frac{\partial P_N}{\partial t} Q_N = \sum_{NM} W_{N\to M}\,(P_M - P_N) Q_N = \sum_{NM} W_{M\to N}(P_N - P_M) Q_M$$
$$= \sum_{NM} W_{N\to M}\,(P_M - P_N)(-Q_M) = \frac{1}{2}\sum_{NM} W_{N\to M}\,(P_M - P_N)(Q_N - Q_M) \quad \text{(A1-1)}$$

The key point is that the last expression is invariant when one exchanges (P_N) and (Q_N). At thermodynamical equilibrium, P_N has relaxed to a maxwellian value $P_{0N} = X_0 \exp(\frac{-h_N}{T_0})$. At a given time, a small deviation of the system from the thermodynamical equilibrium is specified by the set u (t) = $(u_N$ (t)) such that $P_N(t) = X_0 \exp(\frac{-h_N + u_N(t)}{T_0})$. By using the certainty equations $\sum_N P_N = \sum_N P_{0N} = 1$ and the equation $\sum_N P_N h_N = \sum_N P_{0N} h_N$ expressing the energy conservation during the relaxation, the entropy $S = -\sum_N (P_N \ln(P_N) - P_{0N} \ln(P_{0N}))$ is readily identified at second order in u with $S(u,u)$, where $S(u,\underline{u})$ is the symmetrical bilinear form $S(u,\underline{u}) = \sum_N \frac{-P_{0N}}{2T_0^2} u_N \underline{u}_N$ in $u = (u_N)$ and $\underline{u} = (\underline{u}_N)$. At first order in u , one may replace in the equation (A1-1) $\frac{\partial P_N}{\partial t}$ by $P_{0N}\frac{\partial u_N}{T_0 \partial t}$. Taking into account the energy conservation $h_N = h_M$ for the allowed transitions $W_{N\to M} \neq 0$, one may replace also

$W_{N\to M}(P_M - P_N)$ by $\frac{P_{0N}}{T_0} W_{N\to M}(u_M - u_N)$. We put $Q_N = \frac{u_N}{T_0}$ and define the bilinear form $\dot{S}(u,\underline{u}) = \sum_{NM} \frac{P_{0N}}{2T_0^2} W_{N\to M}(u_M - u_N)(\underline{u}_M - \underline{u}_N)$ in $u, \underline{u}$, which is symmetrical since $P_{0N}\ W_{N\to M} = P_{0M}\ W_{M\to N}$. The equation (A2-1) finally means that

$$-2S(\frac{\partial u}{\partial t}, \underline{u}) + \dot{S}(u, \underline{u}) \qquad \text{(A2-2)}$$

is an extremum for all the variations of $\underline{u}$. It thus appears that the vector $\Lambda(t)$ representing the set u (t), verifies the variational principle (4). The quadratic form $\dot{S}(u,u)$ is of course identified to the entropy production rate induced py the transitions.

The quasilinear theory leads to similar results for a classical system of N particles, numbered I,II, ..., whose individual unperturbed motion is integrable, described by 3 angular variables Φ conjugate to 3 action variables **J**. The 3 action variables **J** label the various stationary states of an individual particle. The unperturbed system of the particles is descibed by the 3N angular variables $\boldsymbol{\Phi} = (\Phi_I, \Phi_{II}, ...)$ conjugate to the 3N action variables $\boldsymbol{J} = (\mathbf{J}_I, \mathbf{J}_{II}, ...)$. The unperturbed hamiltonian is h(**J**) for a particle and $h(\boldsymbol{J}) = h(\mathbf{J}_I) + h(\mathbf{J}_{II}) + ...$ for the system. A small resonant hamiltonian perturbation induces a diffusion of the system in its action space $\boldsymbol{J} = (J_k\ ,\ k = 1,...,3N)$. The quasilinear theory, if it applies, predicts that the probability density $P(\boldsymbol{J},t)$ of finding the system in state $\boldsymbol{J}$ satisfies the autonomous evolution law $\frac{\partial P(\boldsymbol{J},t)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial J_k}\left(D_{kl}(\boldsymbol{J})\frac{\partial P(\boldsymbol{J},t)}{\partial J_l}\right)$, with the symmetry $D_{kl} = D_{lk}$. Multiplying by the auxiliary function $Q(\boldsymbol{J},t)$ and integrating, one obtains that at each time

$$\int \frac{\partial P}{\partial t} Q\ d_{3N}J = \int \frac{-1}{2} D_{kl} \frac{\partial P}{\partial J_k}\frac{\partial Q}{\partial J_l}\, d_{3N}J \qquad \text{(A2-3)}$$

That equation, where the RHS is invariant when one exchanges P and Q since $D_{kl} = D_{lk}$, is the classical version of the equation (A2-1). The coefficients D_{kl} , identifiable with the diffusion coefficients $\langle \delta J_k\ \delta J_l \rangle$ in the space $\boldsymbol{J}$, are given for a static resonant perturbation $\sum_{\boldsymbol{n}} H_{\boldsymbol{n}}(\boldsymbol{J}) \exp(i\boldsymbol{n}\cdot\boldsymbol{\Phi})$, where $\boldsymbol{n}$ is a set of integers $(n_k\ ,\ k = 1,...,3N)$, by the well known formula $D_{kl} = \sum_{\boldsymbol{n}} n_k n_l |h_{\boldsymbol{n}}|^2 2\pi\delta(n_p \frac{\partial h}{\partial J_p})$. For a static perturbation the diffusion preserves the energy, so that $D_{kl}\ \frac{\partial h(J)}{\partial J_k} = 0$. A small deviation of $P(\boldsymbol{J},t)$ from the maxwellian distribution $P_0(\boldsymbol{J}) = X_0 \exp(\frac{-h(\boldsymbol{J})}{T_0})$ which applies at the thermodynamical equilibrium is specified by the function $u(\boldsymbol{J},t)$ such that

$P(J,t) = X_0 \exp(\frac{-h(J)+u(J,t)}{T_0})$. One readily pass from the equation (A2-3) to the principle (A2-2), where the symmetrical bilinear forms $S(u,\underline{u})$ and $\dot{S}(u,\underline{u})$ are now $S(u,\underline{u}) = \int \frac{-P}{2T_0^2} u\,\underline{u}\, d_{3N}J$ and $\dot{S}(u,\underline{u}) = \int \frac{P}{2T_0^2} D_{kl} \frac{\partial u}{\partial J_k} \frac{\partial \underline{u}}{\partial J_l} d_{3N}J$. In fact the state of the system (assuming to simplify a single particle species) is determined by the distribution function $f(\mathbf{J},t)$ of the $\mathbf{N}$ particles in the 3 dimensional action space $\mathbf{J} = (J_1,J_2,J_3)$, and the deviation from the thermodynamical equilibrium is determined by the function $u(\mathbf{J},t)$ such that $f(\mathbf{J},t) = X_0 \exp(\frac{-h(\mathbf{J}) + u(\mathbf{J},t)}{T_0})$, if $X_0 \exp(\frac{-h(\mathbf{J})}{T_0})$ applies at the thermodynamical equilibrium. The probability density $P(J,t)$ of the system in the space $J = (\mathbf{J}_I, \mathbf{J}_{II}, ...)$ is the normalized product $f(\mathbf{J}_I,t)\, f(\mathbf{J}_{II},t)...$ and the deviation $u(J,t)$ is the sum $u(\mathbf{J}_I,t) + u(\mathbf{J}_{II},t) + ...$ The symmetrical bilinear forms $S(u,\underline{u})$ and $\dot{S}(u,\underline{u})$ are readily rewritten as symmetrical bilinear forms $S(u,\underline{u})$ and $\dot{S}(u,\underline{u})$ in $u(\mathbf{J},t)$ and $\underline{u}(\mathbf{J},t)$. The principle (A2-2) then means the principle (4) for the vector $\Lambda(t)$ now representing at each time the function $u(\mathbf{J},t)$. The scheme may be used to build the Fokker-Planck collision term expressing in the kinetic equation the collisional diffusion of the particles in momentum $\mathbf{p}$ at each position $\mathbf{x}$. One then isolates at the considered time the plasma particles located within a small box centered on $\mathbf{x}$. One applies the usual periodicity trick identifying the opposite sides of the box so that the particles indefinitely travel the 3 dimensions of the box. An individual particle is described by 3 angular variables Φ giving its position within the box and 3 action variables $\mathbf{J}$ identified to its momentum $\mathbf{p}$. The Fokker-Planck diffusion coincides with the quasilinear diffusion of the system of the particles within the box due to the interaction hamiltonian. The basic symmetry of the Fokker-Planck operator expressed by the equation (8a) express the above symmetry $\dot{S}(u,\underline{u}) = \dot{S}(\underline{u},u)$.

APPENDIX 3: $\int \delta\mathbf{A}\cdot\mathbf{I}\, d_3x = \int (2\pi)^2(\delta\psi_{pol}\, I_{tor} - \delta\psi_{tor}\, I_{pol})dv$

At a given time t, the Hamada angular coordinates $\theta_{Ha}(\mathbf{x})$, $\varphi_{Ha}(\mathbf{x})$ form with the volume $v(\mathbf{x})$ a coordinate system for the space $(\mathbf{x})$ such that

$$\frac{D(\theta_{Ha}, \varphi_{Ha}, v)}{D(x_1, x_2, x_3)} = (\nabla\theta_{Ha}(\mathbf{x}) \times \nabla\varphi_{Ha}(\mathbf{x})) \cdot \nabla v(\mathbf{x}) = (2\pi)^2 \qquad \text{(A3-1)}$$

and that

$$\mathbf{B} = \nabla\times\mathbf{A}\,;\ \mathbf{A}(\mathbf{x}) = \psi_{tor}(v(\mathbf{x}))\ \nabla\theta_{Ha}(\mathbf{x}) + \psi_{pol}(v(\mathbf{x}))\ \nabla\varphi_{Ha}(\mathbf{x}) \qquad \text{(A3-2)}$$

They are not unique: any set $\theta'_{Ha} = l\,\theta_{Ha} + m\,\varphi_{Ha}$, $\varphi'_{Ha} = p\,\theta_{Ha} + q\,\varphi_{Ha}$ where l,m,p,q are 4 integers such that $l\,q - m\,p = 1$ is receivable as well. An important point is that a formula of type (A3-2), namely the first equation (50b), applies to any vector $\mathbf{I}(\mathbf{x})$ as soon as it satisfies the equations (49). Let us consider a field $\mathbf{B}' = \nabla\times\mathbf{A}'$ close to $\mathbf{B}$, producing magnetic surfaces with a volume function $v'(\mathbf{x})$ # $v(\mathbf{x})$. If $\theta'_{Ha}(\mathbf{x})$ # $\theta_{Ha}(\mathbf{x})$ and $\varphi'_{Ha}(\mathbf{x})$ # $\varphi_{Ha}(\mathbf{x})$ are the Hamada coordinates for the field $\mathbf{B}'$, we have

$$\mathbf{B}'=\nabla\times\mathbf{A}';\ \mathbf{A}'(\mathbf{x})=\psi'_{tor}(v'(\mathbf{x}))\,\nabla\theta'_{Ha}(\mathbf{x})+\psi'_{pol}(v'(\mathbf{x}))\,\nabla\varphi'_{Ha}(\mathbf{x}) \qquad \text{(A3-3)}$$

We introduce the differences $\delta\psi_{tor}(v) = \psi'_{tor}(v) - \psi_{tor}(v)$ and $\delta\psi_{pol}(v) = \psi'_{pol}(v) - \psi_{pol}(v)$ of the flux functions for the same argument v. We introduce also at given $\mathbf{x}$ the differences $\delta\theta(\mathbf{x}) = \theta'_{Ha}(\mathbf{x}) - \theta_{Ha}(\mathbf{x})$, $\delta\varphi(\mathbf{x}) = \varphi'_{Ha}(\mathbf{x}) - \varphi_{Ha}(\mathbf{x})$, $\delta v(\mathbf{x}) = v'(\mathbf{x}) - v(\mathbf{x})$. Those differences are single valued over a magnetic surface, and we may accordingly write, in terms of the coordinates $v = v(\mathbf{x})$, $\theta_{Ha}= \theta_{Ha}(\mathbf{x})$, $\varphi_{Ha} = \varphi_{Ha}(\mathbf{x})$

$$[\delta\theta(\mathbf{x}),\delta\theta(\mathbf{x}),\delta v(\mathbf{x})]=\sum_{m,n}[a_{mn}(v),b_{mn}(v),c_{mn}(v)]\exp(i\,(m\,\theta_{Ha}+n\,\varphi_{Ha})) \qquad \text{(A3-4)}$$

where m ,n are integers. We put $\delta\mathbf{A}(\mathbf{x}) = \mathbf{A}'(\mathbf{x}) - \mathbf{A}(\mathbf{x})$ and use the equation (50b) and the equations (A3-1, A3-2, A3-3) to obtain

$$\int \delta\mathbf{A}(\mathbf{x})\cdot\mathbf{I}(\mathbf{x})\,d_3x=\int (2\pi)^2\big(\delta\psi_{pol}(v)\,I_{tor}(v)-\delta\psi_{tor}(v)\,I_{pol}(v)\big)dv+X+Y \qquad \text{(A3-5)}$$

where

$$X=\int \big(\psi_{tor}(v)\,I_{tor}(v)\,(\nabla\delta\theta(x))\cdot(\nabla v\times\nabla\theta_{Ha})+\psi_{pol}(v)\,I_{tor}(v)\,(\nabla\delta\varphi(x))\cdot(\nabla v\times\nabla\theta_{Ha})+\ldots\big)d_3x$$

$$Y=\int \Big(\frac{\partial\psi_{pol}(v)}{\partial v}\,I_{tor}(v)-\frac{\partial\psi_{tor}(v)}{\partial v}\,I_{pol}(v)\,(2\pi)^2\,\delta v(x)\,d_3x$$

and again v stands for $v(\mathbf{x})$, etc. It appears that X cancels after that $\delta\theta$ and $\delta\varphi$ have been replaced by their expressions (A3-4), by taking into account that the equation (A3-1) allows to replace d_3x by $\dfrac{d\theta\,d\varphi\,dv}{(2\pi)^2}$. In view of the equations (50a,A3-4)), we have $Y=\int \dfrac{\partial p(v(\mathbf{x}))}{\partial v}\,\delta v(\mathbf{x})\,d_3x=\int \dfrac{\partial p(v)}{\partial v}\,c_{00}(v)\,dv$. We may consider the quantity $c_{00}(v)$ as the *average of* $v'(\mathbf{x}) - v(\mathbf{x}) = \delta v(\mathbf{x})$ *for* $\mathbf{x}$ *equally distributed in the space between the* neighbouring magnetic surfaces of equation $v(\mathbf{x}) = v$ and $v(\mathbf{x}) = v + \Delta v$. It is not difficult

to show that such an average, for given Δv, is of order 2 in $\delta v(\mathbf{x})$, i.e. in $\delta \mathbf{A}$. Therefore the equation (A3-5) is valid at first order in δA with $X = 0$, $Y = 0$. This establishes the equation (52). The equation (A3-5) allows to transform the principle (41) as follows: for given functions $I_{tor}(v)$, $I_{pol}(v)$, the field $\mathbf{A}(\mathbf{x})$ makes extremum the functional $\int \frac{-1}{2\mu_0}|\nabla \times \mathbf{A}|^2 + \int (I_{tor}(v)\psi_{pol}(v|\mathbf{A}) - I_{pol}(v)\psi_{tor}(v|\mathbf{A}))(2\pi)^2 dv$.

References

Balescu R 1991 *Phys. Fluids* B **3** 564

Boozer A H 1992 *Phys. Fluids* B **4** 2845

Casimir H B 1945 *Rev. Mod. Phys.* **17** 343

Coleman B D and Truesdell C 1960 *J. Chem. Phys.* **33** 28

Fitts D D 1962 *Non Equilibrium Thermodynamics* (New York: Mc-Graw-Hill) p 152

Garbet X, Laurent L, Samain A 1994, Phys. Plasmas **1** 850

Glansdorff P and Prigogine I 1971 *Structure, Stabilité et Fluctuations* (Paris: Masson and Cie) p 45

Hazeltine R D and Meiss J D 1992 *Plasma Confinement* (New York: Addison-Wesley Publishing Company) p 123

Kemble E C 1937 *Fundamental principles of quantum mechanics* (New York: Mc-Graw-Hill) p 427

Kreuzer H J 1981 *Non Equilibrium Thermodynamics and its Statistical Fundations* (Oxford: Clarendon Press) p 5, p 44

Krommes J A and Hu G 1993 *Phys. Fluids* B **5** 3908

Kruskal M D 1962 *Journal Math. Phys.* **3** 490

Landau L D and Lifshitz E M 1958 *Course of theoretical Physics* (London: Pergamon Press) vol. **5**, p 379

Landau L D and Lifshitz E M 1960 *Course of Theoretical Physics* (London: Pergamon Press) vol. **8**, p 288

Mercier C 1974 *Lectures in Plasma Physics* (Luxembourg: Commission of the European Communities)

Morozov D K, Osipenko M V, Pogutse O P and Shurigin R V 1988 *Fiz. Plasmy* **14** 147

Mynick H E and Duvall R E 1989 *Phys. Fluids* B **1** 750

Nguyen F 1992 *PhD thesis* Université Paris VII n° 7293 (report EUR-CEA-FC-1471, 1992)

Onsager L 1931 *Phys. Rev.* **37** 405, **38** 2265

Onsager L and Machlup S 1953 *Phys. Rev.* **91** 1505, **91** 1512

Rebut P H and Samain A 1969 *C.R. Acad. Sc. Paris* **268** 607

Rechester A B and Rosenbluth M N 1978 *Phys. Rev. Lett.* **40** 38

Rechester A B, Rosenbluth M N and White R B 1981 *Phys. Rev.* A **23** 2664

Robinson B B and Bernstein I B 1962 *Ann. Phys.* **18** 110

Rosenbluth M N, Sagdeev R Z, Taylor J B and Zaslavsky G M 1966 *Nucl. Fusion* **6** 297

Rosenbluth M N, Hazeltine R D and Hinton F L 1972 *Phys. Fluids* **15** 116

Samain A 1970 *Nucl. Fusion* **10** 325

Samain A and Werkoff F 1977 *Nucl. Fusion* **17** 53

Shaing K C 1988 *Phys. Fluids* **31** 2249

Sugama H and Horton W 1995 *Phys. Plasmas* **2** 2989

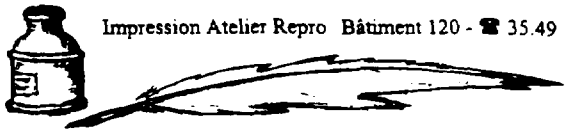
Sugama H, Okamoto M , Horton W and Wakatani M 1996 *Phys. Plasmas* **3** 2379

Sugama H and Horton W 1996 *Phys. Plasmas* **3** 304

Van Kampen N G 1954 *Physica* **20** 603

Van Kampen N G 1957 *Physica* **23** 707, **23** 816

Zaslavsky G M and Chirikov B V 1972 *Sov. Phys. Usp.* **14** 549



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