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Numerical Analysis of State Space Systems Using

Single Term Haar Wavelet Series

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Abstract

In this paper, we present a new method for the solution of state space systems using single term Haar wavelet series (STHWS) method. The effectiveness of this technique is demonstrated by using it to find discrete solutions for any length of time t. We begin by showing how the STHWS method applies to a state space system of differential equations and some examples are illustrated to prove the sufficiency of the method for state systems of differential equations. The method is more general and easy to implement for yielding accurate results.

Keywords: Walsh series, Power series, State system, Haar matrix

1 Introduction

The purpose of this paper is to employ the single term Haar wavelet series (STHWS) method to state space systems of differential equations which are often encountered in many branches of electrical, electronics, physics, chemical and engineering. A variety of methods, exact, approximate and purely numerical are available for the solution of state system of differential equations. Most of these methods are computationally intensive because they are trial and error in nature, or need complicated symbolic computations. Recently, the Haar theory has been innovated and applied to various fields in sciences and engineering applications. In particular, Haar wavelets have been applied extensively for image processing, signal processing in communications and proved to be a useful mathematical tool. The pioneer work in system analysis via Haar wavelets was led by Chen Hui Hsiao [4] who first derived a Haar operational matrix for integrals of the Haar function vector and paved the way for the Haar analysis of the dynamic and control systems.

Chung [3] solved the state space model for linear control systems using the Fourier series expansion. He introduced the Fourier operational matrix of integration and applied computational algorithm to solve linear equations involved in the control problems. Palanisamy [11] introduced a new method for the analysis of optimal control linear singular systems through single-term Walsh series (STWS). Balachandran and Murugesan [1] applied STWS method for stable, unstable, singular and stiff systems. They observed that the method is suitable for singular systems. Marshall and Capehart [10] investigated the relationship between the solutions of state equations obtained by numerical computations of the matrix exponential e^{AT} and the solutions obtained using standard numerical integration techniques.

Yahia and Barrio [16] obtained a numerical method to simplify state space models of thermal systems based on appropriate projection of the thermal field with illustrated examples. Kumar *et al.* [7] derived the state space realizations for the output feedback control of linear, high index differential algebraic equation systems that are not controllable at infinity and for which the control inputs appear explicitly in the underlying algebraic constraints.

A dynamic output feedback compensator is designed that yields a modified system for which the algebraic constraints are independent of the new control inputs and for this feedback modified system and then state space realization is derived. Feng Li and Peng Yung [5] explained a new method for establishing state equations, i.e., the branch replacement and augmented node voltage equations approach. They showed that the new approach is simple and easier for programming when compared with the conventional way of establishing state equations.

Lepik [9] discussed the numerical solution of ODE and PDE by using Haar wavelet techniques. He applied the new technique called the segmentation method and checked the applicability and efficiency of the method. Wu *et al.* [15]

proposed preconditioned AOR iterative method for linear systems for solving the linear system Ax= b. They presented convergence and comparison results along with numerical examples. Blank and Krassnigg [2] studied the matrix algorithms for eigen values to solve homogeneous case of bound state equations and the solution of linear systems in the inhomogeneous case. The efficiency of matrix algorithm was demonstrated by solving the Bethe saltpeter equations. Sundarapandian [14] investigated the computational methods for calculating the canonical forms for linear control systems, computational methods for determining controllability and observability for linear control systems and also the computational methods for solving matrix equations.

Ghomanjani *et al.* [6] presented the Homotopy perturbation method to find the approximate solution of the optimal control of linear systems. They provided some examples with approximate solutions and verified the efficiency of the proposed method. Koyama *et al.* [8] analyzed a non-linear filter for non-linear /non-Gaussian state space models to approximate the states conditional mean and variance together with a Gaussian conditional distribution. They showed the estimation ability of the Laplace Gaussian Filter (LGF) by applying it to the problem of neural decoding. They also proved that the LGF can deliver superior results in a small fraction of the computing time. The numerical solutions of linear singular systems and time varying singular systems have been the subject of intense research activity in the past few years. In order to determine the numerical solutions of singular systems, many efforts have been taken to identify the suitable numerical techniques in literature [12-16]. The main aim of this paper is to determine the effectiveness of the method STHWS and to show the accuracy by applying to solve the state problems on differential equations. The STHWS method can be used to obtain both numerical and analytical solutions of state differential equations.

2 Haar Wavelet and STHWS Technique

2.1 Haar Wavelet Series

The orthogonal set of Haar wavelets $h_i(t)$ is a group of square waves with magnitude of ± 1 in some intervals and zeros elsewhere.

In general,

$$h_n(t) = h_1\left(2^j t - k\right), \tag{1}$$

where $n = 2^j + k$, $j \geq 0$, $0 \leq k < 2^j$, $n, j, k \in Z$

Any function $y(t)$, which is square integrable in the interval [0,1) can be expanded in a Haar series with an infinite number of terms

$$y(t) = \sum_{i=0}^{\infty} c_i h_i(t), \quad \text{with} \quad i = 2^j + k, \tag{2}$$

where the Haar coefficients $j \geq 0, 0 \leq k < 2^j$, $t \in [0,1)$

$$c_i = 2^j \int_0^1 y(t) h_i(t) dt \tag{3}$$

are determined such that the following integral square error ε is minimized

$$\varepsilon = \int_0^1 \left[y(t) - \sum_{i=0}^{m-1} c_i h_i(t) \right]^2 dt, \text{where } m = 2^j,\ j \in \{0\} \cup N$$

Furthermore

$$\int_0^1 h_i(t) h_1(t)\, dt = 2^{-j} \delta_{il} = \begin{cases} 2^{-j}, & i = l = 2^j + k,\ j \geq 0, 0 \leq k \leq 2^j \\ 0, & i \neq l \end{cases}$$

usually, the series expansion in (2) contains an infinite number of terms for a smooth *y(t)*. If *y(t)* is a piecewise constant or may be approximated as a piecewise constant, then the sum in (2) will be terminated after *m* terms, that is

$$y(t) \approx \sum_{i=0}^{m-1} c_i h_i(t) = c_{(m)}^T h_{(m)}(t), t \in [0,1)$$

$$c_{(m)}(t) = \left[c_0 c_1 ... c_{m-1} \right]^T, \qquad h_{(m)}(t) = \left[h_0(t) h_1(t) ... h_{m-1}(t) \right]^T$$

where "T" indicates transposition, the subscript *m* in the parentheses denotes their dimensions, $c_{(m)}^T h_{(m)}(t)$ denotes the truncated sum. Since the differentiation of Haar wavelets results in generalized functions, which in any case should be avoided, the integration of Haar wavelets are preferred. Integration of Haar Wavelets should be expandable in Haar series

$$\int_0^t h_m(\tau) d\tau = \sum_{i=0}^{\infty} C_i h_i(t)$$

If we truncate to $m = 2^n$ terms and use the above vector notation, then integration is performed by matrix vector multiplication and expandable formula into Haar series with Haar coefficient matrix defined by [4]

$$\int_0^1 h_{(m)}(\tau)\, d\tau \approx E_{(m \times m)} h_{(m)}(t), t \in [0,1)$$

where the m-square matrix H is called the Haar matrix of integration which satisfies the following recursive formula [4]

$$H\left(2^K\right) = \begin{bmatrix} H\left(2^{k-1}\right) & H\left(2^{k-1}\right) \\ H\left(2^{k-1}\right) & -H\left(2^{k-1}\right) \end{bmatrix} \tag{4}$$

Then

$$H_{m \times m} = \left[h_n(x_0), h_n(x_1), h_n(x_2), ... h_m(x_{m-1}) \right], \qquad \frac{i}{m} \leq x_i \leq \frac{i+1}{m}$$

$$H_{(m\times m)}^{-1}=\left(\frac{1}{m}\right)H_{(m\times m)}^{T}dia(r)$$

$$r=\left[1,1,2,2,4,4,44....\frac{m}{2},\frac{m}{2}.....\frac{m}{2}\right]^{T},\ m>2$$

Proof of (4) is found in [11].

Since $H_{(m\times m)}$ and $H_{(m\times m)}^{-1}$ contain many zeros.

Let us define $h_{(m)}(t)h_{(m)}^{T}(t)\approx M_{(m\times m)}(t)$ and $M_{(1\times 1)}(t)=h_0(t)$

$M_{(m\times m)}(t)c_{(m)}=C_{(m\times m)}h_{(m)}(t)$, and $C_{(1\times 1)}=c_0$.

2.2 Some nice properties of Haar scaling function

(i) Orthogonality

(a) Scaling functions: $\langle\Phi(x\Leftrightarrow j),\Phi(x\Leftrightarrow k)\rangle=\delta_{j,k},\ j,k\in Z$

(b)Wavelets: $\langle\psi(x\Leftrightarrow j),\psi(x\Leftrightarrow k)\rangle=\delta_{j,k},\ j,k\in Z$

(c)Scaling function and Wavelets: $\langle\psi(x\Leftrightarrow j),\psi(x\Leftrightarrow k)\rangle=0,\ j,k\in Z$

(ii) Compact Support: $Supp\,\Phi(x)=Supp\,\Psi(x)=[0,1]$.

(iii) Symmetry/Anti- Symmetry

(a)Haar scaling function is symmetric.

(b)Haar wavelet is anti-symmetric

(iv) Analytic expressions are available.

(v) Haar vanishing moments $\int_{-\infty}^{\infty}\Psi(x)dx=0$

2.3 Lemma

The non-recursive expression for Haar product matrix defined as $H_m(t)H_m^T(t)$ is expressed as

$$H_m(t)H_m^T(t)=H_m diag\left(B_m(t)\right)H_m^T$$

where $B_m(t)$ are Block Pulse Functions (BPF) defined to be unity in an unit interval of time and zero elsewhere and expressed collectively as $B_m(t)=\left[b_0(t)b_1(t).....,b_{m-1}(t)\right]^T$ where $b_i(t)$ are individual BPF. The product matrix is the non recursive formulation.

2.3.1 Lemma

When the product of two Haar wavelet matrices operate on Haar expansion coefficients of any square integrable function $f(t)$, then it can be expressed as

simple Haar expansion via connection coefficients as $H_m(t)H_m^T(t)c_h = c_h H_m(t)$. where c_h are Haar connection coefficients and $c_h = \left[c_{h0}, c_{h1}, ..., c_{h(m-1)} \right]^T$ are Haar expansion coefficients of $f(t)$ Using the BPF expansion coefficients c_b of $f(t)$, value of Haar connection coefficients c_h can be evaluated non-recursively as

$$c_h = H_m diag(c_b) H_m^{-1}$$

The Haar connection coefficients in Equation (2.3) are the non recursive formulations. These non-recursive formulations have the advantage of computing the Haar connection coefficients directly at the required resolution *m*, there by obviating the need of computing all the matrices at lower resolutions.

The reported advantage of recursive formulations of avoiding inverse of large matrices is of not much relevance today in the period of abundant cheap computing capability at-hand and the need for avoiding recursive computer implementations in general.

2.3.2 Lemma

If $\{u_n\}_{n=0}^{\infty}$ is a sequence of functions which have equiabsolutely continuous integrals, i.e. given $\varepsilon > 0$, there is a δ such that for $E \subseteq [0,1]$, $m(E) < \delta$ implies

$$\left| \int_E u_n(x)\,dx \right| < \varepsilon \qquad \forall n$$

and if $u_n(x) \to f(x)$ in measure, then $f(x)$ is integrable and

$$\lim_{n\to\infty} \int_0^1 u_n(x)\,dx = \int_0^1 f(x)\,dx$$

3 Solution of the State Space System via STHWS

Consider a state equation of the following form

$$x'(t) = A(t)x(t) + B(t)u(t) \tag{5}$$

with $x(0) = x_0$, where *A(t)* is an $n \times n$ matrix, *B (t)* is an $n \times r$ matrix, $x(t)$ is an *n*-state vector and $u(t)$ is an *r*-input vector. With the STHWS approach, the given function is expanded into single term Haar wavelet series in the normalized interval$_{[0,1)}$, which corresponds to $t \in \left[0, \frac{1}{m}\right)$ by defining $mt = \tau$, m being any integer. The Equation (5) becomes, in the normalized interval,

$$x'(\tau)=\frac{A(\tau)}{m}x(\tau)+\frac{B(\tau)}{m}u(\tau)\ ,\ x(0)=x_0 \text{ where } \tau\in[0,1) \tag{6}$$

Now, be expanding (6) by STHWS interval with

$$x'(\tau)=C_n h_0(\tau) \qquad x(\tau)=D_n h_0(\tau) \qquad A(\tau)=E_n h_0(\tau)$$
$$B(\tau)=F_n h_0(\tau) \qquad u(\tau)=H_n h_0(\tau) \qquad \text{n = 0, 1, 2 .....q.}$$

If the $u(t)$ expression on the series of

$$u(\tau)=[u_1(\tau),u_2(\tau).....,u_n(\tau)]^T$$
$$h_0(\tau)=[h_0(\tau),h_0(\tau),.....,h_{m-1}(\tau)]^T$$

$h_0(\tau)$ being the first term of single term Haar wavelet series, the following set of recursive relations has been obtained for the linear stiff systems with.

$$C_n=\left[I-\frac{1}{2m}E_n\right]^{-1}K_n,\ \ D_n(\tau)=\frac{1}{2}C_n+x_n(n-1)$$
$$x_n(n)=C_n+x_n(n-1) \tag{7}$$

where $K_n=\frac{1}{m}\left[E_n x(n-1)+F_n H_n\right]$ and n = 1, 2,3,….. the interval number. $x(n)$ give the discrete value of the state and C_n give the block pulse values of the state to any length of time. The value *m* can be selected as large enough to increase the accuracy of the results.

4. Numerical Examples

4.1 Example

Let us consider $\begin{bmatrix}x_1'(t)\\x_2'(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\-2 & -3\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}+\begin{bmatrix}0\\5\end{bmatrix},\quad x(0)=\begin{bmatrix}1\\0\end{bmatrix}$

The exact solution is

$$x_1(t)=2.5-5e^{-t}+2.5e^{-2t},$$
$$x_2(t)=5e^{-t}-5e^{-2t} \tag{8}$$

The discrete solution can be calculated using the recursive formula given by (7) and the exact solutions for different time *t* are calculated by using (8). The exact and discrete solutions are compared and the error between them is analyzed in Table 1 and Table 2.

Table 1 Solutions of Example 4.1 by Analytic Solution and STHWS Method

Time	Analytic Solution		STHWS Method	
	$x_1(t)$	$x_2(t)$	$x_1(t)$	$x_2(t)$
0	0.00000000	0.00000000	0.00000000	0.00000000
0.1	0.02263979	0.43053332	0.02263979	0.43053332
0.2	0.08214635	0.74205354	0.08214634	0.74205353
0.3	0.16793799	0.96003292	0.16793798	0.96003291
0.4	0.27172218	1.10495541	0.27172216	1.10495539
0.5	0.38704530	1.19325609	0.38704528	1.19325607
0.6	0.50892735	1.23808712	0.50892733	1.23808710
0.7	0.63356589	1.24994170	0.63356585	1.24994167
0.8	0.75809647	1.23716223	0.75809642	1.23716220
0.9	0.88039892	1.20635386	0.88039887	1.20635382
1.0	0.99894100	1.16272079	0.99894095	1.16272074

Table 2 Solutions of Example 4.1 by STWS Methods and Error Analysis

Time	STWS Method		Error Analysis Exact and STHWS	
	$x_1(t)$	$x_2(t)$	$x_1(t)$	$x_2(t)$
0	0.00000000	0.00000000	0	0
0.1	0.02263974	0.43053328	0	0
0.2	0.08214627	0.74205348	1e-08	1e-08
0.3	0.16793790	0.96003283	1e-08	1e-08
0.4	0.27172206	1.10495529	2e-08	2e-08
0.5	0.38704513	1.19325594	2e-08	2e-08
0.6	0.50892703	1.23808689	2e-08	2e-08
0.7	0.63356553	1.24994139	4e-08	3e-08
0.8	0.75809603	1.23716179	5e-08	3e-08
0.9	0.88039842	1.20635336	5e-08	4e-08
1.0	0.99894095	1.16272074	5e-08	5e-08

4.2 Example

Let us consider

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u \qquad x(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The exact solution is

$$x_1(t) = 0.5e^{-t} - e^{-2t} + 0.5e^{-3t}$$
$$x_2(t) = -0.5e^{-t} + 2e^{-2t} - 1.5e^{-3t}$$
$$x_3(t) = 0.5e^{-t} - 4e^{-2t} + 4.5e^{-3t} \quad (9)$$

The discrete solution can be calculated using the recursive formula given by (7) and the exact solutions for different time *t* are calculated by using (9). The exact and discrete solutions are compared and the error between them is analyzed in Table 3 and Table4.

Table 3 Solutions of Example 4.2 by Analytic Solution and STHWS Method

Time	Analytic Solution			STHWS Method		
	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_1(t)$	$x_2(t)$	$x_3(t)$
0	0.00000000	0.00000000	1.00000000	0.00000000	0.00000000	1.00000000
0.25	0.01905301	0.11511110	0.08892724	0.01905301	0.11511110	0.08892724
0.5	0.04695097	0.09779831	-0.16416671	0.04695096	0.09779830	-0.16416669
0.75	0.06575273	0.05197821	-0.18204085	0.06575271	0.05197819	-0.18204083
1	0.07349797	0.01205024	-0.13335960	0.07349794	0.01205022	-0.13335957
1.25	0.07292627	-0.01435902	-0.07925774	0.07292623	-0.01435900	-0.07925770
1.5	0.06733251	-0.02865444	-0.03759271	0.06733247	-0.02865440	-0.03759268
1.75	0.05931335	-0.03436348	-0.01028873	0.05931331	-0.03436343	-0.01028870
2	0.05059138	-0.03475449	0.00555947	0.05059134	-0.03475444	0.00555942

Table 4 Solutions of Example 4.2 by STWS Method and Error Analysis

Time	STWS Method			Error Analysis Exact and STHWS		
	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_1(t)$	$x_2(t)$	$x_3(t)$
0	0.00000000	0.00000000	1.00000000	0	0	0
0.25	0.01905300	0.11511101	0.08892714	0	0	0
0.5	0.04695086	0.09779820	-0.16416654	1e-08	1e-08	2e-08
0.75	0.06575265	0.05197800	-0.18204064	2e-08	2e-08	2e-08
1	0.07349786	0.01205002	-0.13335935	3e-08	2e-08	3e-08
1.25	0.07292603	-0.01435876	-0.07925742	4e-08	2e-08	4e-08
1.5	0.06733217	-0.02865420	-0.03759234	4e-08	4e-08	3e-08
1.75	0.05931301	-0.03436313	-0.01028831	4e-08	4e-08	3e-08
2	0.05059104	-0.03475414	0.00555904	4e-08	5e-08	5e-08

5 Conclusion

The STHWS method is a powerful tool which enables to find analytical solution in case of state system of differential equations. The method has been successfully applied to state system of differential equations. This method is better than numerical methods since it is free from rounding off error and does not

require large computer power. In the present paper, the method yields a series solution which converges faster than the series obtained by the other methods. The numerical results obtained by STHWS method are compared with the analytical solutions. It is shown that the results are found to be in good agreement with each other. The present method is very convenient as it requires only simple computing systems, less computing time and less memory. The STHWS method is very simple and direct which provides the solutions for any length of time *t*. Since state systems of differential algebraic equations involve in many scientific and engineering applications such as circuit analysis, computer aided design, real time simulation of mechanical system, power systems and optimal control systems the STHWS method can be applied in solving such systems to provide useful numerical solutions.

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