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NEUTRON TRANSPORT IN RANDOM MEDIA

by

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REVIEW OF NEUTRON TRANSPORT IN RANDOM MEDIA (Survey)

by M. Makai

ABSTRACT

The survey reviews the methods available in the literature which allow a discussion of corium recriticality after a severe accident and a characterization of the corium. It appears that to date no one has considered the eigenvalue problem, though for the source problem several approaches have been proposed. The mathematical formulation of a random medium may be approached in different ways. Based on the review of the literature, we can draw three basic conclusions. The problem of static, random perturbations has been solved. The static case is tractable by the Monte Carlo method. There is a specific time dependent case for which the average flux is given as a series expansion.

I. INTRODUCTION

The Boltzmann transport equation describes a neutron gas in a host medium. Usually the structure of the host medium is considered as given. The computation assumes the nuclide density distributions of the host nuclei known; hence, also the macroscopic cross sections known. There are situations when the cross-sections are either unknown, or the investigation aims just at characterizing the uncertainty in the neutron gas distribution and its consequences. In reactor physics this is the case in the following problems:

- We are interested in the consequences of the uncertainties of the fuel properties (density, geometry, composition). Each piece of fuel possesses slightly varying individual parameters.
- In the calculation of the effectiveness of a concrete shield, it is reasonable to take into account the local density variations of the concrete.
- Sometimes the material distribution of the host material can only be guessed. This is the case of a melted or heavily damaged core.
- Burnable poison usually has a grain structure. Only the form of the distribution (average, variance) is known.
- The bubble distribution in a moderator is random by nature. When boiling occurs, little information is available on the statistics. (The problems mentioned in the latter two items are often treated by means of homogenization, this can not account for the finer details of the neutron distribution.)
- Spallation is also discussed as particle transport in a random medium.
- Actinide transmutation. The accelerator based transmutation of nuclear waste is also treated by that means.

- The fuel of the high temperature gas cooled reactor (HTGR) is often considered as random.[1,2]

Neutral particle transport in a random medium occurs in other areas as well, including:

- medical science (photon transport in binary mixtures);
- meteorology (interaction of photons with clouds);[3]
- oceanography (water sediment interaction with light);
- geophysics (propagation of acoustic waves in stochastic media);[4]
- Radiative transfer through mixed fusion pellets.[5]
- Turbulent streaming is often described as a random phenomenon.[6]

The above list is far from exhaustive.

To be more specific, we mention a few particular situations where particle transport in a random medium is a possible model.

There are no means to learn the material composition of a melted core after a severe accident. Any estimation on the Chernobyl lava composition, for example, must be based on the random medium approach. The core composition and geometry prior to the accident is known, taking a given scenario the probability of the geometry and the material composition of the molten core can be estimated. From such a model the isotopic composition, the released energy and radiation becomes estimable.

In operating reactors, it is vital to know the consequences of the uncertainties in such quantities as fuel density, core geometry or flow rates. Using the random composition approach,

the expectation values and the variances can be estimated along with the expectation values and variances of such derived quantities as power peaking factors and reactivity coefficients.

When radiation interacts with matter, the resulting material composition becomes inhomogeneous because of the various types of interaction. The lattice imperfections, the impurities, are random by nature, we know only the distribution function. In such problems, the random medium model is, in most cases, the only possible approach.

In the context of the geological disposition of waste, in particular radioactive waste, the possible transport of the material in the geological environment of the burial site needs to be estimated. Due to limited sampling, the composition of the soils, the texture and fractures of the rock, and possible sources of water, are all uncertain. In this context, the random medium model may be very helpful.

II. FORMULATION OF THE PROBLEM

The problem of neutron transport in a random medium can be formulated as the solution of the following general operator equation

$$\mathbf{M}(\xi, x)\Phi(\xi, x) - \lambda(\xi)\Phi(\xi, x) = Q(\xi, x), \quad (1)$$

where ξ is a random parameter. If $Q = 0$ and $\lambda \neq 0$, and we have an eigenvalue problem. If $Q \neq 0$ and $\lambda = 0$, and we have a source problem. In either case, the problem is stochastic, since the operator \mathbf{M} depends on the random variable ξ . When ξ is fixed, we have a particular realization of the operator. Several realizations allow one to compute the averages, variances, correlation matrices, etc. The above outlined procedure would be a rather ineffective method of determining the statistical properties of the eigenvalue and eigenvector. The problem of determining the statistics of the eigenvalue and the eigenvector from the given statistics of the operator occurs in several areas of physics including solid state physics[7] and many body problems.[8]

The above formulation is simultaneously too general and too specific. Too specific, because the material composition is taken as fixed in time. Too general, because when the elements of the matrix are random, then the matrix can easily be singular; and such essential properties as positivity or symmetry can easily be broken. The matrix is random in the sense that it is not individual elements that are random, but rather the matrix as a whole. This is because \mathbf{M} reflects physical properties of the medium, and once the medium is given, all its properties have been fixed. Randomness occurs because we can only guess which material is at a given spacepoint in a given moment of time.

Another model of the randomness of the medium is the following. Let us consider a random binary mixture of two immiscible components; and the cross-sections are assumed to fluctuate between two values that correspond to each component. For this case, the most frequently investigated mixing statistics are Markovian[9] or Gaussian.[10,11,12]

There are three main approaches to the description of particle transport in a random medium. In the text, the original notation has been kept. For orientation, Table I gives a comparison between the basic assumptions in the three approaches. The line "assumption" specifies the additional assumption in the derivation. A detailed explanation is given when a particular method is discussed.

The transport equation describes the interaction between a host material and a field of particles. A cross-section is assigned to every interaction. A random host medium is characterized by a random function rendering a material composition to each space point. Otherwise, a given cross-section as a function of the spatial position is a random function or stochastic process. The basic concepts of probability theory can be found in Ref. [13]. Thus, in general, the random medium is characterized by a random function, or a stochastic process, which assigns a material composition to each space point. Hence, the random material is described by the same characteristics as stochastic processes: correlation functions, averages, variances etc. The transport equation, however, requires specific data, such as conditional probabilities and transition probabilities. This problem is well described in Ref. [14]. The task, therefore, is to find the averages, variances, correlation matrices without running a large number of eigenvalue or source problems. That goal has been approached in several ways.

Table I. Comparison of the Major Descriptions.

Characteristics	Pomraning	Sahni	Sanchez
medium	binary mixture	arbitrary	arbitrary
statistics	Gaussian	no restriction	no restriction
medium description	$P_{ab}, P_{aa}, P_{ba}, P_{bb}$	$p(R, T, \alpha r, t, \beta), \lambda_{\beta \rightarrow \alpha}$	$P_{\alpha}, P_{\alpha\beta}$
formulation	Liouville equation	master equation	Liouville equation
assumption	closure	no	no
dependent variable/	ensemble averaged flux	conditional distribution function	ensemble averaged flux
notation	$\Phi(x)$	$n(R, V, T, \alpha r, v, t, \beta)$	$\Psi_{\alpha}(x)$

III. SOLUTION METHODS

To date only the source problem has been considered in the literature. These works differ in regard to the following three points:

- the description of the random medium and the geometry;
- the sought quantities;
- the applied solution technique.

One approach[15,16] considers the material distribution as an immiscible binary mixture governed by Markovian statistics. In this approach, we find with given probability either material a or b, at any space point r. As a particle travels through the mixture, it passes through alternating packets of material a or b. The stationary Markov process means the following. Given that a particle is in material a at position r, the transition probability P_{ab} of finding itself (in the absence of absorption) in material b at position r+dr is given by dr/λ_a , where λ_a is the transition intensity from material a to material b. Similarly, given that a particle is in material b at position r, the probability of finding itself (in the absence of absorption) in material a at a position r+dr is given by dr/λ_b . The transition balance is known as Kolmogorov equations (forward form) and is given by

$$\frac{\partial P_{ab}}{\partial r} = -\frac{P_{ab}}{\lambda_b} + \frac{P_{aa}}{\lambda_a} \quad (2)$$

$$\frac{\partial P_{aa}}{\partial r} = -\frac{P_{aa}}{\lambda_a} + \frac{P_{ab}}{\lambda_b} \quad (3)$$

$$\frac{\partial P_{ba}}{\partial r} = -\frac{P_{ba}}{\lambda_a} + \frac{P_{bb}}{\lambda_b} \quad (4)$$

$$\frac{\partial P_{bb}}{\partial r} = -\frac{P_{bb}}{\lambda_b} + \frac{P_{ba}}{\lambda_a} \quad (5)$$

The boundary conditions on these differential equations are

$$\begin{aligned} P_{aa}(r,r) &= P_{bb}(r,r) = 1 \\ P_{ab}(r,r) &= P_{ba}(r,r) = 0 \end{aligned} \quad (6)$$

Under the stipulated conditions the correlation between the material distribution at the two points r and r' is given by exponential functions.[6] That approach has proven to be successful in several applications.[15]

In Ref. [17], the mixture of materials is not confined to two, and the material distribution is characterized by three functions. The first is the conditional probability $p(R,T,a|r,t,b)$ which is the probability for the medium at R to be of type a at time T given that the medium at (r,t) was of type b . The second function is related to the motion of the neutron. In the time δt , the neutron moves distance $v\delta t$, and the probability for the medium to change from type b to c is given by $v\delta t\lambda_{bc}$. The third function is defined as

$$\lambda_b(r,t) = \sum_c \lambda_{bc}(r,t) \quad (7)$$

and gives the probability for the material to change in any way in $v\delta t$. When we investigate a time independent case, the independent variable t should be left out in the above defined functions.

Zuchuat and Sanchez and Sanchez,[18,14] considered the cross-sections to depend on a random variable ξ and a mapping associates a material type to ξ at every point. That mapping is heavily exploited in the averaging. An exhaustive discussion of the problem of characterizing the randomness can be found in Ref. [14] where the following notation has been utilized:

- $p_a(x,y)$ = the probability for the interval (x,y) to be in material a.
- $p_{ab}[x,y)$ = the probability of undergoing a transition a**→**b in $[x-dx,x]$ and to have $(x,y)\in b$.
- $p_{ab}(x,y]$ = the probability of undergoing a transition a**→**b in $[y,y+dy]$ and to have $(x,y)\in a$.
- $p_{abc}(x,y]$ = the probability of undergoing a transition a**→**b in $[x-dx,x]$, have $(x,y)\in b$ and undergoing a second transition b**→**c in $[y,y+dy]$.

$p_{ab}(x,y)$ is the conditional probability of entering material b at y, given that $(x,y)\in a$ and that there is a transition at point y from material a into a different material. Point x is a point of transition for the realization ξ if

$$\lim_{\epsilon \rightarrow 0} \xi(x-\epsilon) \neq \lim_{\epsilon \rightarrow 0} \xi(x+\epsilon) . \tag{8}$$

The underlying assumptions are as follows:

- A. The measure of the set of realizations that have two or more transitions within an interval of length dx goes to zero with dx.
- B. The conditional probability for the material distribution to the right of position x, given that x is a transition point, is independent of the material distribution to the left of x.
- C. The conditional probability for the material distribution to the right of an arbitrary position x is independent of the material distribution to the left of x.

Under these conditions, the local densities turn out to be symmetrical:

$$p_{ab}[x] = p_{ab}(x) . \quad (9)$$

For applications to the integral transport formalism, it is more suitable to trace back statistics to chord length distributions. Sanchez et al.[16] have introduced the following additional probabilities:

$Q_a[x,y]$ = conditional probability of having $(x,y) \in a$ given that there is a transition into a at point x .

$Q_a(x,y)$ = conditional probability of having $(x,y) \in a$ given that there is a transition into a at point y in the direction of decreasing x values.

$f_a^+[x,y]$ = conditional density of probability of having a chord of material a of length $y-x$ to the right of point x given that there is a transition into a at point x .

$f_a^-[x,y]$ = conditional density of probability of having a chord of material a of length $y-x$ to the left of point y given that there is a transition into a at point y in the direction of decreasing x values.

These conditional probabilities can be expressed by the P_a etc. probabilities introduced above:

$$Q_a[x,y] = p_{(a)a}[x,y]/p_{a(a)}(x) \quad (10)$$

$$Q_a(x,y) = p_{a(a)}(x,y)/p_{a(a)}(y) \quad (11)$$

$$f_a^+[x,y] = -\partial_y Q_a[x,y], \quad f_a^- = \partial_x Q_a(x,y) . \quad (12)$$

Devooght[19] adopted the stochastic geometry viewpoint. The stochastic geometry, as described in Ref. [20], considers the random media as a distribution of convex, non-overlapping grains, obeying Poisson distribution.

Lovejoy et al.,[21] described the material as multifractal. In the cloud-photon interaction, which is the topic of Lovejoy et al., a large structure is broken up into smaller substructures. The density is modulated multiplicatively in each substructure by a random factor. Another equivalent model is to specify the scaling of the statistical moments of the density.

A simple powerful model is the rod (slab) model discussed e.g., in Refs. [9] and [10], utilized in Refs. [12,11].

The general problem allows for varying random composition in time.[22,23] That renders the problem extremely difficult since while a neutron flies towards the next collision, the material composition of the medium may randomly change.

The recriticality problem is formulated as follows. We consider a volume V filled out by a random material but the composition is constant in time. The random distribution is representable in two fashions. The first one assumes that a given infinitesimal volume dV may be filled by one of a given set of materials. Hence every space point is associated with a given material. The second one assumes that the material at every point r is a mixture of N materials and the ratios of the nuclide concentrations are random. Each material possesses a given microscopic cross-section. The unknowns are the expected value of the eigenvalue λ or k_{eff} , and its variance.

Before dealing with the posed problem, a survey of the technique available in the literature is given.

1. Master equation

It is the average neutron number or neutron flux that obeys the transport equation. When one is interested in the statistical aspects, an equation governing the probability distribution is needed. As shown by L. Pál,[24] an equation can be obtained for the generating function of the neutron number. Later that technique has been applied to noise analysis problems by Williams[25] and Lewins.[26] Pázsit[27,28] has derived simultaneous master equations for the forward and adjoint problem with given cross-sections. It has been Sahni who applied that technique in Ref. [22] to the

derivation of the equation for the generating function of the neutron density in a stochastic medium. Muñoz-Cobo et al.,[28] derive the equation for the ensemble average by means of the master equation.

In Ref. [17], the joint probability (Q_n) for exactly n neutrons to be present in a volume element $d\mathbf{R}$ with velocities in an element $d\mathbf{V}$ around the phase point (\mathbf{R},\mathbf{V}) at time T and the medium at (\mathbf{R},T) to be of type α when one source neutron is found at the space point \mathbf{r} with velocity \mathbf{v} at an instant of time $t \leq T$, the medium at (\mathbf{r},t) being of type β . This probability merges the properties of the material and of the neutron field. This is clearly seen from the relationship

$$\sum_{n=0}^{\infty} Q_n(\mathbf{R},\mathbf{V},T,\alpha|\mathbf{r},t,\beta) = p(\mathbf{R},T,\alpha|\mathbf{r},t,\beta) . \quad (13)$$

Making use of the considerations in Ref. [24], Sahni arrives at the following set of Chapman-Kolmogorov equations:

$$\begin{aligned} & \left[- \left(\frac{\partial}{\partial t} + \mathbf{v} \nabla_{\mathbf{r}} \right) + \mathbf{v} \left(\sigma_b^t(\mathbf{r},\mathbf{v},t) + \lambda_{\beta}(\mathbf{r},t) \right) \right] Q_n(\mathbf{R},\mathbf{V},T,\alpha|\mathbf{r},\mathbf{v},t,\beta) d\mathbf{R}d\mathbf{V} = \\ & \mathbf{v} \sum_{\gamma} \lambda_{\beta-\gamma}(\mathbf{r},t) Q_n(\mathbf{R},\mathbf{V},T,\alpha|\mathbf{r},\mathbf{v},t,\beta) d\mathbf{R}d\mathbf{V} + \mathbf{v} \sigma_{\beta}^c(\mathbf{r},\mathbf{v},t) \delta_{n0} p(\mathbf{R},T,\alpha|\mathbf{r},t,\beta) \\ & + \mathbf{v} \int \sigma_{\beta}^s(\mathbf{r},\mathbf{v},t) f_{\beta}(\mathbf{v}'-\mathbf{v}) d\mathbf{v}' Q_n(\mathbf{R},\mathbf{V},T,\alpha|\mathbf{r},\mathbf{v},t,\beta) d\mathbf{R}d\mathbf{V} \\ & + \mathbf{v} \sum_{\nu=1}^{\infty} \sigma_{\beta}^r(\mathbf{r},\mathbf{v},t) p_{\nu,\beta}(\mathbf{v}) \int \dots \int \chi_{\beta}(\mathbf{v}:\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_{\nu}) d\mathbf{v}_1 d\mathbf{v}_2 \dots d\mathbf{v}_{\nu} \times \\ & \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots \sum_{n_{\nu}=0}^{\infty} Q_{n_1}^*(\mathbf{v}_1) Q_{n_2}^*(\mathbf{v}_2) \dots Q_{n_{\nu}}^*(\mathbf{v}_{\nu}) (d\mathbf{R}d\mathbf{V})^{\nu} . \end{aligned} \quad (14)$$

In the last term,

$$Q_n^*(\mathbf{R},\mathbf{V},T,\alpha|\mathbf{r},\mathbf{v},t,\beta) d\mathbf{R}d\mathbf{V} \quad (15)$$

is the conditional probability for exactly n particles to be present in $d\mathbf{R}d\mathbf{V}$ around the phase point (\mathbf{R},\mathbf{V}) at time T given that the medium at (\mathbf{R},T) is of type α ; $(\mathbf{r},\mathbf{v},t)$ are the source coordinates. Only

the velocity coordinates have been written out explicitly in the two terms of Eq. (14). The second equation reads as

$$\left[-\left(\frac{\partial}{\partial t} + \mathbf{v}\nabla \right) + v\lambda_{\beta}(\mathbf{r},t) \right] p(\mathbf{R},T,\alpha|\mathbf{r},t,\beta) = v \sum_{\gamma} \lambda_{\beta-\gamma}(\mathbf{r},t) p(\mathbf{R},T,\alpha|\mathbf{r},t,\gamma) . \quad (16)$$

Q_n can be looked upon as a Green's function.[27,28] By means of Q_n , the generating function can be calculated, and we obtain the following equation for the average value of the flux:

$$\bar{N}(\mathbf{R},\mathbf{V},T,\alpha) = \sum_{\beta} \iint S(\mathbf{r},\mathbf{v},t) p_{\beta}(\mathbf{r},t) \bar{n}(\mathbf{R},\mathbf{V},T,\alpha|\mathbf{r},\mathbf{v},t,\beta) d\mathbf{r}d\mathbf{v} . \quad (17)$$

Here, the Green's function satisfies the following equation:

$$\begin{aligned} & \left[\left(\frac{\partial}{\partial T} + \mathbf{V}\nabla \right) + V \left(\sigma_{\alpha}^i(\mathbf{R},\mathbf{V},T) + \lambda_{\alpha}(\mathbf{R},T) \right) \right] \bar{n}(\mathbf{R},\mathbf{V},T,\alpha|\mathbf{r},\mathbf{v},t,\beta) \\ & = V \sum_{\lambda} \lambda_{\gamma-\alpha}(\mathbf{R},T) \bar{n}(\mathbf{R},\mathbf{V},T,\gamma|\mathbf{r},\mathbf{v},t,\beta) \\ & + \int \mathbf{V}' \sigma_{\beta}^s(\mathbf{R},\mathbf{V}',T) f_{\beta}(\mathbf{V}' - \mathbf{V}) \bar{n}(\mathbf{R},\mathbf{V}',T,\alpha|\mathbf{r},\mathbf{v},t,\beta) d\mathbf{V}' \\ & + \int \mathbf{V}' \sigma_{\alpha}^f(\mathbf{R},\mathbf{V}',T) \bar{v}_{\alpha}(\mathbf{V}') \int \chi_{\beta}(\mathbf{V}':\mathbf{V}) \bar{n}(\mathbf{R},\mathbf{V}',T,\alpha|\mathbf{r},\mathbf{v},t,\beta) d\mathbf{V}' \\ & + \delta_{\alpha\beta} \delta(\mathbf{r}-\mathbf{R}) \delta(\mathbf{v}-\mathbf{V}) \delta(T-t) . \end{aligned} \quad (18)$$

If the random medium is stationary, i.e., is constant in time, the time derivatives should be canceled and all time variables should be left out from all the argument lists. Now the Chapman -Kolmogorov equation for the medium correlation reads as

$$\left[-\mathbf{v}\nabla + v\lambda_{\beta}(\mathbf{r}) \right] p(\mathbf{R},\alpha|\mathbf{r},\beta) = v \sum_{\gamma} \lambda_{\beta-\gamma}(\mathbf{r}) p(\mathbf{R},\alpha|\mathbf{r},\gamma) . \quad (19)$$

From that equation, the velocity cancels out. The remainder expresses that the effect of a small change in the source coordinate \mathbf{r} is identical to the change due to the material correlations. The

author believes that the formalism elaborated by Sahni is suitable for further analysis, and it is hard to understand why the people involved in stochastic transport have not pursued further that line.

2. Liouville equation

The distribution of the average density of the neutron gas is described by the Liouville equation in phase space. An equation for the ensemble average or higher moments of the neutron gas density distribution is given by R. Sanchez, G. Pomraning, and A. Prinja.

The basic problem with an approach based on the Liouville equation is related to the appearance of cross-correlations in the equation; and, it is not obvious how to assure the consistency of the closed system and derive a feasible algorithm for computation.

Below we consider the source problem, i.e., $\lambda=0$ in Eq. (1). We introduce the following notation. Let \mathbf{M} , Φ and Q denote the average values:

$$\begin{aligned}\mathbf{M} &= \langle \mathbf{M} \rangle = \sum_{\xi} p(\xi) \mathbf{M}(\xi), \\ \Phi &= \langle \Phi \rangle = \sum_{\xi} p(\xi) \Phi(\xi) \\ \mathbf{Q} &= \langle \mathbf{Q} \rangle = \sum_{\xi} p(\xi) \mathbf{Q}(\xi),\end{aligned}\tag{20}$$

and we write $\mathbf{M}(\xi) = \mathbf{M} + \Delta \mathbf{M}(\xi)$; $\Phi(\xi) = \Phi + \Delta \Phi(\xi)$ and $Q(\xi) = Q + \Delta Q(\xi)$. Now, by averaging Eq. (1), we get

$$\mathbf{M}\Phi + \langle \Delta \mathbf{M} \Delta \Phi \rangle = \mathbf{Q}.\tag{21}$$

To express the cross-correlation term, we solve Eq. (1) for $\Delta \Phi$:

$$\Delta \Phi = \Phi + (\mathbf{M} + \Delta \mathbf{M})^{-1} \mathbf{Q} + (\mathbf{M} + \Delta \mathbf{M})^{-1} \Delta \mathbf{Q},\tag{22}$$

and multiply it by $\Delta\mathbf{M}$, after averaging, we get

$$\langle \Delta\mathbf{M}\Delta\Phi \rangle = \langle \Delta\mathbf{M}(\mathbf{M} + \Delta\mathbf{M})^{-1} \rangle \mathbf{Q} + \langle \Delta\mathbf{M}(\mathbf{M} + \Delta\mathbf{M})^{-1} \Delta\mathbf{Q} \rangle. \quad (23)$$

Substituting this correlation into Eq. (21), we get an expression for the average flux:

$$\Phi = \mathbf{M}^{-1} \left(1 - \langle \Delta\mathbf{M}(\mathbf{M} + \Delta\mathbf{M})^{-1} \rangle \right) \mathbf{Q} - \mathbf{M}^{-1} \langle \Delta\mathbf{M}(\mathbf{M} + \Delta\mathbf{M})^{-1} \Delta\mathbf{Q} \rangle. \quad (24)$$

The average flux has two components. The first one depends on the statistics (average and higher moments) of the transport operator that are applied immediately on the average source. If $\Delta\mathbf{M}=0$ then the first term reduces to $\mathbf{M}^{-1}\mathbf{Q}$, the solution of the non-stochastic medium problem. The second term involves cross-correlations of the source and the transport operator. If the source and the surroundings are statistically independent, we get

$$\langle \Delta\mathbf{M}(\mathbf{M} + \Delta\mathbf{M})^{-1} \Delta\mathbf{Q} \rangle = \langle \Delta\mathbf{M}(\mathbf{M} + \Delta\mathbf{M})^{-1} \rangle \langle \Delta\mathbf{Q} \rangle = 0. \quad (25)$$

The average flux, from Eq. (24), is given as an infinite series.[6] Using the fact that the inverse to the streaming operator, which occurs in \mathbf{M} , is an integral, the average flux is given by an integral equation along with correlation functions of the stochastic cross-sections. Using the Fokker-Planck approximation, the integral operator can be localized.

The ensemble averaged flux can be evaluated in a simple case, see Ref. [22]. Write the neutron transport equation as

$$\Phi(\mathbf{r}) = \int_{\mathbf{v}} \left(F(\mathbf{r}') + Q(\mathbf{r}') \right) \frac{e^{-\tau(\mathbf{r},\mathbf{r}')}}{4\pi|\mathbf{r}-\mathbf{r}'|^2} d\mathbf{r}', \quad (26)$$

where $\Phi(\mathbf{r})$ is the scalar flux at \mathbf{r} and $F(\mathbf{r}')$ is the collision density at \mathbf{r}' :

$$F(\mathbf{r}') = \Sigma_s(\mathbf{r}')\Phi(\mathbf{r}'). \quad (27)$$

$\tau(r,r')$ is the optical distance between r and r' . The flux is given by the Neumann series

$$\begin{aligned}
 \Phi(r) &= \int_{\mathcal{V}} Q(r') \frac{e^{-\tau(r,r')}}{4\pi|r-r'|^2} dr' \\
 &+ \sum_{n=1}^{\infty} \int_{\mathcal{V}} dr_1 \Sigma_s(r_1) \frac{e^{-\tau(r,r_1)}}{4\pi|r-r_1|^2} \times \\
 &\int_{\mathcal{V}} dr_2 \Sigma_s(r_2) \frac{e^{-\tau(r_1,r_2)}}{4\pi|r_1-r_2|^2} \times \\
 &\dots \int_{\mathcal{V}} dr' Q(r') \frac{e^{-\tau(r_n,r')}}{4\pi|r_n-r'|^2} .
 \end{aligned} \tag{28}$$

We have to average the RHS of Eq. (28) over the ensemble. Let us assume no absorption and take $\Sigma_b=0$. Now, the contribution of the n -fold integral is finite only if the points $(r, r_1, r_2, \dots, r_{n-1})$ are in material a while material at r_n can be of either type. Likewise in the first term r must be in material a and there is no restriction at r' . If the free flights are uncorrelated, the ensemble average will be

$$\begin{aligned}
 &\langle \Sigma_s(r) \Sigma_s(r_1) \dots \Sigma_s(r_{n-1}) e^{-\tau(r,r_1)} \times e^{-\tau(r_1,r_2)} \dots e^{-\tau(r_n,r_{n-1})} \rangle \\
 &= p_a \Sigma_{sa}^n \langle e^{-\tau(r,r_1)} | r, r_1 \in A \rangle \times \langle e^{-\tau(r_1,r_2)} | r_1, r_2 \in A \rangle \times \langle e^{-\tau(r_n,r_{n-1})} | r_n, r_{n-1} \in A \rangle ,
 \end{aligned} \tag{29}$$

where A denotes the set of space points with material a . The factor p_a accounts for the probability that the point lies in medium a . With the help of Eqs. (26) and (27), the ensemble averages can be evaluated, they will contain exponential functions. If the flights are correlated, the ensemble average will be a product of averages, and we may need assumptions, such as closure, to make the equations solvable.

As we have seen, the Liouville equation involves a term with the correlation $\langle \Delta M \Delta \Phi \rangle$. Prinja[10] has proposed a closure based on the formal solution to the space dependent problem

$$\frac{d\Phi(x)}{dx} + M(x)\Phi(x) = 0 , \quad (30)$$

in a half space, because as Pomraning has pointed out, there the flux at $x=0$ is deterministic. In that case a proper closure can be obtained in the forward-backward scattering rod model (see Ref. [6]).

When the medium is large and static and only scattering occurs, the flux will be the sum of a large number of independent terms. Mello[30] used the central limit theorem and showed that the flux distribution is then normal.

3. Perturbation Theory

A perturbation is small if

$$\|\Delta M M^{-1}\| < 1. \quad (31)$$

Then, the solution to the source problem takes the following form

$$\Phi = M^{-1}Q - \left\langle \sum_n (-1)^n R^{n+1} \right\rangle M^{-1}Q - \left\langle \sum_n (-1)^n R^{n+1} \Delta Q \right\rangle . \quad (32)$$

Here, $R = M\Delta M$. When $\Delta M = 0$, the solution reduces to $M^{-1}Q$. The second term depends solely on the correlation of the operator, whereas the third term contains cross-correlations between source and the operator. Reference [6] suggests selecting M as the streaming term plus the average value of the cross-section matrix:

$$Mx = -\Omega \nabla x + \Sigma x .$$

Since the inverse to that operator is known, the solution is an integral. Stopping after the linear terms in Eq. (24), we get

$$\frac{d\langle\Phi\rangle}{ds} + \langle\Sigma\rangle\langle\Phi\rangle + \int_0^s ds e^{(-\langle\sigma\rangle-\langle\tau_1\rangle)s} [\langle M(s)\Delta Q(s_1)\rangle - \langle M(s)M(s_1)\rangle\langle\Phi(s_1)\rangle] = \langle Q\rangle. \quad (34)$$

The expression can be simplified by the Fokker-Planck approximation, when

$$\langle\Phi(s_1)\rangle \approx \sum_{n=0}^N \frac{1}{n!} (s_1-s)^n \frac{d^n\langle\Phi(s)\rangle}{ds^n}, \quad (35)$$

is substituted into Eq. (34). The resulting equations have been studied for two component mixture of Markovian statistics.[6]

As to the eigenvalue problem, an exhaustive discussion based on linear perturbation theory can be found in Ref. [31]. Write

$$\mathbf{M}\Phi = (\mathbf{A} + \lambda\mathbf{F})\Phi = 0, \quad (36)$$

where \mathbf{F} is the fission (production) operator and \mathbf{A} contains the remaining terms of the operator. If the average state is assumed critical:

$$\langle\mathbf{M}\rangle\Phi = (\mathbf{A}_0 + \lambda_0\mathbf{F}_0)\Phi = 0. \quad (37)$$

The effect of fluctuations in the medium is twofold. The global effect manifests itself in the new eigenvalue:

$$\lambda(\xi) = \lambda_0 + \rho, \quad (38)$$

where the stochastic quantity ρ is the reactivity perturbation. The local effect manifests itself in the flux perturbation. Szatmáry's goal is to determine the autocorrelation function

$$R_{\Phi\Phi}(r',E',r,E) = \langle \Delta\Phi(r',E')\Delta\Phi(r,E) \rangle. \quad (39)$$

Introducing the notation

$$\phi = \Delta M \langle \Phi \rangle, \quad (40)$$

we get the following correlation functions:

$$R_{\phi\phi}, R_{\phi\rho}, R_{\Phi\phi}, R_{\Phi\rho}, R_{\Phi\Phi}. \quad (41)$$

In the context of linear perturbation theory, the following relationships hold:

$$\langle M \rangle R_{\Phi\Phi}(r',E',r,E) = -R_{\Phi\phi}(r',E',r,E) + R_{\Phi\rho}(r',E') F_0 \langle \Phi(r,E) \rangle \quad (42)$$

$$\langle M' \rangle R_{\Phi\phi}(r',E',r,E) = -R_{\phi\phi}(r',E',r,E) + R_{\phi\rho} F'_0 \langle \Phi(r',E') \rangle \quad (43)$$

$$R_{\phi\rho}(r,E) = \frac{(\langle \Phi^+ \rangle; R_{\phi\phi})}{(\langle \Phi^+ \rangle; F_0 \langle \Phi \rangle)} \quad (44)$$

$$R_{\phi\rho}(r',E') = \frac{(\langle \Phi^+ \rangle; R_{\phi\phi}(r',E',r,E))}{(\langle \Phi^+ \rangle; F_0 \langle \Phi \rangle)} \quad (45)$$

Here, an operator with prime denotes an operator acting on the variables with primes. The equations form a closed system, and the correlations can be determined. The scheme starts with $R_{\phi\phi}$ which is given and from Eqs. (44), (43), (45) and (42); sequentially, we determine $R_{\phi\rho}$, $R_{\Phi\phi}$, $R_{\Phi\rho}$, and $R_{\Phi\Phi}$. Szatmáry has remarked that the traditional calculational model is applicable for the determination of the correlation functions after minor modifications are made.

IV. AVERAGING

According to Sanchez,[23] a rigorous derivation of the equation for the ensemble averaged flux has only been obtained for the collisionless, stationary case with stationary Markovian[6] statistics. In the general case, the ensemble averaging leads to an infinite set of equations. The structure of the equations depends on the material properties. When scattering is allowed, only approximations (closure) are proposed.

It is advantageous in a number of cases to average over a given material type. Each ξ represents a physical realization of the system when the cross-sections and the sources are given. Each state ξ can be considered as a mapping that associates a material to each point. Let $\mathbf{P}=(\mathbf{r},t)$ denote a point in the phase space. Now we can write $\alpha=\xi(\mathbf{P})$ for the material at point \mathbf{P} . The set of realizations which result in the presence or absence of material α at \mathbf{P} is complete, hence the average flux is

$$\Psi(\mathbf{x}) = \sum_{\alpha} p_{\alpha}(\mathbf{P})\Psi_{\alpha}(\mathbf{x}) \quad (46)$$

where $\mathbf{x} = (\mathbf{P},\mathbf{v})$. The projector $A_{\alpha}(\mathbf{P})$ projects to those states where material α is in point \mathbf{P} :

$$M_{\alpha}\Psi = \int_{X(\mathbf{P})=\alpha} d\xi p_t(\xi)\Psi(\xi) \quad (47)$$

$$A_X\Psi = \frac{M_X\Psi}{M_X 1} \quad (48)$$

where X denotes a set (e.g., the states where at \mathbf{P} we have material α) and $\mathbf{x}\in X$.

V. TECHNIQUES

There are many different techniques for analyzing neutron transport in non-random media. Criticality problems are usually considered in the integro-differential form, source problems are often formulated by integral transport equation. There are several numerical techniques as well. The tricks applied in neutron transport in random medium often borrow the tools of the above mentioned methods. In his works, Sanchez invoked integral transport tools. In Ref. [23], two formulae are derived for the ensemble averaged flux. The first one is recursive, higher and higher moments are involved. In the second, higher and higher derivatives are included.

Below the renewal equations are reproduced from Ref. [23]. The aim is to write the integral transport equation in a random medium. Before the derivation we set forth the notation. Let $P=(r,t)$ where r is the space coordinate, t is the time. A point in the phase space is $x=(P,v)$, where v is the velocity. The angular flux at x is expressed by means of the local sources along the past trajectory. Let the trajectory be

$$\text{Tr}(\theta_1, x) = \{P_\theta = (r - \theta v, t - \theta) | \theta \in [0, \theta_1]\} , \quad (49)$$

where

$$\theta_1(x) = \min[\theta_{bd}(r, v), \theta_{in}(t)] . \quad (50)$$

Here, θ_{bd} is the time required for a neutron of velocity v to travel from the boundary to location r , and $\theta_{in}=t-t_0$. Assume that ξ assigns a material α to P , then the local behavior of ξ along the past trajectory is characterized by the maximum length of time during which a neutron of velocity v travels until it reaches point P so that it traverses continuously in material α . Let us define the function $\theta(x)$ mapping the set of representations of the random medium into the set of positive numbers such that

$$\theta_\omega(x) = \max\{\theta \in \mathfrak{R}_+ | \omega(P') = \omega(P), \forall P' \in \text{Tr}(\theta, x)\} . \quad (51)$$

Then, for a state ξ assigning material α to point P , the integral transport equation will be

$$\begin{aligned}
 \psi_{\xi}(\mathbf{x}) &= H(\theta_{\xi} - \theta_1) e^{-\tau_{\alpha}(\theta_1, \mathbf{x})} \Phi_{\xi}(\mathbf{x}_1) \\
 &+ H(\theta_1 - \theta_{\xi}) e^{-\tau_{\alpha}(\theta_{\xi}, \mathbf{x})} \psi_{\xi}(\mathbf{x}_{\xi}) \\
 &+ \int_0^{\min(\theta_{\xi}, \theta_1)} d\theta e^{-\tau_{\alpha}(\theta, \mathbf{x})} F_{\xi}(\mathbf{x}_{\theta}) .
 \end{aligned} \tag{52}$$

Here, H is Heaviside's function, τ_{α} is the optical distance along the past trajectory $\text{Tr}(\theta, \mathbf{x})$. $F_{\xi}(\mathbf{x})$ is the emissivity (the external + internal source) at position \mathbf{x} . The first term is the contribution from the boundary (or initial) source, the second from the interface where material α changes to another material. The third term is the contribution from the point wise source along the trajectory through material α .

The next step is ensemble averaging. The set of the representations having material α at \mathbf{r} is the sum of two sets, the first set includes representations assigning material α to every point of a past trajectory, the second set consists of representations assigning material α to point \mathbf{P} , but the material changes somewhere along the past trajectory. Formally, that is expressed by

$$M_{\alpha}(\mathbf{P}) = M_{\alpha\alpha}(\theta, \mathbf{x}) + \int_0^{\theta} d\theta' M_{\alpha}(\theta', \mathbf{x}) . \tag{53}$$

Here,

$$M_{\alpha\alpha}(\theta, \mathbf{x}) f = \int_{X_{\alpha\alpha}(\theta, \mathbf{x})} d\xi p_i(\xi) f_{\xi} \tag{54}$$

with support

$$X_{\alpha\alpha}(\theta, \mathbf{x}) = \{ \xi \in X \mid \xi(\mathbf{P}') = \xi(\mathbf{P}), \forall \mathbf{P}' \in \text{Tr}(\theta, \mathbf{x}) \} . \tag{55}$$

Furthermore,

$$M_{\alpha}(\pm\theta, \mathbf{x}) \equiv \mp \partial_{\theta} M_{\alpha\alpha}(\mp\theta, \mathbf{x}) . \tag{56}$$

Let $R_\alpha(\theta, \mathbf{x})$ be the probability for the change of material type to take place after θ . Furthermore, let

$$Q_\alpha(\theta, \mathbf{x}) = -\partial_\theta R_\alpha(\theta, \mathbf{x}) . \quad (57)$$

Now, the ensemble average can be written as

$$\begin{aligned} \psi_\alpha(\mathbf{x}) &= e^{-\tau(\theta, \mathbf{x})} R_\alpha(\theta, \mathbf{x}) \Phi_{\alpha\alpha}^{\theta, \mathbf{x}}(\mathbf{x}_1) \\ &+ \int_0^{\theta_1} e^{-\tau_\alpha(\theta, \mathbf{x})} \times [Q_\alpha(\theta, \mathbf{x}) \psi_{\alpha\alpha}^{\theta, \mathbf{x}} + R_\alpha(\theta, \mathbf{x}) F_{\alpha\alpha}^{\theta, \mathbf{x}}(\mathbf{x}_\theta)] \end{aligned} \quad (58)$$

where

$$F_{\alpha\alpha}^{\theta, \mathbf{x}}(\mathbf{x}_\theta) = q_\alpha(\mathbf{x}_\theta) + (H_\alpha \psi_{\alpha\alpha}^{\theta, \mathbf{x}})(\mathbf{x}_\theta) . \quad (59)$$

Here, the following ensemble averaged fluxes have been introduced:

$$\psi_{\alpha\alpha}^{\theta, \mathbf{x}} = \frac{M_\alpha(\theta, \mathbf{x}) \psi}{M_{\alpha\alpha}(\theta, \mathbf{x})} \quad (60)$$

$$\psi_{\alpha\alpha}^{\theta, \mathbf{x}} = \frac{M_{\alpha\alpha}(\theta, \mathbf{x}) \psi}{M_{\alpha\alpha}(\theta, \mathbf{x})} . \quad (61)$$

Equation (58) gives ψ_α in terms of $\psi_{\alpha\alpha}^{\theta, \mathbf{x}}$ and $\psi_{\beta\alpha}^{\theta, \mathbf{x}}$. The same equation can be used to derive equations for $\psi_{\alpha\alpha}^{\theta, \mathbf{x}}$ and $\psi_{\beta\alpha}^{\theta, \mathbf{x}}$ and each of the two new equations introduces two further new ensemble averaged fluxes. The resulting system is called a renewal (or renewal like) set of equations. No proof is given in Ref. [23] that the set of equations can be solved.

The second approach is obtained by ensemble averaging the kinetic equation. The averaged equation will be

$$\begin{aligned} A_\alpha(\mathbf{P})L\psi + v\Sigma_\alpha\psi_\alpha &= q_\alpha + H_\alpha\psi_\alpha && \text{in } V \\ \psi_\alpha &= \Phi_\alpha && \text{on } \partial V . \end{aligned} \quad (62)$$

Here, $A_\alpha(\mathbf{P})$ is the ensemble averaging operator associated with $X_\alpha(\mathbf{P})$. As can be shown, operators $A_\alpha(\mathbf{P})$ and L do not commute, thus, we have

$$\begin{aligned} A_\alpha(\mathbf{P})L - LA_\alpha(\mathbf{P}) &= Q_\alpha^-(\mathbf{x})[A_\alpha^-(\mathbf{x}) - A_\alpha(\mathbf{P})] \\ &- Q_\alpha^+(\mathbf{x})[A_\alpha^+(\mathbf{x}) - A_\alpha(\mathbf{P})]. \end{aligned} \quad (63)$$

The quantities $Q_\alpha^\pm(\mathbf{x})$ measure the ratio at which states enter (+) or leave (-) the set $X_\alpha(\mathbf{P})$ as one moves with velocity v along the trajectory at \mathbf{P} . In terms of the probability defined in Eq. (57):

$$Q_\alpha^\pm(\mathbf{x}) = \lim_{\theta \rightarrow 0} Q_\alpha(\pm\theta, \mathbf{x}). \quad (64)$$

Operators $A_\alpha^\pm(\mathbf{x})$ indicate ensemble averaging over the states that locally enter (+) or leave (-) the set $X_\alpha(\mathbf{P})$. With

$$A_\alpha^\pm(\theta, \mathbf{x})f = \frac{M_\alpha^\pm(\theta, \mathbf{x})f}{M_\alpha^\pm(\theta, \mathbf{x})1} \quad (65)$$

we get

$$A_\alpha^\pm(\mathbf{x})\times = \frac{M_\alpha^\pm(\mathbf{x})\times}{M_\alpha^\pm(\mathbf{x})1}. \quad (66)$$

Finally, we arrive at the following ensemble averaged integral equation:

$$\begin{aligned} \psi_\alpha(\mathbf{x}) &= e^{-\tau(\theta, \mathbf{x})} \Phi_\alpha(\mathbf{x}_1) \\ &+ \int_0^{\theta_1} d\theta e^{-\tau(\theta, \mathbf{x})} \left[Q_\alpha^+(\mathbf{x}_\theta) \psi_\alpha^+(\mathbf{x}_\theta) - Q_\alpha^-(\mathbf{x}_\theta) \psi_\alpha^-(\mathbf{x}_\theta) + F_\alpha(\mathbf{x}_\theta) \right]. \end{aligned} \quad (67)$$

Here, F_α is the local emissivity. These equations have a similar structure to that of the renewal equations: we have an infinite system as Ψ_α is defined in terms of Ψ_α^- and Ψ_α^+ , likewise in the previous method.

The Monte-Carlo method[32] is another widely applied tool to determine neutron distribution in complex geometries. The application of the method to random medium problems needs further considerations.

Consider the stationary transport equation in a realization of a random medium:[32]

$$\Phi(P,\xi) = \int Q(P')T(P',P,\xi)dP' + \int \Phi(P',\xi)K(P,P',\xi)dP' . \quad (68)$$

That problem is tractable by the Monte Carlo method. The ensemble average flux is obtained from the above equation as

$$\int \rho(\xi)\Phi(P,\xi)d\xi = \iint Q(P')T(P',P,\xi)dP'p(\xi)d\xi + \iint \Phi(P',\xi)K(P',P,\xi)p(\xi)d\xi dP' . \quad (69)$$

If the transfer probabilities $T(P',P,\xi)p(\xi)$ and $K(P',P,\xi)p(\xi)$ are known, the above integrals can be evaluated by the standard Monte Carlo technique. The only difference is that ξ is an additional coordinate and $T(P',P,\xi)p(\xi)$ and $K(P',P,\xi)p(\xi)$ are replaced by $T(R',R)$ and $K(R',R)$, respectively, where $R=(P,\xi)$, $R'=(P',\xi)$. Fixing ξ is the same as fixing the cross-section at every point. Below we investigate the calculation of the probabilities.

It can be shown that an unbiased estimate for the flux is obtained, if the integral is constrained to a homogeneous region, and the boundary of the region is taken into account as a source.[32] In order to apply the above observation, the track (P,P') is decomposed into tracks in homogeneous regions. Following Ref. [23], details of an integral transport theory formulation can be worked out, though its practical usefulness is not clear. Another choice is to treat randomness as an additional coordinate in the phase space, and to apply the standard Monte Carlo technique. The latter approach has been adopted in Refs. [2] and [33], where results for high temperature gas cooled

reactor (HTGR) fuel have been demonstrating the practical applicability of the Monte Carlo technique.

In the investigated double heterogeneous structure,[34,2] two approaches have been tried. The first one replaces the heterogeneous structure by an equivalent homogeneous mixture, which, as emphasized by Sanchez and Pomraning, and under certain assumptions can be exact. The homogenization, however, has been reported to result in serious errors in some cases.[2,33,34]

As to the recriticality problem, a conservative estimate is to search for the largest eigenvalue. It means, we have to find the ξ that leads to the largest eigenvalue. That problem leads us to seek that material distribution from among the possible ones where the eigenvalue is maximum. This is an optimization problem, often discussed in connection with core reload pattern optimization. If the maximum eigenvalue excludes recriticality the job has been done. In the opposite case, however, we have to estimate the probability of the worst case and give some estimation for the probability of recriticality. That approach requires further investigations.

VI. PROPOSAL: CRITICALITY IN RANDOM MEDIUM

By a random medium, we mean that the cross-sections depend on a random parameter ξ and $-1 \leq \xi \leq +1$. The corresponding notation is $\Sigma(r, \xi)$. Let ξ_1 be a realization of the random variable ξ then the corresponding cross-section is $\Sigma(r, \xi_1)$. Discrete materials can be realized by a step function in the random variable ξ , e.g., a two component material is $\Sigma(r, \xi) = \Sigma_1(r)$ if $\xi < 0$ and $\Sigma(r, \xi) = \Sigma_2(r)$ if $\xi \geq 0$.

First, we formulate the criticality problem in a random medium. To this end, we consider the formal solution of the time dependent transport equation with a random operator, in which, additionally, the fission term is multiplied with a random scalar to make the ensemble averaged flux time independent. This formulation permits one to set up a set of equations for the ensemble averaged flux and the criticality parameter. The derivation of the equations start with the usual definition of k , when the media is not random. It should be emphasized, however, that the accepted

definition of criticality allows for sub- or super-critical states. The critical parameter has, thus, two meanings. First, its expectation value tells us the size of the non-stochastic adjustment we need to make in order to assure criticality. Second, the random part of the eigenvalue reflects on the stochastic aspect of criticality. As the reader might have noted, the criticality defined in the Introduction has been abandoned. In Eq. (1), one parameter is sought for each realization to make the given realization of the random medium critical. In that formalism, each of the realizations is critical i.e., constant in time.

An equation can be derived to determine the ensemble averaged flux by considering the structure of the time dependent problem

$$\frac{\partial \Psi}{\partial t} = \mathbf{M}\Psi(0). \quad (70)$$

The solution of which is written as

$$\Psi(t) = e^{\mathbf{M}t} \cdot \Psi(0), \quad (71)$$

where $\Psi(0)$ is the initial distribution at $t=0$. In a random medium \mathbf{M} is random, $\mathbf{M}=\mathbf{M}(\xi)$, where ξ is a random variable. Taking the expectation value of both sides we get

$$\langle \Psi(t) \rangle = \langle e^{\mathbf{M}t} \rangle \cdot \Psi(0). \quad (72)$$

The condition for $\Psi(t)=\Psi(0)$ is

$$\langle \mathbf{M}^n \rangle \Psi(0) = 0, \quad n = 0, 1, \dots \quad (73)$$

Let $\mathbf{M} = -\mathbf{A} + k\mathbf{B}$. By substituting this expression into Eq. (5), we get a set of relations of which the first equations are

$$\begin{aligned} \langle \mathbf{A} \rangle \Psi(0) &= \langle k\mathbf{B} \rangle \Psi(0); \\ \langle \mathbf{A}^2 \rangle \Psi(0) + \langle (k\mathbf{B})^2 \rangle \Psi(0) &= 2\langle k\mathbf{A}\mathbf{B} \rangle \Psi(0) \end{aligned} \quad (74)$$

When k is expanded into a Taylor series of the random variable ξ as

$$k(\xi) = k_0 + k_1\xi + \dots, \quad (75)$$

the set of equations can be solved for k_0, k_1 etc. in terms of the autocorrelation functions $\langle \mathbf{A}^n \rangle, \langle \mathbf{B}^n \rangle$ and the cross-correlation functions $\langle \mathbf{A}^n \mathbf{B}^m \rangle, n, m = 0, 1, 2, \dots$. As an example, k_0 and k_1 are determined by

$$\langle \mathbf{A} \rangle \Psi(0) = k_0 \langle \mathbf{B} \rangle \Psi(0) + k_1 \langle \xi \mathbf{B} \rangle \Psi(0) \quad (76)$$

$$\langle \mathbf{A}^2 \rangle \Psi(0) + k_0^2 \langle \mathbf{B}^2 \rangle \Psi(0) + 2k_1^2 \langle \xi^2 \mathbf{B}^2 \rangle \Psi(0) = 2k_0 \langle \mathbf{A} \mathbf{B} \rangle \Psi(0) + 2k_1 \langle \xi \mathbf{A} \mathbf{B} \rangle \Psi(0). \quad (77)$$

Even the simplest equation with random k has lead to a non-linear eigenvalue problem. It can be seen that for causal k , we get the traditional eigenvalue problem. Unfortunately, it is not possible to construct Eq. (77) when the random component of the matrix is essential. The problem is in stopping at the second order term, which can be substantial. To proceed with more terms is rather complicated; the equations are non-linear; the coefficients require the knowledge of higher order cross-correlations of the distributions.

The problem of finding the eigenvalues and eigenvectors of random linear operators is too intricate. In practice it suffices to investigate the linear operators in a finite dimensional space where matrices replace the linear operators. The problem of finding eigenvalues of random linear matrices has been studied.[8,35] To understand the nature of the problem, let us consider a case when \mathbf{M} is a symmetric, positive matrix. If the perturbation is sufficiently small, those properties remain valid for the random matrices as well. When the perturbations are large, the random matrix may even be singular. All attempts have failed to find the distribution of the largest eigenvalue of a random matrix. The following result is available.[35] If \mathbf{M} is a symmetric matrix with probability density $g(\lambda_1, \dots, \lambda_n)$ where $\lambda_1 > \dots > \lambda_n > 0$, then the joint density distribution of the eigenvalues is

$$\frac{\pi^{n(n+1)/4} g(\lambda_1, \dots, \lambda_n) \prod_{i < j} (\lambda_i - \lambda_j)}{\Gamma\left(\frac{1}{2}(n-i+1)\right)} . \quad (78)$$

That distribution is of limited use, since the density distribution of the matrix has been formulated with the unknown eigenvalues. When the matrix is a Gaussian distribution, its density depends solely on the determinant and the trace; both expressible with the eigenvalues. Other available results are confined to large symmetric matrices (e.g., Refs. [37,38,38]).

Brody et al.[8] has investigated many particle systems, however, most of these results are too specific. Girko[35] has studied the problem of finding the statistical properties of the eigenvalues and eigenvectors for several matrix types. To give the flavor of this approach, we give the results for non-symmetric real matrices quoted from Ref. [35]. The goal is to solve

$$\mathbf{X}(\xi)\Lambda(\xi)\mathbf{X}^{-1}(\xi) = \mathbf{M}(\xi) , \quad (79)$$

where Λ is a diagonal matrix, \mathbf{X} is a non-singular square matrix. To make the solution (i.e., \mathbf{X} and Λ) unique and measurable, the columns in matrix \mathbf{X} are required to be normalized and the first non-zero element is required to have positive sign, if the element is real; or is required to have a given phase, if it is complex. The eigenvalues are ordered in increasing modulus. The method to be presented relies on calculating the Jacobian associated with the change of variables. Let the equation $f(\mathbf{X})=\mathbf{M}$ be given, where f is a one-to-one function on a set L . Furthermore, assume that elements of matrix \mathbf{M} have a joint probability density distribution $p(\mathbf{Z})$, $\mathbf{Z} \in L$. Then the distribution of matrix \mathbf{X} is $p(f(\mathbf{Z}))J(\mathbf{Z})$, where $J(\mathbf{Z})$ is the Jacobian of the mapping $f(\mathbf{Z})$.

Let $\mathbf{M}=(m_{ij})$ be a real random matrix with density $p(\mathbf{X})$, where $\mathbf{X}=(x_{ij})$ is a real matrix. Let the eigenvalues of matrix \mathbf{M} be $(\lambda_k+i\mu_k, \lambda_k-i\mu_k, k=1,s)$ and $(\lambda_k, k=s+1,n-2s)$. Let $(x_k+iy_k, x_k-iy_k, k=1,s)$ and $(z_k, k=s+1,n-2s)$ be the eigenvectors. The eigenvalues are ordered according to increasing magnitude. Matrix \mathbf{M} can be cast, with probability 1, into the form

$$\mathbf{M} = \mathbf{T} \operatorname{diag} \left[\begin{pmatrix} \lambda_1 & \mu_1 \\ -\mu_1 & \lambda_1 \end{pmatrix}, \dots, \begin{pmatrix} \lambda_s & \mu_s \\ -\mu_s & \lambda_s \end{pmatrix}, \lambda_{s+1}, \dots, \lambda_{n-2s} \right] \mathbf{T}^{-1}, \quad (80)$$

where \mathbf{T} is a non-singular matrix with probability 1. Let K be the group of non-singular real matrices B , a σ -algebra of Borel subsets of group K , and v_i the eigenvalues of matrix \mathbf{M} . Function $\varphi(\mathbf{Y}_s)$ equals to 1, if the eigenvalue $(x_k+iy_k, x_k-iy_k), k=1, \dots, s$, is among the ordered eigenvalues, and equals to zero otherwise.

$$\mathbf{Y}_s = \operatorname{diag} \left[\begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix}, \dots, \begin{pmatrix} x_s & y_s \\ -y_s & x_s \end{pmatrix}, x_{s+1}, \dots, x_{n-2s} \right], \quad (81)$$

and

$$J_s(\mathbf{Y}_s) = \left| \prod_{p=1}^s (q_p - q_l) \right| \quad (82)$$

$$dX_n = \prod_{i=1 \div n, j=2 \div n} dx_{ij} \quad (83)$$

$$dY_s = \prod dx_s dy_s. \quad (84)$$

Here $q_p, (p=1, n)$ are the eigenvalues of matrix \mathbf{Y}_s . K denotes the following matrix set:

$$K = \{ (x_{ij})_{i,j=1}^n : x_{li} = \left[\sum_{j=2}^n x_{ji}^2 \right]^{1/2}, \sum_{j=2}^n x_{ji}^2 < 1, i=1 \div n \}. \quad (85)$$

If there exists the density distribution $p(\mathbf{X}_n)$ of the random matrix \mathbf{M} , then for any subset $E \in B$ and for any real α_i and $\beta_i, (i=1, n)$

$$P\{T_n \in E, \operatorname{Re} \theta_i < \alpha_i, \operatorname{Im} \theta_i < \beta_i, i=1, n\} = \sum_{s=0}^{[n/2]} c_s \int_{K_s} p(\mathbf{X}_n \mathbf{Y}_s \mathbf{X}_n^{-1}) J_s(\mathbf{Y}_s) \varphi(\mathbf{Y}_s) |\det \mathbf{X}_n|^{-n} \times \prod_{i=1}^n \left(1 - \sum_{j=2}^n x_{ji}^2 \right)^{-1/2} dX_n dY_s, \quad (86)$$

where the range of the integral K_s is

$$\{X_n \in E \cap K, x_1 < \alpha_1, y_1 < \beta_1, \dots, x_s < \alpha_{2s-1}, -y_s < \alpha_{2s}, x_{s+1} < \alpha_{2s+1}, 0 < \beta_{2s+1}, \dots, x_{n-2s} < \alpha_n, 0 < \beta_n\}. \quad (87)$$

Constants c_s are obtained from a set of linear equations that are given in Ref. [35], where also the proof is given.

It should be noted that the density distribution of matrix M , and that of matrix X_n , is not the same. The evaluation of the integral is feasible only in certain simple cases. To explore the usefulness of this requires further careful analysis.

VII. A PARTICULAR PROBLEM

This section is devoted to a particular problem when matrix M has a specific form. There are situations when we have stationary, but random surroundings, and a maneuver is performed which is a random, time dependent process. This is the case, for example, when the effect of bubbles are investigated experimentally, or when a control rod moves with a random trajectory and we are interested in the neutron field. That situation is described by decomposing the matrix M as

$$M = A_0 + f(t)A_1(\xi). \quad (88)$$

Here, both A_0 and A_1 are random operators. The discussion presented below is based on the works by Isidori and Fliess.[40,41] We assume the initial condition to be causal and given. The ensemble averaged flux is expressed by ensemble averages (correlations) of the random operators. The method is applicable to any statistics.

The transport operator is separated as the sum of a time independent and a time dependent term. In the latter, the time dependence is in a function $f(t)$ multiplied by an operator A_1 :

$$\frac{1}{v} \frac{\partial \Psi(r,t)}{\partial t} = [A_0(r) + f(t)A_1(r)] \Psi(r,t). \quad (89)$$

In a widely studied noise analysis problem involving a vibrating absorber, $f(t)$ is the strength of the absorber rod, $A_1(r)=H(x-x_0)\partial/\partial x$ (here H is the step function). The theory is capable of treating simultaneously more functions $f(t)$. If the operators $A_0(r)$ and $A_1(r)$ are smooth enough, Eq. (89) has a unique solution; and there are several methods for finding an approximate solution.

According to Fliess, the solution to Eq. (89) is given by

$$\Psi(r,t) = \Psi_0(r) + \sum_{k \geq 0} \sum_{j_k, \dots, j_0=0}^1 A_{j_k} \dots A_{j_0} \int_0^t d\xi_{j_k} \dots d\xi_{j_0} \Psi_0(r), \quad (90)$$

where the indices j_0, \dots, j_k may have the values 0 or 1. Let subscript 0 corresponds to $k=0$, this is the case where the length of the product is zero. In Eq. (90), the solution is given by a sum in which each term is derived from the initial state by means of a transition operator. The transition operators are classified according to length k . Each operator comprises a space dependent part, which is the product of k operators A_i and a time dependent part, which is given as an integral. The space dependent part is the unit operator when $k=0$. The integral representing the time dependent part is called iterated integral, and is determined by the following relations:

$$\xi_0(t) = t \quad (91)$$

$$\xi_1(t) = \int_0^t f(s) ds \quad (92)$$

$$\int_0^t d\xi_{i_k} \dots d\xi_{i_0} = \int_0^t d\xi_{i_k}(s) \int_0^s d\xi_{i_{k-1}} \dots d\xi_{i_0}; \quad k > 1 \quad (93)$$

The iterated integral corresponding to the length $k=0$ is the real number 1. Let us denote the iterated integrals by $E_{i_k, \dots, i_0}(t)$. Then the solution is given as

$$\Psi(r,t) = \sum_{k=0}^{\infty} \sum_{i_k, \dots, i_0=0}^1 E_{i_k, \dots, i_0}(t) C_{i_k, \dots, i_0}(r) \Psi_0(r) \quad (94)$$

and the first few terms are

$$E_0 = 0; E_0 = t; E_{00} = t^2/2; E_{000} = t^3/3!;$$

$$E_1 = \int_0^t f(s)ds; E_{11} = \int_0^t f(s) \int_0^s f(p)dpds \quad (95)$$

$$E_{01} = \int_0^t \int_0^s f(p)dpds;$$

$$E_{10} = \int_0^t f(s)sds . \quad (96)$$

If $A_0(r)$ and $A_1(r)$ are smooth, expression (90) can be evaluated. We defer evaluation of operator C .

That technique is easy to apply to the solution of Eq. (89) with the initial condition $\Psi(r,0)=\Psi_0(r)$, because from Eq. (89) the following coefficients of the Taylor expansion are obtained for the flux. Comparing that to the expression from the Taylor series Eq. (94), we have :

$$\Psi(r,0) = \Psi_0(r); \quad \frac{\partial \Psi}{\partial t}(r,0) = (A_0(r) + f(0)A_1(r))\Psi_0(r);$$

$$\frac{\partial^2 \Psi}{\partial t^2}(r,0) = [A_0^2 + f(0)(A_0A_1 + A_1A_0) + \dot{f}(0)A_1 + f^2(0)A_1^2] . \quad (97)$$

The series Eq. (94) converges for any finite t according to Isidori. The above technique is applicable even if A_0 or A_1 are nonlinear. The transient flux is given as a sum, and each additive component is a product of a time- and a space dependent term. The time dependent term is given as integrals of the $f(t)$ functions. Those integrals are called iterated integrals.

$$C_{0...0} = A_0^n; \quad C_1 = A_1; \quad C_{01} = A_0A_1;$$

$$C_{001} = A_0^2A_1; \quad C_{10} = A_1A_0; \quad C_{11} = A_1A_1. \quad (98)$$

The transient solution will contain terms like $A_0^m(A_1)^n\Psi_0(x)$. Those terms are to provide the space dependence of the transient. Since the method has been developed for smooth operators A and B , its application to cases when either operator is singular may cause problems. This is the case when A_1 contains a Dirac's delta function.

Now, let us form the ensemble average of the flux to arrive at

$$\langle \Psi(r,t) \rangle = \sum_{k=0}^{\infty} \sum_{i_k, \dots, i_0=0}^1 \langle E_{i_k, \dots, i_0}(t) C_{i_k, \dots, i_0} \rangle \Psi_0(r) . \quad (99)$$

The terms of the above series are arranged into three groups. The first group includes the terms in which all subscripts take the zero value, and that part of the series is

$$e^{\langle A_0 \rangle t} \Psi_0(r) = \Psi_0 + \langle t A_0 \rangle \Psi_0 + \langle t^2 / 2 A_0^2 \rangle \Psi_0(r) + \dots . \quad (100)$$

The second group involves terms with all indices equal to one:

$$\langle E_1(t) A_1 \rangle \Psi_0(r) + \langle E_{11}(t) A_1^2 \rangle \Psi_0 + \dots . \quad (101)$$

And the third group contains terms with mixed indices:

$$\langle E_{01}(t) A_0 A_1 \rangle \Psi_0(r) + \langle E_{10}(t) A_1 A_0 \rangle \Psi_0(r) + \dots , \quad (101)$$

if $f(t)=\cos\omega t$, the first group depicts the time dependent process corresponding to operator A_0 . If we assume the state at $t=0$ to be critical, the result of the summation of all terms in the first group is Ψ_0 . The second term describes the higher temporal harmonics, it includes the modes $\cos(n\omega t), n=1,2,\dots$. The third group involves powers of ω .

The final result is that the ensemble averaged flux is obtained as ensemble averaged operators applied to the initial distribution. As we see, the average flux is expressed with the help of moments of operators A_0 and A_1 . That relation is useful in two ways. First, when the average flux and its

correlations are known, we can determine the average and correlations of the involved operator. controversially, when the statistics of the involved operator are known, statistics of the flux can be predicted.

VIII. CONCLUSIONS

The present work has surveyed investigations concerning the particle (mostly neutron) transport in a random medium. Most of the available works have dealt with the source problem.

The available results can be summarized as follows. Two types of approaches exist. The first approach aims at finding master equations.[22,10,27] There are equations for a general time and space dependent stochastic medium in Ref. [17]. The second approach endeavors to write down an equation for the ensemble averaged flux (e.g., Refs. [9] and [23]). Here, the problem is that the momenta equations have led to an infinite system. That system is cut to a solvable size by assumptions called closure. Verified closures are available for specific statistics. One of the best studied cases is the binary mixture with Markovian statistics.[9,23,17] Prinja studied also normal statistics. Other approaches can also be found.[19] Elaborated is the integral transport theory approach by R. Sanchez that has helped to understand the depth of the problem. The noninitiated to the field, as the author, has got the impression that the present status of the area endeavors to provide means for understanding better the problem but at present it is hard to see the practical applicability of the results.

The eigenvalue problem has been formulated in a random medium as a natural extension of the static eigenvalue problem. Now the eigenvalue is random, its expectation value is always greater than zero. The problem is nonlinear as soon as we have a stochastic eigenvalue. The obtained problem worth further study.

There are different models to specify the randomness of the transport operator and to account for the correct physical nature of the problem. There are two specific cases when the general particle

transport in a random medium can be solved. The first one is when linear perturbation theory is applicable.[31] In that case, the autocorrelation function of the flux is determined from a set of equations which are tractable with the usual numerical transport methods (e.g., S_n , P_n , collision probability method). The problem discussed by Szatmáry[31] is static. There are several questions that Szatmáry's formalism can answer: consequences of uncertainty in material properties (density, enrichment, geometry). The material properties in the previous problems changes slowly (e.g., due to burnup). The second specific case is when we have two random operators in the transport equation, and those operators may be nonlinear as well, the first one (A_0) is constant in time, the other changes with time as a given (random) function of the time (called here $f(t)$) multiplied by a static random operator A_1 . That problem can be solved explicitly by the iterated integrals. The result is given as a series involving three groups of terms. The first group is the evolution due to A_0 , the second involves higher modes of $f(t)$, and the third group contains mixed terms.

Since the Monte Carlo technique has been designed to solve stochastic problems, we investigated also the applicability of standard Monte Carlo programs to describe (static) random medium phenomena. The Monte Carlo technique has turned out to be applicable when the operators do not depend on time, the randomness of the medium enlarges only the size of the problem but no essentially new phenomenon occurs. That conclusion has been confirmed by some HTGR calculations as well.[1,2] The Monte Carlo method raises the possibility of a conservative estimation of the criticality. To this end the maximum criticality medium has to be selected from among the available material distributions. That leads to an optimization problem emerging in core design.

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