$$q(\rho)=\frac{0.702B_0}{\sqrt{2\mu_0 p_{e0}}}\left(\frac{\rho_0}{R_0}\right)^{3/4}x^{-1/4}(1-9x^2/13)^{-1/2} \tag{43}$$

In region $x \leqslant 0.537$, the magnetic shear is negative, the minimum value of q is $q_{\min} = 0.725q(l)$.

The Shafranov shift and the deformations of the cross-section can be obtained analytically or by a little numerical work.

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3. 4 Analytic Study of Propagation and Absorption of Nearly Perpendicular Injected Electron Cyclotron Ordinary Wave[1)]

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Key words: O-X mode coupling, Relativistic resonance, Nearly perpendicular propagation

Electron cyclotron resonance heating (ECRH), such as the fundamental heating and the second harmonic heating, is a basic and powerful method to heat the plasma in tokamak and stellarator devices. Theoretical studies of this heating have been done in rather early literatures[1~3], however, the understanding of some important problems is still uncertain. These include: the coupling of the O-mode and the E-mode and the role of this coupling in wave damping, the O-mode damping mechanism, the evolution of the electron distribution function during O-mode damping, the synergetic effect of the O-mode heating with other wave processes, etc. For the linear dispersion study, we have recently obtained a refined result for pure

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O-mode[4]. Here we give result for nearly perpendicular propagation.

At first we consider the non-relativistic case. The coupling of the O-mode and the X-mode only happens in the narrow region near the resonance point with a width of about $2 k_z R\rho_e$, with k_z being the parallel wave number, R the major radius, ρ_e the electron gyro-radius. Outside this region, the wave polarization vector is parallel to the magnetic field.

In the near resonance region, the dispersion relation of the O-mode is[4~7]

$$n_\perp^2 = \varepsilon_3\left[1-\chi_3+\frac{\chi_{13}^2(2-\varepsilon_3)}{\varepsilon_1(2-\varepsilon_3)-1}\right] \tag{1}$$

where

$$\begin{aligned}
&\varepsilon_1=\varepsilon_{11}=\varepsilon_{22}=1-\frac{\alpha}{4}+\alpha\zeta_0 Z(\zeta_1)/2\\
&\varepsilon_{12}=-\varepsilon_{21}=-\mathrm{i}\alpha\zeta_0 Z(\zeta_1)/2=-\mathrm{i}(\varepsilon_1-1)\\
&\varepsilon_{33}=\varepsilon_3-\chi_3 n_\perp^2\\
&\varepsilon_{13}=\varepsilon_{31}=\chi_{13}n_\perp\\
&\varepsilon_{23}=-\varepsilon_{32}=\mathrm{i}\chi_{13}n_\perp
\end{aligned} \tag{2}$$

where $n_\perp$ is the perpendicular refractive while n_z is the parallel refractive,

$$\alpha=\omega_p^2/\omega^2$$

and

$$\begin{aligned}
&\chi_{13}=\alpha(\omega/\Omega_e)Z'(\zeta_1)4n_z\\
&\varepsilon_3=1-\alpha\\
&\chi_3=-\alpha(\omega/\Omega_e)^2(v_{eT}/c)\zeta_{-1}Z'(\zeta_1)/4n_z
\end{aligned} \tag{3}$$

Z and Z' are the plasma dispersion function and its derivative with respect to the argument $\zeta_1=\frac{\omega-\Omega_e}{k_z v_{eT}}$. Ω_e is the electron cyclotron frequency, which is inversely proportional to the major radius R. Condition for the validity of non-relativistic approximation is

$$1 \gg n_z \gg v_{eT}/c \tag{4}$$

Outside this region, the dispersion relation of the O-mode is approximately equal[1]

$$n_\perp^2 = 1-\alpha\left(1-\frac{1}{4}\frac{v_{\mathrm{eT}}^2}{c^2}\frac{\Omega_\mathrm{e}}{\omega-\Omega_\mathrm{e}}\right)+\mathrm{i}(1-\alpha)\alpha\frac{\sqrt{\pi}}{2n_\mathrm{z}}\zeta_1^2\exp(-\zeta_{-1}^2) \tag{5}$$

The real part of this relation gives $n_\perp^2 = 1-\alpha$. In the inner region of $\left|\zeta_1 < 2\right|$, the wave is nearly right circular polarized and we have

$$\frac{E_x - \mathrm{i}E_y}{E_z} = \frac{\chi_{13}n_\perp}{\varepsilon_1} = \frac{\varepsilon_3 - n_\perp^2(1-\chi_3)}{\chi_{13}n_\perp} \tag{6}$$

The imaginary part of the wave number can be obtained from Eq. (1):

$$(k_\perp)_\mathrm{i} = -\left(\frac{\omega}{c}\right)\left[\frac{\alpha(1-\alpha)^{1/2}}{8n_\mathrm{z}}\frac{v_{\mathrm{eT}}}{c}\right]\mathrm{Im}\left(\frac{Z'^2}{2Z}+\zeta_1 Z'\right) \tag{7}$$

We note that this quantity is inversely proportional to the parallel refractive n_z, implying that the smaller the parallel refractive, the narrower the width of the resonance region and the more peaked value of the absorption.

Now we go to the relativistic analysis for $n_\mathrm{z} \leqslant v/c$. Modification only happens to the elements ε_{11}, ε_{13}, ε_{33}, other elements do not need to change. Expressions of Eqs. (1) and (6) are still correct. In the weakly relativistic case, we have

$$n_\perp^2 = \frac{(1-\alpha)\mathrm{G}(0,\ 1)}{\alpha\mathrm{G}^2(1,\ 1)/4+\mathrm{G}(0,\ 1)[1-\alpha\mathrm{G}(2,\ 1)/4]} \tag{8}$$

where function G(s, n) was defined in Ref. [5], its general form is

$$\mathrm{G}(s,n) = 2\pi\omega\left(\frac{mc}{\theta}\right)^2(mc)^{-s-2|n|}\int_0^\infty \mathrm{d}p_\perp p_\perp^{2|n|+1}\int_{-\infty}^{\infty}\mathrm{d}p_z\frac{f_0 p_z^s}{k_z p_z/m-\gamma\omega+n\Omega_\mathrm{e}} \tag{9}$$

and $f_0 = (2\pi)^{-3/2}\theta^3\exp(-p^2/2\theta^2)$ is the distribution function with $\theta = (mT_\mathrm{e})^{1/2}$. It is sufficient to calculate the three quantities G(0, 1), G(1, 1), G(2, 1) with the relativistic formulae while other quantities can use the non-relativistic results, even the cold plasma results, such as G(2, 0)= −1. Now we obtain

$$\mathrm{G}(0,1) = \left(\frac{2}{\pi}\right)^{1/2}\left(\frac{mc}{\theta}\right)\frac{1}{n_\mathrm{z}}\int_0^\infty \mathrm{d}\tau 2\pi\tau\exp(-\tau^2)\int_{-\tau}^{\tau}\mathrm{d}\tau_z\frac{\tau^2-\tau_\mathrm{z}^2}{\tau_\mathrm{z}-\tau_{\mathrm{z}0}} \tag{10}$$

$$\mathrm{G}(1,1) = \frac{2}{\sqrt{\pi}}\frac{1}{n_z}\int_0^\infty \mathrm{d}\tau 2\pi\tau\exp(-\tau^2)\int_{-\tau}^{\tau}\mathrm{d}\tau_z\frac{(\tau^2-\tau_z^2)\tau_z^2}{\tau_z-\tau_{z0}} \tag{11}$$

$$\mathrm{G}(2,1)=2\sqrt{\frac{2}{\pi}}\left(\frac{\theta}{mc}\right)\int_{0}^{\infty}\mathrm{d}\tau\,2\tau\exp(-\tau^{2})\int_{-\tau}^{\tau}\mathrm{d}\tau_{z}\,\frac{(\tau^{2}-\tau_{z}^{2})\tau_{z}^{2}}{\tau_{z}-\tau_{z0}} \tag{12}$$

where

$$\tau=\left(\frac{mc}{\sqrt{2\theta}}\right)p,\ \ \tau_{z}=\left(\frac{mc}{\sqrt{2\theta}}\right)p_{z} \tag{13}$$

$$\tau_{z0}=\left(\frac{mc}{\sqrt{2\theta}}\right)\frac{1}{n_{z}}\left[\Delta+\left(\frac{\theta}{mc}\right)^{2}\tau^{2}\right] \tag{14}$$

$$\Delta=1-\frac{n\Omega}{\omega} \tag{15}$$

The imaginary parts of these quantities can be easily obtained, the real parts of them (the principal parts) are rather complicated.

The imaginary part of G (s, n): in condition

$$|\tau_{z0}|\le\tau \tag{16}$$

we obtain

$$\mathrm{Im}\{\mathrm{G}(s,\ n)\}=\frac{\sqrt{\pi}}{4n_{z}}\left(\frac{mc}{\theta}\right)\exp\left[-\left(\frac{mc}{\theta}\right)^{2}(n_{z}^{2}-\Delta)\right]\times\int_{-t_{1}}^{t_{1}}\mathrm{d}t\exp(-t)\left[t+t_{0}-\frac{1}{2}\left(\frac{mc}{\theta}\right)^{2}\left(n_{z}+\left(\frac{\theta}{mc}\right)^{2}\frac{t}{n_{z}}\right)^{2}\right]^{n}\left(n_{z}+\left(\frac{\theta}{mc}\right)^{2}\frac{t}{n_{z}}\right) \tag{17}$$

and

$$\mathrm{Im}\{\mathrm{G}(0,\ 1)\}=\left[-\frac{\sqrt{2\pi}}{n_{z}}\left(\frac{\theta}{mc}\right)(\exp(t_{1})-\exp(-t_{1}))+2\sqrt{\pi}\left(\frac{mc}{\theta}\right)\sqrt{\frac{n_{z}^{2}}{2}-\Delta}(\exp(t_{1})+\exp(-t_{1}))\right]\frac{1}{n_{z}^{2}}\exp(-t_{0}) \tag{18}$$

$$\mathrm{Im}\{\mathrm{G}(1,\ 1)\} = \frac{\sqrt{2\pi}}{n_z}\left(\frac{mc}{\theta}\right)\exp(-t_0)\left\{\left(\frac{n_z}{2}-\frac{\Delta}{n_z}\right)[(\exp(t_1)-\exp(-t_1))-t_1(\exp(t_1)+\exp(-t_1))]-\right.$$
$$\frac{1}{n_z}\left(\frac{\theta}{mc}\right)^2[(\exp(t_1)-\exp(-t_1))-t_1(\exp(t_1)+\exp(-t_1))]- \tag{19}$$
$$\left.\frac{1}{2n_z^3}[-t_1(\exp(t_1)+\exp(-t_1))+3t_1^2(\exp(t_1)-\exp(-t_1)+6(\exp(t_1)-\exp(-t_1))]\right\}$$

The expression of $\mathrm{Im}\{\mathrm{G}(2,\ 1)\}$ is very complicated and we do not write it. For very small n_z, we have

$$\mathrm{Im}\{\mathrm{G}(1,\ 1)\} \approx \frac{8\sqrt{\pi}}{3}n_z\left(\frac{mc}{\theta}\right)^5\left(\frac{n_z^2}{2}-\Delta\right)^{3/2}\exp\left[-\left(\frac{mc}{\theta}\right)^2(n_z^2-\Delta)\right] \tag{20}$$

$\mathrm{Im}\{\mathrm{G}(0,\ 1)\}$ and $\mathrm{Im}\{\mathrm{G}(2,\ 1)\}$ approximately have their values of case $n_z=0$ [1]:

$$\mathrm{Im}\{\mathrm{G}(0,\ 1)\} \approx -\frac{8\sqrt{\pi}}{3}\left(\frac{mc}{\theta}\right)^5\left(\frac{n_z^2}{2}-\Delta\right)^{3/2}\exp\left[-\left(\frac{mc}{\theta}\right)^2(n_z^2-\Delta)\right] \tag{21}$$

$$\mathrm{Im}\{\mathrm{G}(0,\ 1)\} \approx -\frac{16\sqrt{\pi}}{15}\left(\frac{mc}{\theta}\right)^5\left(\frac{n_z^2}{2}-\Delta\right)^{5/2}\exp\left[-\left(\frac{mc}{\theta}\right)^2(n_z^2-\Delta)\right] \tag{22}$$

The above results show that absorption of the O-mode happens in the inner side of the resonance.

The real part of $\mathrm{G}(s,\ n)$:in some literatures the real part of these function are replaced by the cold plasma results. This is not correct. We need to calculate these quantities more accurately. The dominant role is played by the principal value of the quantity G(1, 1), from Eq. (13), we get

$$\mathrm{Re}\{\mathrm{G}(1,\ 1)\} = \frac{4}{\sqrt{\pi}n_z}\int_0^\infty \mathrm{d}\tau\exp(-\tau^2)\left[\frac{4}{3}\tau^4-2\tau^2\tau_{z0}^2+\tau\tau_{z0}(\tau^2-\tau_{z0}^2)\ln\left|\frac{\tau-\tau_{z0}}{\tau+\tau_{z0}}\right|\right] \tag{23}$$

Detailed analysis shows that in above integral, contribution from the region of $|\tau_1|\le\tau\le\tau_2$ is almost cancelled by contribution from outside this region. The final result is proportional to n_z. We write it as

$$\mathrm{Re}\{\mathrm{G}(1,\ 1)\} = \left(\frac{mc}{\theta}\right)^2 n_z g_{11}(n_z,\ \Delta) \tag{24}$$

where function $g_{11}(n_z, \varDelta)$ basically only depends on $\varDelta$. Now we understand that for the wave absorption, the main role is played by the above three imaginary quantities while for the mode coupling of the O-mode and the E-mode, the real parts of these quantities are also important. We note that in region of width $\left(\frac{T_e}{mc^2}\right)R$, the real parts and the imaginary parts of these relevant quantities are comparable.

Now we consider the coupling of the O-mode and the X-mode. In the inner region we have

$$\left|\frac{E_x - \mathrm{i}E_y}{E_z}\right| \approx \left|\frac{\mathrm{G}(1,\ 1)n_\perp}{\mathrm{G}(0,\ 1)}\right| \tag{25}$$

For nearly perpendicular propagation, the value of both the real part and the imaginary part of the function G(0, 1) is approximately the same as in case of n_z=0.

$$\mathrm{G}(0,\ 1) = -\frac{8}{3}\left(\frac{mc}{\theta}\right)^2 I_{3/2}(\hat{\varDelta}) \tag{26}$$

where

$$I_{3/2}(\hat{\varDelta}) = \frac{1}{\sqrt{\pi}}\int_0^\infty \mathrm{d}t \frac{t^{3/2}\exp(-t)}{t+\hat{\varDelta}} \tag{27}$$

Both the real and imaginary parts of this function have values of order unity, the perpendicular refractive index of the O-mode is also unity. This means that in the inner region, there is a vertical component of the electric field, which is proportional to n_z $n_\perp$. For Fokker-Planck study, it is necessary to know the following quantity:

$$\frac{\left|E_x - \mathrm{i}E_y\right|^2}{\left|E_z\right|^2} = n_\perp^2 n_z^2 \frac{(9/64)g_{11}^2 + \pi X_0^3 \exp(-2X_0)}{[PI_{3/2}(X_0)]^2 + \pi X_0^3 \exp(-2X_0)} \tag{28}$$

here $$X_0 = -\hat{\varDelta} = \left(\frac{mc^2}{T_e}\right)\left(\frac{R-R_0}{R_0}\right).$$

Noting that in the inner region, three functions related to the damping rate are in same order of magnititude, we obtain

$$k_{\perp i}=\left(\frac{\omega}{c}\right)\alpha\sqrt{1-\alpha}\left\{\frac{\mathrm{Im}\,\mathrm{G}(2,1)}{8}+\frac{3}{16}\left(\frac{mc^2n_z^2}{T_e}\right)\mathrm{Im}\left[\frac{\left[g_{11}+\mathrm{i}\left(\frac{8\sqrt{\pi}}{3}\right)X_0^{3/2}\exp(-X_0)\right]^2}{I_{3/2}(X_0)}\right]\right\} \tag{29}$$

The function G(2, 1) can be replaced by its value in case $n_z=0$ [7]:

$$\mathrm{G}(2,\ 1)=-\frac{16}{15}I_{5/2}(\hat{\Delta}) \tag{30}$$

The above result can continuously transforms into that of case $n_z = 0$[7]. The non-relativistic analysis can not transform in this way. Expressions of Eqs. (28) and (29) are the main results of our study. They show that the absorption of relativistic resonance comes from two parts, one is the direct O-mode damping, the second is the X-mode damping coupled with the O-mode.

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