Decay Rates for Spherical and Deformed Proton Emitters[1]

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Abstract. Using Green's function techniques, we derive expressions for the width of a proton decaying state in spherical and deformed nuclei. We show that the proton decay widths calculated by the "exact" expressions of Maglione et al. are equivalent to the distorted wave expressions of Bugrov et al., and that of Åberg et al. in the spherical case.

INTRODUCTION

The calculation of the decay rates for ground state proton emitters is of current interest. Several authors have presented expressions for the proton decay width of spherical nuclei [1–3], and deformed nuclei [1,4–6]. The proton-emitting states are extremely narrow, with observable widths not exceeding 10^{-10} eV.

Two apparently different methods have been used for deriving the decay rates for proton emitters. In both cases one first determines the wave function for the relative motion of the proton and the daughter nucleus in the resonant state. In this work we treat the decaying states as stationary states. The decay of the states is imposed by imposing an outgoing wave Green's function to solve the Schroedinger equation. We shall consider the spherical and deformed cases separately.

SPHERICAL NUCLEI

Maglione, Ferreira, and Liotta [1,4], using what we will call the direct (Dir) method, describe the parent nucleus as a single nucleon moving in the potential set up by the daughter nucleus. They give the single particle radial wavefunction at a large distance R outside the nucleus to be an outgoing Coulomb wave:

$$\psi_{\ell j}^{out}(r) = \frac{u_{\ell j}(r)}{r} = \frac{N_{\ell j} O_\ell(kr)}{r} \quad \text{at } r = R, \tag{1}$$

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DISCLAIMER

where $O_\ell(kr) = [G_\ell(kr) + iF_\ell(kr)]$, $N_{\ell j}$ is a normalization constant, k is the wave number, and F_ℓ and G_ℓ are the regular and irregular Coulomb functions, respectively. They calculate the probability flux penetrating a sphere at large distances to obtain an expression for the mean lifetime τ:

$$\frac{1}{\tau} = |N_{\ell j}|^2 v,$$

where v is the velocity. One may then obtain the decay width

$$\Gamma_{\ell j}^{\mathrm{Dir}} = \frac{\hbar^2 k}{\mu}|N_{\ell j}|^2 \tag{2}$$

where μ is the reduced mass.

Åberg et al. [3] derive an expression for the width using what they refer to as the DWBA method. We prefer to call it the Distorted Wave (DW) method, since it uses distorted waves but not the Born approximation. The width is given by

$$\Gamma_{\ell j}^{\mathrm{DW}} = \frac{4\mu}{\hbar^2 k}\left|\int_0^\infty F_\ell(kr)[V(r) - V_C^0(r)]u_{\ell j}(r)dr\right|^2$$

where $V(r)$ is the total potential between the outgoing proton and the daughter nucleus, $V_C^0(r)$ is the point source Coulomb potential, and $u_{\ell j}(r)/r$ is again the radial wavefunction obtained by numerically integrating the Schroedinger equation with a one-body potential.

To compare the two widths $\Gamma_{\ell j}^{\mathrm{Dir}}$ and $\Gamma_{\ell j}^{\mathrm{DW}}$, we note that the wavefunctions $F_\ell(kr)$ and $u_{\ell j}(r)$ are both solutions of the radial Schroedinger equation:

$$\frac{d^2 u_{\ell j}(r)}{dr^2} + [k^2 - \frac{\ell(\ell+1)}{r^2} - \frac{2\mu}{\hbar^2}V(r)]u_{\ell j}(r) = 0 \tag{3}$$

$$\frac{d^2 F_\ell(kr)}{dr^2} + [k^2 - \frac{\ell(\ell+1)}{r^2} - \frac{2\mu}{\hbar^2}V_C^0(r)]F_\ell(kr) = 0. \tag{4}$$

Multiply equation (3) by $-F_\ell(kr)$ and equation (4) by $u_{\ell j}(r)$ and add the two equations:

$$u_{\ell j}(r)\frac{d^2 F_\ell(kr)}{dr^2} - F_\ell(kr)\frac{d^2 u_{\ell j}(r)}{dr^2} + \frac{2\mu}{\hbar^2}F_\ell(kr)[V(r) - V_C^0(r)]u_{\ell j}(r) = 0.$$

Integrate over r from 0 to a large radius R, well outside the nucleus.

$$\begin{aligned}\frac{2\mu}{\hbar^2}\int_0^R F_\ell(kr)[V(r) - V_C^0(r)]u_{\ell j}(r)dr &= \int_0^R \left[F_\ell(kr)\frac{d^2 u_{\ell j}(r)}{dr^2} - u_{\ell j}(r)\frac{d^2 F_\ell(kr)}{dr^2}\right]dr \\ &= \int_0^R \frac{d}{dr}\left[F_\ell(kr)\frac{du_{\ell j}(r)}{dr} - u_{\ell j}(r)\frac{dF_\ell(kr)}{dr}\right]dr \\ &= \left[F_\ell(kr)\frac{du_{\ell j}(r)}{dr} - u_{\ell j}(r)\frac{dF_\ell(kr)}{dr}\right]_0^R\end{aligned}$$

At the lower limit $r = 0$, both $F_\ell(kr)$ and $u_{\ell j}(r)$ vanish. For R well outside the nucleus, from equation (1), substitute $u_{\ell j}(R) = N_{\ell j}[G_\ell(kR) + iF_\ell(kR)]$. Therefore

$$\frac{2\mu}{\hbar^2}\int_0^R F_\ell(kr)[V(r) - V_C^0(r)]u_{\ell j}(r)dr = N_{\ell j}\left[F_\ell(kR)\frac{dG_\ell(kR)}{dr} - G_\ell(kR)\frac{dF_\ell(kR)}{dr}\right].$$

The square bracket on the right hand side of the equation equals $-k$ times the Wronskian of the Coulomb functions, which has the value 1. In addition, we may safely extend the upper bound of the integral on the left hand side from R to ∞, since the right hand side is radius-independent for large R. So

$$\left|\int_0^\infty F_\ell(kr)[V(r) - V_C^0(r)]u_{\ell j}(r)dr\right|^2 = \left|\frac{-\hbar^2 k N_{\ell j}}{2\mu}\right|^2$$

and

$$\Gamma_{\ell j}^{\rm DW} = \frac{4\mu}{\hbar^2 k} \times \frac{\hbar^4 k^2 |N_{\ell j}|^2}{4\mu^2} = \frac{\hbar^2 k}{\mu}|N_{\ell j}|^2 = \Gamma_{\ell j}^{\rm Dir}.$$

It is of interest to compare the numerical results for the half-lives ($t_{1/2} = \hbar\ln(2)/\Gamma$) obtained with the direct method and the distorted wave method for spherical nuclei. We show in Table 1 the calculated half-lives for three spherical decaying states having orbital angular momentum $\ell = 0$, 2, and 5, respectively. The radial wavefunctions $u_{\ell j}(r)$ were calculated by integrating the radial Schroedinger equation, using for the proton-daughter nucleus potential the real part of the Becchetti-Greenlees optical model potential [7]. The potential depth was adjusted to match the energy eigenvalue to the proton decay Q-value, corrected for recoil and atomic screening:

$$Q_{p,n} = E_p\frac{m_p}{\mu} + E_{sc},$$

where m_p is the proton mass and E_{sc} is the atomic screening correction [8].

TABLE 1. Comparison of the proton half-lives of three spherical proton radioactivities calculated with the direct (Dir) and distorted wave (DW) methods.

Nucleus	E_p(keV)	j_p	ℓ_p	$t_{1/2,p}^{\rm Dir}$	$t_{1/2,p}^{\rm DW}$
^{167}Ir^g	1064(6)	$\frac{1}{2}^+$	0	35.7687 ms	35.7671 ms
^{147}Tm^m	1119(5)	$\frac{3}{2}^+$	2	171.332 μs	171.327 μs
^{167}Ir^m	1238(7)	$\frac{11}{2}^-$	5	1.99360 s	1.99352 s

It is seen that the calculated half-lives agree to $< 0.005\%$ between the two methods. Since the two methods should give identical results, the difference must reflect the accuracy of the numerical techniques that have been used.

DEFORMED NUCLEI

We consider a deformed odd-A nucleus, consisting of a single particle that is strongly coupled to an axially-symmetric even-even core. This is also known as the adiabatic limit, where the excitation energy of the daughter nucleus is considered to be zereo. To calculate the outgoing proton wavefunction we use the exact Gell-Mann-Goldberger transformation and the distorted-wave Green's function with outgoing Coulomb wave boundary conditions [9]. At large distances,

$$\Psi^{(+)}_{KIM}(\mathbf{r}) = -\frac{2\mu}{\hbar^2 k}\sum_{\ell jR}\frac{O_\ell(kr)}{r}|(\ell jR)IM\rangle\left\langle\langle(\ell jR)IM|\frac{F_\ell(kr')}{r'}\left|V(\mathbf{r}')-V^0_C(r')\right|\Psi_{KIM}\right\rangle. \tag{5}$$

Here $V(\mathbf{r}')$ is the total deformed potential acting between the proton and the core nucleus. The angular momentum part of the Green's function includes the rotational states $|RM_R\rangle$ $(R = 0, 2\ldots)$ of the core and the single-particle state $|\ell jm\rangle$, which are coupled to the total spin (IM):

$$|(\ell jR)IM\rangle = \sum_{mM_R}\langle jmRM_R|IM\rangle|\ell jm\rangle|RM_R\rangle.$$

The total wavefunction of the initial state is of the form [10]

$$\Psi_{KIM} = \sqrt{\frac{\hat{I}}{16\pi^2}}[D^I_{MK}(\omega')\phi_K + (-1)^{I+K}D^I_{M-K}(\omega')\phi_{\overline{K}}], \tag{6}$$

where $\hat{I} = 2I + 1$. The single-particle wavefunction ϕ_K is described (as in the Nilsson model) in terms of the intrinsic (body-fixed) coordinates of the daughter nucleus. It can be expanded in spherical components

$$\phi_K(\mathbf{r}') = \sum_{\ell j}\phi^{(i)}_{\ell j}(r')|\ell jK\rangle_0, \tag{7}$$

where the sum is over $j \geq |K|$ and the subscript "0" denotes a state in the intrinsic frame. For the final state, the rotational state wavefunction of the daughter is

$$|RM_R\rangle = \sqrt{\frac{\hat{R}}{8\pi^2}}D^R_{M_R0}(\omega').$$

We pick out a particular outgoing channel, in which the proton carries off angular momentum $\ell_p j_p$, with projection m_p, leaving the daughter nucleus with angular momentum R and projection M_R. For the evaluation of the matrix element in equation (5), the final state $|(\ell_p j_p R)IM\rangle$ must be expressed in terms of the single-particle wavefunction in the intrinsic system:

$$|\ell_p j_p m_p\rangle = \sum_{K'}D^{j_p}_{m_pK'}(\omega')|\ell_p j_p K'\rangle_0.$$

Combining these, we obtain

$$|(\ell_p j_p R) IM\rangle = \sqrt{\frac{\hat{R}}{8\pi^2}} \sum_{K' m_p M_R} \langle j_p m_p R M_R | IM\rangle D^R_{M_R 0}(\omega') D^{j_p}_{m_p K'}(\omega') |\ell_p j_p K'\rangle_0$$
$$= \sqrt{\frac{\hat{R}}{8\pi^2}} \sum_{K'} \langle j_p K' R 0 | I K'\rangle D^I_{MK'}(\omega') |\ell_p j_p K'\rangle_0, \quad (8)$$

where we have used a well-known relation involving a sum of D-functions [11].

Evaluation of the Matrix Element

We can now calculate the matrix element found in equation (5). Integrating over the orientation of the daughter nucleus ω' produces an expression that is diagonal in the quantum numbers IMK. This is evident from the first part of (6). The second term will select the value $-K$ from the sum (8) over final state K'-values. In fact, the two terms are of equal magnitude, so we obtain

$$\left\langle \langle (\ell_p j_p R) IM | \frac{F_{\ell_p}(kr')}{r'} \left| V(\mathbf{r}') - V_C^0(r') \right| \Psi_{KIM} \right\rangle = \sqrt{\frac{2\hat{R}}{\hat{I}}} \langle j_p K R 0 | I K\rangle \mathcal{M}_{\ell_p j_p K}, \quad (9)$$

where

$$\mathcal{M}_{\ell_p j_p K} = \left\langle \ell_p j_p K \left| \frac{F_{\ell_p}(kr')}{r'} [V(\mathbf{r}') - V_C^0(r')] \right| \phi_K \right\rangle_0 \quad (10)$$

is evaluated in the intrinsic frame. It is noted that the matrix element (9) is independent of the M-quantum number.

Partial Decay Width

We can now write the outgoing wavefunction (5) for a specific channel $(\ell_p j_p R I K)$ as

$$\Psi^{(+)}_{\ell_p j_p R, KIM}(\mathbf{r}) = -\frac{2\mu}{\hbar^2 k} \sqrt{\frac{2\hat{R}}{\hat{I}}} \langle j_p K R 0 | I K\rangle \frac{O_{\ell_p}(kr)}{r} |(\ell_p j_p R) IM\rangle \mathcal{M}_{\ell_p j_p K}.$$

The deformed decay width follows as

$$\Gamma^{\mathrm{DW}}_{\ell_p j_p R I K} = \frac{4\mu}{\hbar^2 k} \frac{2\hat{R}}{\hat{I}} \langle j_p K R 0 | I K\rangle^2 \left| \mathcal{M}_{\ell_p j_p K} \right|^2. \quad (11)$$

Apart from a pairing term, this expression is identical to that obtained by Kadmensky and Bugrov [5,6]. This can be shown by inserting the expansion (7) of the initial state into the matrix element (10) and using the expression

$$|\ell j m\rangle = \sum_{m_\ell m_s} \langle \ell m_\ell \tfrac{1}{2} m_s | j m\rangle Y_\ell^{m_\ell}(\hat{\mathbf{r}}')\chi(m_s)$$

for the single-particle states. Since the interaction does not change the proton spin, the single-particle matrix element will be diagonal in m_s. It is also diagonal in K. Thus one obtains

$$\mathcal{M}_{\ell_p j_p K} = \sum_{\ell j m_s} \langle \ell_p m_{\ell_p} \tfrac{1}{2} m_s | j_p K\rangle \langle \ell m_\ell \tfrac{1}{2} m_s | j K\rangle$$

$$\times \left\langle Y_{\ell_p}^{m_{\ell_p}}(\hat{\mathbf{r}}') \frac{F_{\ell_p}(kr')}{r'} \left| V(\mathbf{r}') - V_C^0(r') \right| \phi_{\ell j}^{(i)}(r') Y_\ell^{m_\ell}(\hat{\mathbf{r}}') \right\rangle .$$

This expression has been simplified by noting that $m_{\ell_p} = m_\ell = K - m_s$, thus eliminating sums over those variables. The $Y_2^0(\hat{\mathbf{r}}')$ term in the deformed potential $V(\mathbf{r}')$ apparently allows for an angular momentum exchange at the nuclear surface between the outgoing proton and the daughter nucleus, leading to non-diagonal terms in the matrix element.

The DW method can also be used to calculate the decay width to an exited state of the daughter nucleus. This is accomplished by substituting in equation (11) the correct angular momentum R, as well as the appropriate wave number k, which also appears in the argument of the Coulomb function in the matrix element (10).

Direct Method

We can also use the direct method to determine the decay width of a deformed proton emitter. To do this we expand the wavefunction $\Psi_{KIM}(\mathbf{r})$ of the initial state on the complete set of angular momentum basis states:

$$\Psi_{KIM}(\mathbf{r}) = \sum_{\ell j R} |(\ell j R) I M\rangle \langle (\ell j R) I M | \Psi_{KIM}\rangle .$$

Using equations (9) and (10) with $F_{\ell_p}(kr)/r[V(\mathbf{r}) - V_C^0(r)]$ replaced by 1 we can immediately write the overlap matrix element as:

$$\langle (\ell j R) I M | \Psi_{KIM}\rangle = \sqrt{\frac{2\hat{R}}{\hat{I}}} \langle j K R 0 | I K\rangle \langle \ell j K | \phi_K\rangle .$$

Inserting the expansion (7) for the outgoing channel $\ell_p j_p$ we obtain

$$\langle \ell_p j_p K | \phi_K\rangle = \phi_{\ell_p j_p}^{(i)}(r) \to A_{\ell_p j_p} \frac{O_{\ell_p}(kr)}{r}, \text{ for } r \to \infty, \tag{12}$$

assuming that the intrinsic states are matched to outgoing Coulomb waves as in equation (1). The outgoing wave is therefore

$$\Psi^{(+)}_{\ell_p j_p R,KIM}(\mathbf{r}) = N^{\rm Dir}_{\ell_p j_p RIK}\frac{O_{\ell_p}(kr)}{r}|(\ell_p j_p R)IM\rangle,$$

where

$$N^{\rm Dir}_{\ell_p j_p RIK} = \sqrt{\frac{2\hat{R}}{\hat{I}}}\langle j_p K R0|IK\rangle A_{\ell_p j_p}.$$

This gives the decay width, according to equation (2), as

$$\Gamma^{\rm Dir}_{\ell_p j_p RIK} = \frac{\hbar^2 k}{\mu}\frac{2\hat{R}}{\hat{I}}\langle j_p K R0|IK\rangle^2|A_{\ell_p j_p}|^2. \tag{13}$$

This expression is consistent with the result given in equation (1) of ref. [4]. As it stands, this expression cannot be used to calculate the decay width to an excited state of the daughter, because it doesn't take into account the lower proton energy.

It can also be demonstrated explicitly that the distorted wave method and the direct method give identical results for the decay width of a deformed proton emitter. This can be seen by replacing the interactions in the matrix element (10) by the associated single-particle Hamiltonian minus the kinetic energy operators,

$$V(\mathbf{r}) - V^0_C(r) \rightarrow \left[H + \hbar^2\nabla^2/2\mu\right]_{right} - \left[H_0 + \hbar^2\nabla^2/2\mu\right]_{left}. \tag{14}$$

The subscripts 'right' and 'left' indicate that the operators must act to the right and to the left, respectively, when inserted in the matrix element of Eq. (10).

Making the substitution (14) in equation (10), the contributions from the two single-particle Hamiltonians must cancel because the two wave functions have the same energy. Thus the matrix element (10) can be expressed as

$$\mathcal{M}_{\ell_p j_p K} = \frac{\hbar^2}{2\mu}\left\langle \ell_p j_p K\left|\frac{F_{\ell_p}(kr)}{r}(\nabla^2_{right} - \nabla^2_{left})\right|\phi_K\right\rangle.$$

Using Green's theorem and the expansion (7) for the initial state one obtains

$$\mathcal{M}_{\ell_p j_p K} = \frac{\hbar^2}{2\mu}\left(F_{\ell_p}(kr)\frac{d[r\phi^{(i)}_{\ell_p j_p}(r)]}{dr} - [r\phi^{(i)}_{\ell_p j_p}(r)]\frac{dF_{\ell_p}(kr)}{dr}\right)_{r\rightarrow\infty}$$

$$= -\frac{\hbar^2 k}{2\mu}A_{\ell_p j_p},$$

where we again have used the asymptotic form (12) of $\phi^{(i)}_{\ell_p j_p}(r)$. Inserting this into the DW decay width expression (11) we see that it becomes identical to the Dir width, equation (13).

For the deformed case we show in Table 2 the calculated half-lives for three decaying states ($\beta_2 = 0.3$) and total angular momentum $j = 3/2^+, 5/2^+$, and $7/2^-$, respectively.

TABLE 2. Comparison of the proton half-lives of deformed proton radioactivities calculated with the direct (Dir) and distorted wave (DW) methods.

Nucleus	E_p(keV)	j_p	ℓ_p	β_2	$C_{\ell j}$[a]	$t^{Dir}_{1/2,p}$	$t^{DW}_{1/2,p}$
^{131}Eu	932(7)	$\frac{3}{2}^+$	2	0.3	-0.208	27.92 ms	24.09 ms
^{131}Eu	932(7)	$\frac{5}{2}^+$	2	0.3	-0.0999	176.2 ms	214.8 ms
^{141}Ho^g	1169(8)	$\frac{7}{2}^-$	3	0.3	0.240	4.087 ms	3.266 ms

[a] A. A. Sonzogni, private communication (1999)

Here the calculated half-lives only agree to within 20% between the two methods. The discrepancy is probably due to the truncation in the eigenfunction space, such that only the nearest spherical states were included. The initial state is therefore not the exact or complete solution to the deformed Hamiltonian, and the Gell-Mann-Goldberger transformation method will therefore not provide exactly the same result as the direct method. The comparison of the results of the two methods is therefore a test of how close the truncated solution comes to being correct. Further investigation in this area is needed. Perhaps a coupled-channels approach, such as that developed by [12], will offer closer agreement between the direct and distorted wave methods.

CONCLUSIONS

We have shown that the distorted wave method and the direct method of calculating the width of spherical and deformed proton emitter are equivalent. In the spherical case numerical agreement is demonstrated to better than 0.05%, while for the deformed case the agreement is only within about 20%. Improved methods of calculating the wavefunctions should reduce this discrepancy. We recommend using either of these methods in place of the WKB method, which has certain problems related to the frequency factor (see ref. [3] for a discussion of this point). For the cases where the radial wavefunction is known over the $0 < r < 25$ fm range, the direct method is preferred for its calculational simplicity. It is valid only for calculating ground state-ground state decay. However, if the radial wavefunction is known reliably only in the region of the nuclear surface, the distorted wave method is to be preferred. In addition, it can be used to calculate decay branches to excited states of the daughter.

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