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# ON $(p,q)$-ANALOGUE OF DIVIDED DIFFERENCES AND BERNSTEIN OPERATORS

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**Abstract.** In this paper, $(p,q)$-calculus is applied to construct $(p,q)$-analogue of divided differences. Another equivalent form of $(p,q)$-Bernstein operators which generalize the Phillips $q$-Bernstein polynomials are defined in terms of $(p,q)$-divided differences. It is shown that these operators reproduce constant as well as linear test functions. Further, we show that the difference of two consecutive $(p,q)$-Bernstein polynomials of a function $f$ can be expressed in terms of second-order divided differences of $f$.

**Keywords.** $(p,q)$-approximation; $(p,q)$-integer; $(p,q)$-Bernstein operator; $(p,q)$-divided difference.

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## 1. Introduction-Preliminaries

Recently, Mursaleen *et al.* introduced $(p,q)$-calculus in approximation theory. They applied it first to construct the $(p,q)$-analogue of the classical Bernstein operators [18]. Most recently, the $(p,q)$-analogues of several operators and related approximation theorems has been studied extensively; see [1, 2, 3, 4, 7, 10, 11, 12, 16, 17, 21, 22] and the references therein.

One of its advantage of using the extra parameter $p$ has been mentioned in [20] to study $(p,q)$-approximation by Lorentz operators in compact disk. Very recently, another nice application is given by Khan *et al.* [13, 14] in computer-aided geometric design in which they applied these

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Bernstein basis for construction of $(p,q)$-Bézier curves and surfaces based on $(p,q)$-integers, which generalize $q$-Bézier curves and surfaces; see [6, 23, 26, 27] and the references therein.

Motivated by the above mentioned work on $(p,q)$-approximation and its application, we apply $(p,q)$-calculus to construct $(p,q)$-analogue of divided differences. We give another equivalent form of $(p,q)$-Bernstein operators in terms of $(p,q)$-divided differences and show that these reproduce constant as well as linear test functions. We state a remark under which these operators also preserve quadratic test functions. We define $(p,q)$-Bernstein polynomials which generalize the $q$-Bernstein polynomials, and show that the difference of two consecutive $(p,q)$-Bernstein polynomials of a function $f$ can be expressed in terms of second-order divided differences of $f$.

The $(p,q)$-analogue of Bernstein operators introduced by Mursaleen *et al.* [18] for $0 < q < p \leq 1$ are defined as follows:

$$B_{n,p,q}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1}(p^s - q^s x)\, f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right),\ x \in [0,1], \quad (1.1)$$

where

$$(1-x)^n_{p,q} = \prod_{s=0}^{n-1}(p^s - q^s x) = (1-x)(p-qx)(p^2-q^2x)...(p^{n-1}-q^{n-1}x)$$
$$= \sum_{k=0}^{n}(-1)^k p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^k.$$

When $p = 1$, $(p,q)$-Bernstein Operators given by (1.1) turns out to be Phillips $q$-Bernstein operators [26].

Let us recall certain notations on $(p,q)$-calculus.

For any $p > 0$ and $q > 0$, the $(p,q)$ integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + ... + pq^{n-2} + q^{n-1} = \begin{cases} \frac{p^n - q^n}{p-q}, \text{ when } p \neq q \neq 1, \\ n\, p^{n-1}, \text{ when } p = q \neq 1, \\ [n]_q, \text{ when } p = 1, \\ n, \text{ when } p = q = 1, \end{cases}$$

where $[n]_q$ denotes the $q$-integers and $n = 0, 1, 2, \cdots$. Obviously, it may be seen that $[n]_{p,q} = p^{n-1}[n]_{\frac{q}{p}}$.

The formula for $(p,q)$-binomial expansion is as follows:

$$(ax+by)^n_{p,q} := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} a^{n-k}b^k x^{n-k}y^k,$$

$$(x+y)^n_{p,q} = (x+y)(px+qy)(p^2x+q^2y)\cdots(p^{n-1}x+q^{n-1}y),$$

$$(1)^n_{p,q} = (1)(p)(p^2)\cdots(p^{n-1}) = p^{\frac{n(n-1)}{2}}.$$

Therefore, we have

$$\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) \; = p^{\frac{n(n-1)}{2}}, \; x \in [0,1], \tag{1.2}$$

where $(p,q)$-binomial coefficients are defined by

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

Also we have the following relation

$$q^k[n-k+1]_{p,q} = [n+1]_{p,q} - p^{n-k+1}[k]_{p,q}, \tag{1.3}$$

$$[n+1]_{p,q} = q^n + p[n]_{p,q} = p^n + q[n]_{p,q}. \tag{1.4}$$

For details on $q$-calculus and $(p,q)$-calculus, one can refer to [5, 8, 18, 28] and the references therein.

One can easily verify by induction that

$$(1+x)(p+qx)(p^2+q^2x)\cdots(p^{n-1}+q^{k-1}x) = \sum_{r=0}^{k} p^{\frac{(k-r)(k-r-1)}{2}} q^{\frac{r(r-1)}{2}} \left[\begin{array}{c} k \\ r \end{array}\right]_{p,q} x^r. \tag{1.5}$$

## 2. $(p,q)$-Bernstein polynomials

For any real function $f$, we define $(p,q)$-differences recursively

$$\triangle^0_{p,q} f_i = f_i, \text{ for all } i \in \mathbb{N}\cup\{0\}, \tag{2.1}$$

$$\triangle^{k+1}_{p,q} f_i = p^k \triangle^k_{p,q} f_{i+1} - q^k \triangle^k_{p,q} f_i, \tag{2.2}$$

for $k=0,1,\cdots,n-i-1$, where $f_i$ denotes $f\left(\frac{p^{n-i}[i]_{p,q}}{[n]_{p,q}}\right)$. For $p=1$, these reduces to $q$-forward differences. Now it is easily established by induction that the $(p,q)$-differences satisfy the following:

$$\triangle_{p,q}^{k} f_i = \sum_{r=0}^{k} (-1)^r p^{\frac{(k-r)(k-r-1)}{2}} q^{\frac{r(r-1)}{2}} \left[\begin{array}{c} k \\ r \end{array}\right]_{p,q} f_{i+k-r}. \tag{2.3}$$

**Definition 2.1.** For each positive integer $n$, we have

$$B_{p,q}^{n}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{r=0}^{n} f_r \left[\begin{array}{c} n \\ r \end{array}\right]_{p,q} p^{\frac{r(r-1)}{2}} x^r \prod_{s=0}^{n-r-1} (p^s - q^s x), \tag{2.4}$$

where an empty product denotes 1. For $p=1$, we obtain the Phillips $q$-Bernstein polynomials [26]. We observe immediately from (2.4) that

$$B_{p,q}^{n}(f;0) = f(0),\ B_{p,q}^{n}(f;1) = f(1), \tag{2.5}$$

for all functions $f$.

We now state a generalization of the well-known forward difference form [25]. Let us write the interpolating polynomial for $f$ at points $x_0, x_1, \cdots, x_n$ in the Newton divided difference form as

$$P_{p,q}^{n}(x) = \sum_{r=0}^{n} \left( \prod_{s=0}^{r-1} (x - x_s) \right) f[x_0, x_1, \cdots, x_r], \tag{2.6}$$

where the empty product denotes 1. For the choice of points $x_r = \frac{p^{n-r}[r]_{p,q}}{[n]_{p,q}}$, $0 \leq r \leq n$, we can express the divided differences in the form of $(p,q)$-differences.

**Theorem 2.1.** *The Newton divided difference in the $(p,q)$-difference form can be written as*

$$f[x_i, x_{i+1}, \cdots x_{i+k}] = \left(\frac{q}{p}\right)^{\frac{-k(2i+k-1)}{2}} [n]_{p,q}^{k} \frac{\triangle_{p,q}^{k} f_i}{[k]_{p,q}!}. \tag{2.7}$$

**Proof.** We may verify by induction on $k$. Let us choose the points $x_i = \frac{p^{1-i}[i]_{p,q}}{[n]_{p,q}}$, $\frac{x_i}{x_n} = \frac{p^{n-i}[i]_{p,q}}{[n]_{p,q}}$, where $x_{i+1}-x_i = \left(\frac{q}{p}\right)^i \frac{[1]_{p,q}}{[n]_{p,q}}$,

$$x_{i+2}-x_i = \left(\left(\frac{q}{p}\right)^{i+1} + \left(\frac{q}{p}\right)^{i}\right)\frac{1}{[n]_{p,q}} = \left(\frac{q}{p}\right)^{i}\frac{[2]_{p,q}}{p[n]_{p,q}},$$

$$\cdots$$

$$x_{i+k}-x_i = \left(\left(\frac{q}{p}\right)^{i+k-1} + \left(\frac{q}{p}\right)^{i+k-2} + \cdots + \left(\frac{q}{p}\right)^{i}\right)\frac{1}{[n]_{p,q}} = \left(\frac{q}{p}\right)^{i}\frac{[k]_{p,q}}{p^{k-1}[n]_{p,q}},$$

where

$$f[x_i,x_{i+1}] = \frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i} = \left(\frac{p}{q}\right)^{i}[n]_{p,q}\triangle_{p,q}f(x_i),$$

$$f[x_i,x_{i+1},x_{i+2}] = \frac{f[x_{i+1},x_{i+2}]-f[x_i,x_{i+1}]}{x_{i+2}-x_i} = \frac{\left(\frac{p}{q}\right)^{2i+1}[n]^2_{p,q}\triangle^2_{p,q}f(x_i)}{[2]_{p,q}!},$$

and

$$f[x_i,x_{i+1},x_{i+2},x_{i+3}] = \frac{f[x_{i+1},x_{i+2},x_{i+3}]-f[x_i,x_{i+1},x_{i+2}]}{x_{i+3}-x_i} = \frac{\left(\frac{p}{q}\right)^{3i+3}[n]^3_{p,q}\triangle^3_{p,q}f(x_i)}{[3]_{p,q}!}.$$

Clearly from (2.7) it is true for $k=1$. Suppose it is true for $k=m$, i.e.,

$$\begin{aligned} f[x_i,x_{i+1},\cdots,x_{i+m}] &= \frac{f[x_{i+1},\cdots,x_{i+m}]-f[x_i,\cdots,x_{i+m-1}]}{x_{i+m}-x_i} \\ &= \frac{\left(\frac{q}{p}\right)^{\frac{-m(2i+m-1)}{2}}[n]^m_{p,q}\triangle^m_{p,q}f(x_i)}{[m]_{p,q}!}. \end{aligned} \tag{2.8}$$

For $k=m+1$, we have

$$\begin{aligned} f[x_i,x_{i+1},\cdots,x_{i+m+1}] &= \frac{f[x_{i+1},\cdots,x_{i+m+1}]-f[x_i,\cdots,x_{i+m}]}{x_{i+m+1}-x_i} \\ &= \frac{\left(\frac{p}{q}\right)^{j-\frac{m(2i+m-1)}{2}}[n]^{m+1}_{p,q}p^m}{[m+1]_{p,q}!}\left(\left(\frac{q}{p}\right)^{-m}\triangle^m_{p,q}f(x_{i+1})-\triangle^m_{p,q}f(x_i)\right) \\ &= \frac{\left(\frac{q}{p}\right)^{\frac{-(m+1)(2i+m)}{2}}[n]^{m+1}_{p,q}\triangle^{m+1}_{p,q}f(x_i)}{[m+1]_{p,q}!}. \end{aligned}$$

Hence (2.7) holds.

**Theorem 2.2.** *The generalized Bernstein polynomial defined by (2.4) can be expressed in the $(p,q)$-difference form as follows*

$$B^n_{p,q}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{r=0}^{n} \left[\begin{array}{c} n \\ r \end{array}\right]_{p,q} p^{\frac{(n-r)(n-r-1)}{2}} \triangle^r_{p,q} f_0 x^r, \tag{2.9}$$

*where $\triangle^r_{p,q}$ defined by (2.2).*

**Proof.** Clearly from (2.4), the coefficient of $x^k$

$$\frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{\nu=0}^{\infty} f_{k-\nu} \left[\begin{array}{c} n \\ k-\nu \end{array}\right]_{p,q} p^{\frac{(k-\nu)(k-\nu-1)}{2}} (-1)^\nu p^{\frac{(n-\nu)(n-\nu-1)}{2}} q^{\frac{\nu(\nu-1)}{2}} \left[\begin{array}{c} n-k+\nu \\ \nu \end{array}\right]_{p,q}$$

$$= \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{\nu=0}^{k} (-1)^\nu p^{\frac{(n-\nu)(n-\nu-1)}{2}} q^{\frac{\nu(\nu-1)}{2}} \left[\begin{array}{c} k \\ \nu \end{array}\right]_{p,q} p^{\frac{(k-\nu)(k-\nu-1)}{2}} f_{k-\nu}.$$

Now we see immediately from the expansion of the $(p,q)$-difference (2.3) that the coefficients of $x^k$ in (2.4) simplifies to give

$$\frac{1}{p^{\frac{n(n-1)}{2}}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} \triangle^k_{p,q} f_0,$$

which verifies (2.9). This completes the proof.

From the uniqueness of interpolating polynomial it is clear that if $f$ is a polynomial of degree $m$, then $\triangle^r_{p,q} f_0 = 0$ for $r > m$ and $\triangle^m_{p,q} f_0 \neq 0$. Thus it follows from (2.9) that if $f$ is a polynomial of degree $m$, then $B^n_{p,q}(f;x)$ is a polynomial of degree $\min(m,n)$. In particular, we will evaluate $B^n_{p,q}(f;x)$ explicitly for $f(x) = 1,\ x,\ x^2$. If $f(x) = 1$, then $f(0) = 1$, which implies from (2.3) that $\triangle^0_{p,q} f_0 = f_0$. Hence we have

$$B^n_{p,q}(1;x) = 1. \tag{2.10}$$

For $f(x) = x$, we compute from (2.3) $\triangle^0_{p,q} f_0 = f_0 = 0$ and

$$\triangle_{p,q} f_0 = f_1 - f_0 = \frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}} - \frac{p^n[0]_{p,q}}{[n]_{p,q}} = \frac{p^{n-1}}{[n]_{p,q}}.$$

Therefore, one has

$$B^n_{p,q}(x;x) = \left[\begin{array}{c} n \\ 0 \end{array}\right]_{p,q} \triangle^0_{p,q} f_0 + \left[\begin{array}{c} n \\ 1 \end{array}\right]_{p,q} \frac{1}{p^{n-1}} \triangle_{p,q} f_0 x.$$

Since $\begin{bmatrix} n \\ 1 \end{bmatrix}_{p,q} = [n]_{p,q}$, we deduce that

$$B^n_{p,q}(x;x) = x. \tag{2.11}$$

For $f(x) = x^2$, we compute $f_0 = 0$ and

$$\triangle_{p,q} f_0 = \left(\frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}}\right)^2 - \left(\frac{p^n[0]_{p,q}}{[n]_{p,q}}\right)^2 = \frac{p^{2(n-1)}}{[n]^2_{p,q}}.$$

Using (2.3), we have

$$\begin{aligned}
\triangle^2_{p,q} f_0 &= \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{r=0}^{2} (-1)^r p^{\frac{(2-r)(2-r-1)}{2}} q^{\frac{r(r-1)}{2}} \begin{bmatrix} 2 \\ r \end{bmatrix}_{p,q} f_{2-r} \\
&= p f_2 - [2]_{p,q} f_1 + q f_0 \\
&= p\left(\frac{p^{n-2}[2]_{p,q}}{[n]_{p,q}}\right)^2 - [2]_{p,q}\left(\frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}}\right)^2 + q\left(\frac{p^n[0]_{p,q}}{[n]_{p,q}}\right)^2 \\
&= p^{2n-3}\frac{[2]^2_{p,q}}{[n]^2_{p,q}} - p^{2n-2}\frac{[2]_{p,q}}{[n]^2_{p,q}} \\
&= p^{2n-3}\frac{[2]_{p,q}}{[n]^2_{p,q}}\left([2]_{p,q} - p\right) \\
&= p^{2n-3} q[2]_{p,q}\frac{1}{[n]^2_{p,q}}.
\end{aligned}$$

It follows from (2.9) that

$$\begin{aligned}
B^n_{p,q}(x^2;x) &= \begin{bmatrix} n \\ 0 \end{bmatrix}_{p,q} \triangle^0_{p,q} f_0 + \begin{bmatrix} n \\ 1 \end{bmatrix}_{p,q} \frac{1}{p^{n-1}} \triangle_{p,q} f_0 x + \begin{bmatrix} n \\ 2 \end{bmatrix}_{p,q} \frac{1}{p^{2n-3}} \triangle^2_{p,q} f_0 x^2 \\
&= p^{n-1}\frac{x}{[n]_{p,q}} + \frac{q[n-1]_{p,q}}{[n]_{p,q}} x^2.
\end{aligned}$$

In view (1.4), one has

$$q[n-1]_{p,q} = [n]_{p,q} - p^{n-1}, \tag{2.12}$$

$$B^n_{p,q}(x^2;x) = x^2 + p^{n-1}\frac{x(1-x)}{[n]_{p,q}}. \tag{2.13}$$

Note that the relations (2.10), (2.11) and (2.13) are identical to $(p,q)$-Bernstein polynomials [18]. In case of $p = 1$ these results are identical to results of Phillips $q$-discrete Bernstein polynomials. It follows directly from the definition that $(p,q)$-Bernstein polynomials possess the end point interpolation property, i.e.,

$$B^n_{p,q}(f;0)=f(0),\ B^n_{p,q}(f;1)=f(1) \text{ for all } 0<q\leq p \text{ and all } n=1,2,\cdots. \tag{2.14}$$

The following representation of $(p,q)$-Bernstein polynomials is called the $(p,q)$-difference form and we denote it as

$$B^n_{p,q}(f;x)=\sum_{r=0}^{n}\left[\begin{array}{c} n \\ r \end{array}\right]_{p,q}\mathscr{D}^r f_0 x^r, \tag{2.15}$$

where $\mathscr{D}^r f_0$ is expressed as

$$\mathscr{D}^r f_0=\frac{[r]_{p,q}!}{[n]^r_{p,q}}p^{\frac{(n-r)(n-r-1)}{2}}q^{\frac{r(r-1)}{2}}f\left[0,\frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}},\cdots,\frac{p^{n-r}[r]_{p,q}}{[n]_{p,q}}\right], \tag{2.16}$$

and $f[x_0,x_1,\cdots,x_i]$ denote the usual divided difference, i.e.,

$$f[x_0]=f(x_0),\ f[x_0,x_1]=\frac{f(x_1)-f(x_0)}{x_1-x_0},\cdots,$$

$$f[x_0,x_1,\cdots,x_i]=\frac{f[x_1,\cdots,x_i]-f[x_0,x_1,\cdots,x_{i-1}]}{x_i-x_0}.$$

Using (2.15) and (2.16), we write

$$B^n_{p,q}(f;x)=\sum_{r=0}^{n}\lambda^n_{p,q}f\left[0,\frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}},\cdots,\frac{p^{n-r}[r]_{p,q}}{[n]_{p,q}}\right]x^r, \tag{2.17}$$

where

$$\begin{aligned}\lambda^n_{p,q}&=\left[\begin{array}{c} n \\ r \end{array}\right]_{p,q}\frac{[r]_{p,q}!}{[n]^r_{p,q}}p^{\frac{(n-r)(n-r-1)}{2}}q^{\frac{r(r-1)}{2}}\\&=\left(1-\frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}}\right)\left(1-\frac{p^{n-2}[2]_{p,q}}{[n]_{p,q}}\right)\cdots\left(1-\frac{p^{n-r+1}[r-1]_{p,q}}{[n]_{p,q}}\right).\end{aligned} \tag{2.18}$$

From (2.18) and

$$\lambda^0_{p,q}=\lambda^1_{p,q}=1, \tag{2.19}$$

we have

$$0\leq\lambda^n_{p,q}\leq 1,\ r=0,1,\cdots,n. \tag{2.20}$$

It follows that

$$\mid B^n_{p,q}(f;x)\mid\leq\sum_{r=0}^{n}\left|f\left[0,\frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}},\cdots,\frac{p^{n-r}[r]_{p,q}}{[n]_{p,q}}\right]\right|\mid x\mid^k. \tag{2.21}$$

This estimates will be used in the sequel. It follows immediately from (2.17) and (2.19) that the $(p,q)$-Bernstein polynomials leave invariant linear functions, that is,

$$B_{p,q}^{n}(at+b;x)=ax+b, \text{ for all } n=1,2,\cdots. \tag{2.22}$$

If $f$ is a polynomial of degree $m$, then all its divided differences of order $>m$ vanish, and (2.17) implies that $B_{p,q}^{n}(f;x)$ is a polynomial of degree $\min(m,n)$. In other words this means that $(p,q)$- Bernstein operator is degree reducing. We set

$$Q_{p,q}^{n}(x)=\frac{1}{p^{\frac{n(n-1)}{2}}}\left[\begin{array}{c} n \\ r \end{array}\right]_{p,q} p^{\frac{r(r-1)}{2}}x^{r}\prod_{s=0}^{r-k-1}(p^{s}-q^{s}x),\ r=0,1,\cdots,n;\ n=1,2,\cdots. \tag{2.23}$$

By taking $a=0,b=1$ in (2.22), we conclude that

$$\sum_{r=0}^{n}Q_{p,q}^{n}(x)=1, \text{ for all..... } n=1,2,\cdots. \tag{2.24}$$

Obviously,

$$B_{p,q}^{n}(f;x)=\sum_{r=0}^{n}f\left(\frac{p^{n-r}[r]_{p,q}}{[n]_{p,q}}\right)Q_{p,q}^{n}(x). \tag{2.25}$$

We note that $B_{p,q}^{n}$ defined by (2.4), is a monotone linear operator for any $0<q<p\leq 1$ and $B_{p,q}^{n}$ reproduces linear functions, that is,

$$B_{p,q}^{n}(ax+b;x)=ax+b,\ a,b\in\mathbb{R}. \tag{2.26}$$

It also satisfies the end point interpolation conditions $B_{p,q}^{n}(f;0)=f(0)$ and $B_{p,q}^{n}(f;1)=f(1)$.

The generalized Bernstein polynomial $B_{p,q}^{n}$ defined by (2.4) shares the well-known shape-preserving properties of the classical Bernstein polynomial. Thus when the function $f$ is convex then (see [24]) $B_{p,q}^{n-1}(f;x)\geq B_{p,q}^{n}$ for $n\geq 2$ and any $0<q<p\leq 1$. As a consequence of this we can show that the approximation to a convex function by its $(p,q)$-Bernstein polynomial is one sided.

**Theorem 2.3.** *If $f$ is a convex function on $[0,1]$, then $B_{p,q}^{n}(f;x)\geq f(x)$ for $0<q<p\leq 1$.*

**Proof.** Let $l(x)=ax+b$ be any line. Also let $l$ be tangent at an arbitrary point $t\in[0,1]$ so that $l(t)=f(t)$ and $f-l\geq 0$. By using (2.26) and the fact that $B_{p,q}^{n}$ is a monotone linear operator, we see that

$$B_{p,q}^{n}(f-l)=B_{p,q}^{n}(f)-l\geq 0.$$

Thus, at any tangent point $t$ we have $B^n_{p,q}(f;t) \geq l(t) = f(t)$. By continuity, we deduce that $B^n_{p,q}(f) \geq f$. This completes the proof.

**Theorem 2.4.** *Let $B^n_{p,q}$ be the operators defined by (2.4). Then for $n = 2,3,\cdots$, we have*

$$\begin{aligned} &B^{n-1}_{p,q}(f;x) - B^n_{p,q}(f;x) \\ &= \frac{1}{p^{\frac{n(n-1)}{2}}} \frac{x(1-x)}{[n]_{p,q}[n-1]_{p,q}} \sum_{r=0}^{n-2} p^{2n-r-1} q^{n+r-1} \left[ \begin{array}{c} n-2 \\ r \end{array} \right]_{p,q} \\ &\times f\left[ p^{n-r-1}\frac{[r]_{p,q}}{[n-1]_{p,q}},\ p^{n-r-1}\frac{[r+1]_{p,q}}{[n]_{p,q}},\ p^{n-r-2}\frac{[r+1]_{p,q}}{[n-1]_{p,q}} \right] x^r \prod_{s=1}^{n-r-2}(p^s - q^s x). \end{aligned}$$

**Proof.** It is clear that

$$\begin{aligned} f[x_0,x_1,x_2] = &\frac{1}{(x_2-x_0)(x_1-x_0)} f(x_0) - \frac{1}{(x_2-x_1)(x_1-x_0)} f(x_1) \\ &+ \frac{1}{(x_2-x_1)(x_2-x_0)} f(x_2). \end{aligned} \tag{2.27}$$

Take $x_0 = p^{n-r}\frac{[r-1]_{p,q}}{[n-1]_{p,q}}$, $x_1 = p^{n-r}\frac{[r]_{p,q}}{[n]_{p,q}}$, $x_2 = p^{n-r-1}\frac{[r]_{p,q}}{[n-1]_{p,q}}$. Using

$$[j+k+1]_{p,q} - p^{k+1}[j]_{p,q} = q^j[k+1]_{p,q}, \tag{2.28}$$

we get

$$\begin{aligned} &f\left[ p^{n-r}\frac{[r-1]_{p,q}}{[n-1]_{p,q}},\ p^{n-r}\frac{[r]_{p,q}}{[n]_{p,q}},\ p^{n-r-1}\frac{[r]_{p,q}}{[n-1]_{p,q}} \right] \\ &= \frac{[n]_{p,q}[n-1]^2_{p,q}}{[n-r]_{p,q}p^{2n-r-2}q^{2r-2}} f\left( p^{n-r}\frac{[r-1]_{p,q}}{[n-1]_{p,q}} \right) \\ &\quad - \frac{[n]^2_{p,q}[n-1]^2_{p,q}}{[r]_{p,q}[n-r]_{p,q}p^{2n-r-2}q^{n+r-2}} f\left( p^{n-r}\frac{[r]_{p,q}}{[n]_{p,q}} \right) \\ &\quad + \frac{[n]_{p,q}[n-1]^2_{p,q}}{[r]_{p,q}p^{2n-2r-2}q^{n+r-2}} f\left( p^{n-r-1}\frac{[r]_{p,q}}{[n-1]_{p,q}} \right). \end{aligned} \tag{2.29}$$

Define

$$a_r = \lambda f\left( p^{n-r-1}\frac{[r]_{p,q}}{[n-1]_{p,q}} \right) + (1-\lambda) f\left( p^{n-r}\frac{[r-1]_{p,q}}{[n-1]_{p,q}} \right) - f\left( p^{n-r}\frac{[r]_{p,q}}{[n]_{p,q}} \right) \geq 0$$

and let $\lambda = p^r \frac{[n-r]_{p,q}}{[n]_{p,q}}$. Using $p^r[n-r]_{p,q} = [n]_{p,q} - q^{n-r}[r]_{p,q}$, we get $1-\lambda = q^{n-r}\frac{[r]_{p,q}}{[n]_{p,q}}$. It follows that

$$\begin{aligned} a_r = &\, p^r \frac{[n-r]_{p,q}}{[n]_{p,q}} f\left(p^{n-r-1}\frac{[r]_{p,q}}{[n-1]_{p,q}}\right) \\ &+ q^{n-r}\frac{[r]_{p,q}}{[n]_{p,q}} f\left(p^{n-r}\frac{[r-1]_{p,q}}{[n-1]_{p,q}}\right) - f\left(p^{n-r}\frac{[r]_{p,q}}{[n]_{p,q}}\right). \end{aligned} \tag{2.30}$$

From (2.29) and (2.30), we get

$$\begin{bmatrix} n \\ r \end{bmatrix}_{p,q} a_r = \frac{p^{2n-r-2}q^{n+r-2}}{[n]_{p,q}[n-1]_{p,q}} \begin{bmatrix} n-2 \\ r-1 \end{bmatrix}_{p,q} f\left[p^{n-r}\frac{[r-1]_{p,q}}{[n-1]_{p,q}},\ p^{n-r}\frac{[r]_{p,q}}{[n]_{p,q}},\ p^{n-r-1}\frac{[r]_{p,q}}{[n-1]_{p,q}}\right]. \tag{2.31}$$

Now we have

$$B^{n-1}_{p,q}(f;x) - B^n_{p,q}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{r=1}^{n-1} \begin{bmatrix} n \\ r \end{bmatrix}_{p,q} x^r a_r \prod_{s=0}^{n-r-1}(p^s - q^s x), \tag{2.32}$$

where $a_r$ is defined in (2.30). Therefore by using (2.31) and (2.32) we get

$$\begin{aligned} &B^{n-1}_{p,q}(f;x) - B^n_{p,q}(f;x) \\ &= \frac{1}{p^{\frac{n(n-1)}{2}}} \frac{x(1-x)}{[n]_{p,q}[n-1]_{p,q}} \sum_{r=1}^{n-1} p^{2n-r-2}q^{n+r-2} \begin{bmatrix} n-2 \\ r-1 \end{bmatrix}_{p,q} \\ &\times f\left[p^{n-r}\frac{[r-1]_{p,q}}{[n-1]_{p,q}},\ p^{n-r}\frac{[r]_{p,q}}{[n]_{p,q}},\ p^{n-r-1}\frac{[r]_{p,q}}{[n-1]_{p,q}}\right] x^{r-1} \prod_{s=1}^{n-r-1}(p^s - q^s x). \end{aligned}$$

By shifting the limits, we get the desired results. This completes the proof.

**Remark 2.5.** For $q \in (0,1)$ and $p \in (q,1]$, it is obvious that $\lim_{n\to\infty}[n]_{p,q} = 0$ or $\frac{1}{p-q}$. In order to reach to convergence results of the operator $B^n_{p,q}(f;x)$, we take a sequence $q_n \in (0,1)$ and $p_n \in (q_n,1]$ such that $\lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} p_n^n = 1$, $\lim_{n\to\infty} q_n^n = 1$. So we get $\lim_{n\to\infty}[n]_{p_n,q_n} = \infty$.

Clearly $(p,q)$-Bernstein operators are defined for all $q \in (0,1)$ and $p \in (q,1]$, however we cannot approximate every continuous function from the space of all continuous function $C[0,1]$ by these operators for all $q \in (0,1)$ and $p \in (q,1]$. Hence we state a theorem which guarantees this approximation process based on Korovkin's type approximation theorem.

**Theorem 2.6.** *Let $0 < q_n < p_n \leq 1$ such that $\lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} p_n^n = 1$, $\lim_{n\to\infty} q_n^n = 1$. Then for each $f \in C[0,1]$, $B^n_{p,q}(f;x)$ converges uniformly to $f$ on $C[0,1]$.*

**Proof.** Let us recall the following Korovkin's theorem (see [15], [19]):

Let $(T_n)$ be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then $\lim_n \|T_n(f,x) - f(x)\|_{C[a,b]} = 0$, for all $f \in C[a,b]$ if and only if $\lim_n \|T_n(f_i,x) - f_i(x)\|_{\mathscr{C}[a,b]} = 0$, for $i = 0,1,2$, where $f_0(t) = 1$, $f_1(t) = t$ and $f_2(t^2) = t^2$.

We need to show if operators converge for the test function $1, t$ and $t^2$, then any continuous function can be approximated with the help of these positive linear operators. Since $B^n_{p,q}(f,x)$ define positive linear operators, the Korovkin's theorem implies that $B^n_{p,q}(f;x) \to f(x)$ if and only if $B^n_{p,q}(t^m,x) \to x^m$ for all $x \in [0,1]$ and $m = 0,1,2$. For $m = 0,1$ this is true. It follows from (2.13) and Remark 2.5 that $B^n_{p_n,q_n}(f,x) \to f(x)$ for $x \in [0,1]$ if and only if

$$B^n_{p_n,q_n}(x^2;x) = x^2 + p_n^{n-1}\frac{x(1-x)}{[n]_{p_n,q_n}} \to x^2.$$

If we choose sequence $p_n$ and $q_n$ satisfying Remark 2.5, then $\lim_{n\to\infty} [n]_{p_n,q_n} \to \infty$. This completes the proof.

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