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Solution Process of a Class of Differential Equation Using Homotopy Analysis Wiener Hermite Expansion and Perturbation Technique

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Abstract

In this paper, we construct a new method based on the Homotopy analysis method (HAM) linked to Wiener Hermite expansion perturbation (WHEP) technique and it is called HAM WHEP and then apply it to solve the generalized stochastic nonlinear diffusion equation with square or cubic nonlinear losses by obtaining the average and variance of the solution process. The aim of applying this new technique is to overcome the difficulties arising from the Homotopy perturbation method (HPM). Accordingly, applying HPM linked to WHEP in [6] may lead to divergence. This disadvantage is overcome by using the HAM which guarantees the convergence of the series solution. In this direction, this paper revisits and solves the stochastic nonlinear diffusion equation in [6] by applying the HAM WHEP technique. All test problems reveal the accuracy and the convergence of the suggested method.

Mathematics Subject Classification: 65L05; 34A25; 34B15

Keywords: Stochastic nonlinear diffusion equation; Homotopy analysis method; Wiener hermite expansion; Homotopy analysis wiener hermite expansion and perturbation technique

1 Introduction

The mathematical modeling of many real-life phenomena by reason of random perturbation are not possible by ordinary differential equations, and hence are often modeled by using stochastic differential equations in order for the model to become more realistic [16, 22]. Because such differential equations cannot usually be solved analytically, the study of numerical methods is required and these must be designed to perform with a certain order of accuracy. Many authors investigated the stochastic diffusion equation under different views [5, 21].
Recently, M.A. El-Tawil used the Wiener Hermite expansion together with Perturbation theory (WHEP) technique to solve a perturbed nonlinear stochastic diffusion equation [4]. The technique has been then developed to be applied on non-perturbed differential equations using the Homotopy perturbation Method linked to Wiener Hermite expansion perturbation technique and it is called Homotopy WHEP [3]. However, as mentioned S.J. Liao [19], Homotopy perturbation method (HPM) is only a special case of the Homotopy analysis method (HAM). The difference is that, the HPM had to use a good enough initial guess, but this is not absolutely necessary for the HAM. This is mainly because the HAM uses a so-called convergence control parameter $\hbar$ to guarantee the convergence of approximation series over a given interval of physical parameters. So, the Homotopy analysis method (HAM) is more general.

In 2010, M.A. El-Tawil and N.A. Al-Mulla [6] used the HPM linked to WHEP technique to solve the stochastic nonlinear diffusion equation with square or cubic nonlinear losses, as follows,

$$\frac{\partial u(t,x;\omega)}{\partial t} = \frac{\partial^2 u(t,x;\omega)}{\partial x^2} - \varepsilon u^n(t,x;\omega) + \sigma.n(t;\omega);\ (t;x) \in (0,\infty)\times(0,L), \qquad (1)$$
$$u(t,0) = u(t,L) = 0, u(0,x) = \phi(x),$$

where the viscosity ε is a deterministic scale for the nonlinear term. The non homogeneity term $\sigma.n(t)$ is a time white noise process scaled by σ.

However, solving the stochastic nonlinear diffusion equation (1) mentioned above did not consider the influence of using the HPM on the convergence of the series solution. In fact, there is absolutely no guarantee that perturbation methods result in a convergent solution. Accordingly, using the HPM linked to WHEP in [6] may lead to divergence. This disadvantage is overcome by using the Homotopy analysis method (HAM) linked to WHEP (HAM WHEP) technique.

In this direction, this paper revisits and solves the stochastic nonlinear diffusion equation in [6] by applying the HAM WHEP technique. All test problems reveal the accuracy and convergence of the suggested new method.

The main aim of this paper is to construct and develop a new approach based on the Homotopy analysis method introduced in WHEP (HAM WHEP) technique and then apply it for solving the diffusion equation under square and cubic nonlinearities and stochastic non-homogenous on a class of differential equations. Some statistical moments are obtained, mainly the ensemble average and variance of the solution process with corresponding figures.

In this study, for our aim we consider the generalized stochastic nonlinear diffusion equation with square or cubic losses εu^2 or εu^3 of interest is of the following form,

$$\frac{\partial u(t,x;\omega)}{\partial t} = \frac{\partial^2 u(t,x;\omega)}{\partial x^2} - \varepsilon u^n(t,x;\omega) + \sigma(t).n(t;\omega); \quad (t;x) \in (0,\infty)\times(0,L), \tag{2}$$
$$u(t,0) = u(t,L) = 0, u(0,x) = \phi(x),$$

where ε is a deterministic scale for the nonlinear term and $n = 2,3$. $\omega \in (\Omega,\sigma,P)$ is a triple probability space with Ω as the sample space, σ is a σ-algebra on events in Ω and P is a probability measure. The physical meaning of the nonlinear term is that there exists a loss proportional to u^2 or to u^3. The non homogeneity term $\sigma(t).n(t,\omega)$ is a time white noise process scaled by $\sigma(t)$: $\sigma(t;\omega)$ is a continuous time part of the random forcing.

2 The Wiener Hermite Expansion and Perturbation Technique

The application of the Wiener Hermite expansion and perturbation (WHEP) technique [7, 8, 9, 10, 11, 14, 23] aims at finding a truncated series solution to the stochastic solution process of stochastic differential equations. The truncated series is composed of two major parts; the first is the Gaussian part which consists of the first two terms, while the rest of the series constitute the non-Gaussian part. In nonlinear cases, there exist always difficulties of solving the resultant set of deterministic integro-differential equations got from the applications of a set of comprehensive averages on the stochastic integro-differential equation obtained after the direct application of WHE.

Due to the completeness of the wiener Hermite set, any random function $G(t;\omega)$

can be expanded as follows,

$$G(t) = G^{(0)}(t) + \int_{\Re} G^{(1)}(t;t_1)H^{(1)}(t_1)dt_1 + \int_{\Re}\int_{\Re} G^{(2)}(t;t_1,t_2)H^{(2)}(t_1,t_2)dt_1 dt_2 + .. \quad (3)$$

Where the first two terms are the Gaussian part of $G(t;\omega)$. The rest of the terms in the expansion represent the non-Gaussian part of $G(t;\omega)$.
The average of $G(t;\omega)$ is

$$\mu_G = EG(t;\omega) = G^{(0)}(t) \text{ with } EH^{(1)}(t_1) = 0, EH^{(1)}(t_1).H^{(1)}(t_2) = \delta(t_1 - t_2), \quad (4)$$

where the time white noise process is $n(t_1) = H^{(1)}(t_1)$.
The covariance of $G(t;\omega)$ is

$$Cov(G(t;\omega), G(\tau;\omega)) = E(G(t;\omega) - \mu_G(t))(G(\tau;\omega) - \mu_G(\tau))$$

$$= \int_{\Re} G^{(1)}(t;t_1)G^{(1)}(\tau;t_1)dt_1 + 2\int_{\Re}\int_{\Re} G^{(2)}(t;t_1,t_2)G^{(2)}(\tau;t_1,t_2)dt_1 dt_2 + ... \quad (5)$$

The variance of $G(t;\omega)$ is

$$\sigma_G^2 = E(G(t;\omega) - \mu_G(t))^2$$
$$= \int_{\Re} \left[G^{(1)}(t;t_1)\right]^2 dt_1 + 2\int_{\Re}\int_{\Re} \left[G^{(2)}(t;t_1,t_2)\right]^2 dt_1 dt_2 + ... \quad (6)$$

The WHEP technique can be applied on linear or nonlinear perturbed systems described by ordinary or partial differential equations. The solution can be modified in the sense that additional parts of the Wiener Hermite expansion can be taken into considerations and the required order of approximations can always be made depending on the computing tool.
The first order solution can be obtained when considering only the Gaussian part of the solution process $u(t;\omega)$ can be expanded as,

$$u(t;\omega) = u^{(0)}(t) + \int_{\Re} u^{(1)}(t;t_1)H^{(1)}(t_1)dt_1 . \quad (7)$$

The WHEP technique uses the following expansion for its deterministic kernels,

$$u^{(i)}(t) = u_0^{(i)} + \varepsilon u_1^{(i)} + \varepsilon^2 u_2^{(i)} + ..., i = 0,1,... \quad (8)$$

3 Basic idea of Homotopy Analysis Method

The Homotopy analysis method (HAM) initially proposed by S.J. Liao in his

Ph.D. thesis [17]. A systematic and clear is exposition on the HAM is given in [18]. In recent years, this method has been successfully employed to solve many types of nonlinear problems in science and engineering [1, 2, 13, 15, 20]. HAM contains a certain auxiliary parameter $\hbar$, which provides with a simple way to adjust and control the convergence region and rate of convergence of the series solution. Moreover, by means of the so-called $\hbar$-curve, a valid region of $\hbar$ can be studied to gain a convergent series solution. It is important to note that, one has great freedom to choose auxiliary objects such as $\hbar$ and L in HAM. Thus, through HAM, explicit analytic solutions of nonlinear problems are possible.

To describe the basic idea of the HAM, we consider the following differential equation,

$$\mathrm{N}[u(x,t)] = 0, \tag{9}$$

where N is a nonlinear operator, x and t denotes the independent variables, $u(x,t)$ is an unknown function, respectively. By means of generalizing the traditional Homotopy method, S.J. Liao [17] construct the so-called zero order deformation equation,

$$(1-q)£[\psi(x,t;q) - u_0(x,t)] = q\hbar H(x,t)N[\psi(x,t;q)], \tag{10}$$

where $q \in [0,1]$ is an embedding parameter, $\hbar$ is the nonzero auxiliary parameter and $H(x,t)$ is the nonzero auxiliary function, £ is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $\psi(x,t;q)$ is an unknown function.

Obviously, when $q = 0$ and $q = 1$ both,

$$\psi(x,t;0) = u_0(x,t) \text{ and } \psi(x,t;1) = u(x,t), \tag{11}$$

respectively hold. Thus as q increases from 0 to 1, the solution $\psi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$.

Expanding $\psi(x,t;q)$ in Taylor series with respect to q, one has,

$$\psi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m, \tag{12}$$

where,

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \psi(x,t;q)}{\partial q^m}\bigg|_{q=0}. \tag{13}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function are so properly chosen, then the series (12) converges at $q = 1$ and,

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t), \tag{14}$$

which is the solution of the original equation, as proved by S.J. Liao [30]. As $H(x,t) = 1$ and $\hbar = -1$, Equation (10) becomes,

$$(1-q)£[\psi(x,t;q) - u_0(x,t)] + qN[\psi(x,t;q)] = 0, \tag{15}$$

which is used mostly in the Homotopy perturbation method (HPM) proving that the HPM is a special case of the Homotopy analysis method (HAM). Comparison between the HAM and HPM can be found in [12, 19].
According to equation (13), the governing equation can be deduced from the zero order deformation equation (10). Define the vector,

$$\overrightarrow{u_n} = \{u_0(x,t), u_1(x,t), \ldots, u_n(x,t)\}. \tag{16}$$

Differentiating (10) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth order deformation equations,

$$£[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t) R_m(\vec{u}_{m-1}), \tag{17}$$

where

$$\chi_m = \begin{cases} 0, m \leq 1 \\ 1, m > 1 \end{cases} \tag{18}$$

and

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\psi(x,t;q)]}{\partial q^{m-1}}\bigg|_{q=0}. \tag{19}$$

For any given nonlinear operator N and the term $R_m(\vec{u}_{m-1})$ can be easily expressed by equation (19). So we can obtain $u_1(x,t), u_2(x,t), \ldots$ by means of solving the linear high order deformation equation (17). The mth order approximation of $u(x,t)$ is given by,

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t), \tag{20}$$

The foregoing approximate solution consist of $\hbar$, which is a cornerstone of the HAM in determining convergence of series solution rapidly. We may adjust and control the convergence region and rate of the solution series (20) by means of the auxiliary parameter $\hbar$. To obtain valid region of $\hbar$, we first plot the so called $\hbar$-curves of $u(x,t)$, $u_t(x,t)\big|_{x=\alpha}$ where $\alpha \in [a,b]$ and so on. According to these $\hbar$-curves, it is easy to discover the valid region of $\hbar$, which corresponds to the line segments nearly parallel to the horizontal axis.

Theorem 3.1 According to S.J. Liao [18], as long as the series (20) converges to $u(x,t)$, where $u_m(x,t)$ is governed by the high-order deformation equation (17) under the definitions (18) and (19), it is must be the exact solution of equation (9).

4 Solving the Langevin equation using some techniques: HAM, WHEP and HAM WHEP technique

This section deals with the Langevin equation by using three techniques, in particular: WHEP, HAM and HAM WHEP technique.
We consider the Langevin equation for $n = 2,3$,

$$\begin{aligned} &\frac{\partial u(t;\omega)}{\partial t} = -\varepsilon u^n(t;\omega) + \sigma(t).n(t;\omega), t \in (0,\infty), \\ &u(0) = 1, \end{aligned} \tag{21}$$

where $n(t;\omega)$ is the time white noise process, $\sigma(t)$ is a continuous function and ε is a constant.

4.1 Using WHEP technique for solving the Langevin equation

At $n = 2$, Eq.(21) recasts as,

$$\begin{aligned} &\frac{\partial u(t;\omega)}{\partial t} = -\varepsilon u^2(t;\omega) + \sigma(t).n(t;\omega), t \in (0,\infty), \\ &u(0) = 1, \end{aligned} \tag{22}$$

where $\sigma(t).\, n(t;\omega)$ is the time white noise process scled by $\sigma(t)$: $\sigma(t)$ is a continuous function and ε is a constant.
Applying the WHEP technique on Eq. (22) and taking the necessary averages, we get the following equations,

$$\frac{\partial u^{(0)}(t)}{\partial t} = -\varepsilon\left[u^{(0)}(t)\right]^2 - \varepsilon\int_0^t \left[u^{(1)}(t;t_1)\right]^2 dt_1, \tag{23}$$

$$\frac{\partial u^{(1)}(t;t_1)}{\partial t} = -2\varepsilon u^{(0)}(t)u^{(1)}(t;t_1) + \sigma(t)\delta(t-t_1). \tag{24}$$

Where,

$$u(t;\omega) = u^{(0)}(t) + \int_0^t u^{(1)}(t;t)H^{(1)}(t_1)dt_1. \tag{25}$$

Applying the perturbation technique, the deterministic kernels can be represented in first order approximation as,

$$u^{(0)}(t) = u_0^{(0)}(t) + \varepsilon u_1^{(0)}(t), \tag{26}$$

$$u^{(1)}(t;t_1) = u_0^{(1)}(t;t_1) + \varepsilon u_1^{(1)}(t;t_1). \tag{27}$$

The solution is to evaluate $u_0^{(0)}$ and $u_0^{(1)}$ and then computing the other two kernels independently. The final results of the first order first correction mean and variance respectively are,

$$\mu_u(t) = u^{(0)}(t), \tag{28}$$

$$\sigma_u^2(t) = \int_0^t \left[u^{(1)}(t;t_1)\right]^2 dt_1. \tag{29}$$

4.2 Using HAM for solving the Langevin equation

In order to solve Eq. (22) by the HAM, we choose the initial approximation,

$$u_0(t) = 1, \tag{30}$$

and the auxiliary linear operator,

$$£[\psi(t;q)] = \frac{\partial \psi(t;q)}{\partial t}, \tag{31}$$

with the property,

$$£[c_1] = 0, \tag{32}$$

where c_1 is an integral constant. Furthermore, Eq. (22) suggests that we define the nonlinear,

$$N[\psi(t;q)] = \frac{\partial \psi}{\partial t} + \varepsilon \psi^2 - \sigma(t)n(t.). \quad (33)$$

Using the above definition, we construct the zero order deformation equation

$$(1-q)£[\psi(t;q) - u_0(t)] = q\hbar H(t)N[\psi(t;q)]. \quad (34)$$

As $H(t)=1$, Eq. (34) becomes,

$$(1-q)£[\psi(t;q) - u_0(t)] = q\hbar N[\psi(t;q)], \quad (35)$$

and the so-called *mth* order deformation equation,

$$£[u_m(t) - \chi_m u_{m-1}(t)] = \hbar R_m(\vec{u}_{m-1}), \quad (36)$$

with the initial condition,

$$u_m(0) = 0, \quad (37)$$

where

$$R_m(\vec{u}_{m-1}) = (u_{m-1})_t + \varepsilon(u_{m-1})^2 - (1-\chi_m)\sigma(t)n(t). \quad (38)$$

Now, the solution of the so-called *mth* order deformation equation Eq. (36) for $m \geq 1$ becomes,

$$u_m(t) = \chi_m u_{m-1}(t) + \hbar \int_0^t R_m(\vec{u}_{m-1}) d\tau + c_1, \quad (39)$$

where the integration constant c_1 is determined by the initial condition (37).
Taking the necessary averages with $\sigma(t)=t$, we get the following results when getting both the fourth order approximation for the mean and the variance respectively,

$$\mu_u = 1 + \hbar t + (2+\hbar)(\hbar + \hbar^2)t + \frac{[3\hbar^4 + 3\hbar^3 + \hbar^2]}{3}t^3 + \frac{(\hbar+\hbar^2)^2}{12}t^4 + \frac{3\hbar^4(1+\hbar)}{10}t^5 - \frac{\hbar^3(\hbar+\hbar^2)}{9}t^6 + \frac{\hbar^6}{14}t^7 + \frac{\hbar^6}{32}t^8 + \frac{\hbar^6}{81}t^9, \quad (40)$$

$$\sigma_u^2(t) = \frac{(\hbar+\hbar^2)^2}{3}t^3 + \frac{2\hbar^4(\hbar+1)}{3}t^5 + \frac{\hbar^6}{3}t^7 + \frac{\hbar^6}{9}t^8. \tag{41}$$

The proper value of $\hbar$ which ensures that the approximation solution is convergent is found from the $\hbar$-curves obtained both from the fourth order HAM approximation of the mean and the variance shown in Figures 1 and 2 respectively. The valid region of $\hbar$ corresponds to the line segments nearly parallel to the horizontal axis.

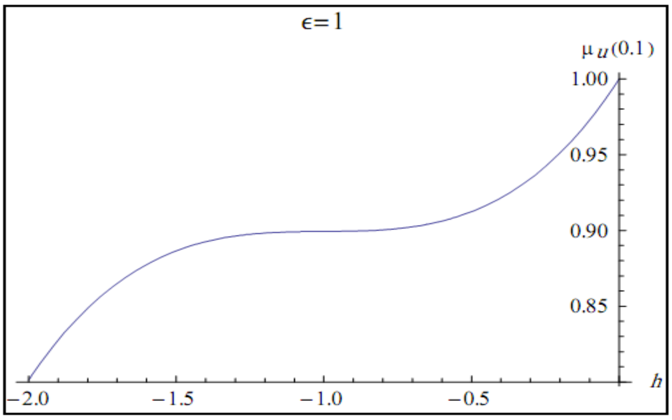

Figure 1: The $\hbar$-curve of the mean based on the fourth order HAM approximation

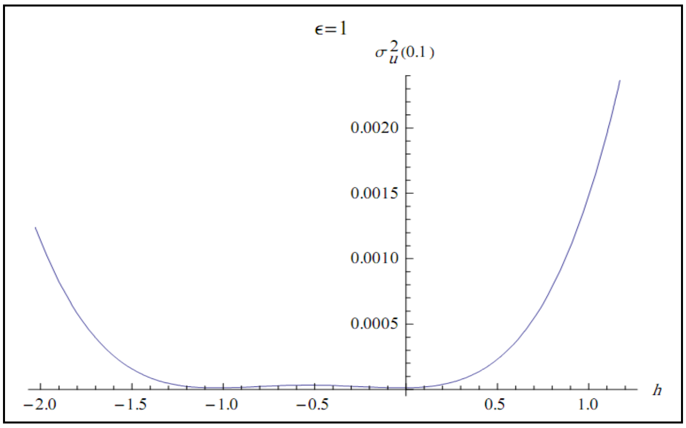

Figure 2: The $\hbar$-curve of the variance based on the fourth order HAM approximation

4.3 Using HAM WHEP technique for solving the Langevin equation

Applying the WHEP technique on the proposed example of the Eq. (22), and taking the necessary averages, we get the following equations,

$$\frac{\partial u^{(0)}(t)}{\partial t} = -\varepsilon\left[u^{(0)}(t)\right]^2 - \varepsilon\int_0^t \left[u^{(1)}(t;t_1)\right]^2 dt_1 \tag{42}$$

$$\frac{\partial u^{(1)}(t;t_1)}{\partial t} = -2\varepsilon u^{(0)}(t)u^{(1)}(t;t_1) + \sigma(t)\delta(t-t_1)\,. \tag{43}$$

In order to solve the Eqs. (42) and (43) by the HAM, we choose the initial approximations

$$u_0^{(0)}(t) = 1, u_0^{(0)}(t;t_1) = t\,. \tag{44}$$

Applying the same approach as in subsection (4.2) with $\sigma(t) = t$. The proper value of $\hbar$ is found from the $\hbar$-curves obtained from the sixth order HAM WHEP approximation of the mean and the fifth order HAM WHEP approximation of the variance shown in Figures 3 and 4 respectively. The valid region of $\hbar$ corresponds to the line segments nearly parallel to the horizontal axis.

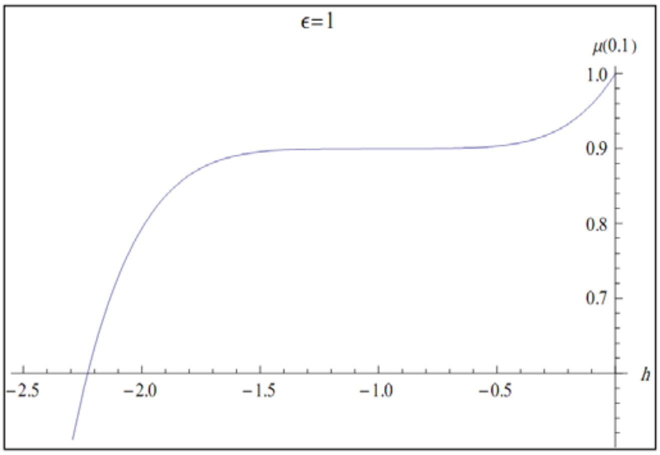

Figure 3: The $\hbar$-curve of the mean obtained from the sixth order HAM WHEP approximation

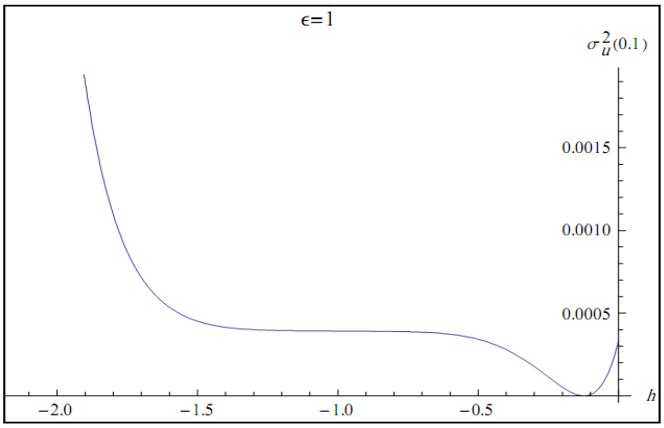

Figure 4: The $\hbar$-curve of the variance obtained from the fifth order HAM WHEP approximation

Now, considering the case $n = 3$, and proceeding in a similar manner as in subsection 4.3, the results are obtained from the $\hbar$-curves both of the fourth order HAM WHEP approximation of the mean and the variance shown in Figures 5 and 6 respectively. The valid region of $\hbar$ corresponds to the line segments nearly parallel to the horizontal axis.

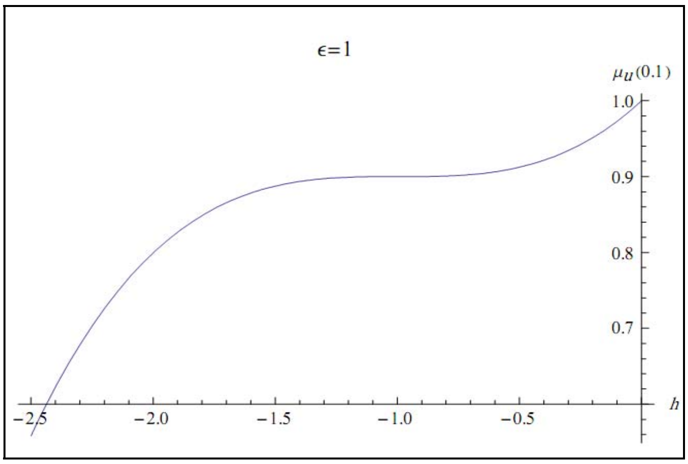

Figure 5: The $\hbar$-curve of the mean obtained from the fourth order HAM WHEP approximation

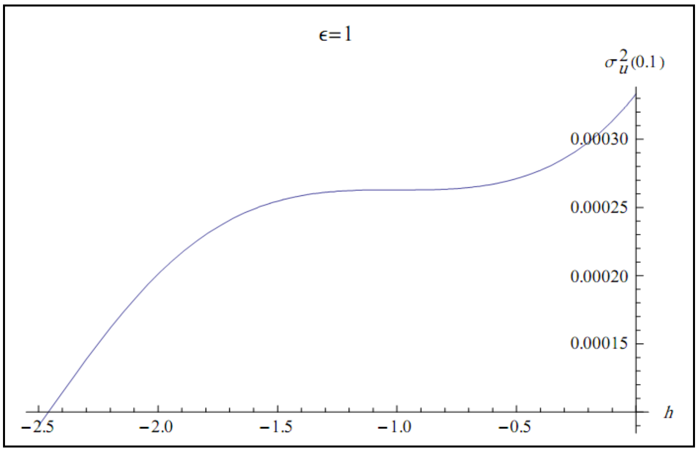

Figure 6: The $\hbar$ -curve of the variance obtained from the fourth order HAM WHEP approximation

Concerning only a first order approximation, it can be noticed both the HAM and HAM WHEP techniques give near results. The HAM WHEP technique seems an efficient one because of its correction possibilities.

5. Solving the Boundary value problem using Homotopy Analysis Wiener Hermite Expansion and Perturbation technique

In the present paper, for our aim, the HAM WHEP technique is applied to solve the generalized stochastic nonlinear diffusion equation with square or cubic losses, εu^2 or εu^3 and it is shown that how one can control the convergence of approximate solution and make the convergence fast.

At $n = 2$, the equation (2) of interest in this paper becomes,

$$\frac{\partial u(t,x;\omega)}{\partial t} = \frac{\partial^2 u(t,x;\omega)}{\partial x^2} - \varepsilon u^2(t,x;\omega) + \sigma(t).n(t;\omega); \quad (t;x) \in (0,\infty) \times (0,L), \tag{45}$$
$$u(t,0) = u(t,L) = 0, u(0,x) = \varphi(x),$$

where $\sigma(t).\, n(t;\omega)$ is the time white noise process scaled by $\sigma(t) : \sigma(t)$ is the continuous time part of the random forcing.

Applying the WHEP technique on the proposed equation in (45), and taking the necessary averages, we get the following equations,

$$\frac{\partial u^{(0)}(t,x)}{\partial t} = \frac{\partial^2 u^{(0)}(t,x)}{\partial x^2} - \varepsilon\left[u^{(0)}(t,x)\right]^2 - \varepsilon\int_0^t \left[u^{(1)}(t,x;t_1)\right]^2 dt_1, \quad (46)$$
$$u^{(0)}(t,0) = u^{(0)}(t,L) = 0, u^{(0)}(0,x) = \varphi(x)$$

$$\frac{\partial u^{(1)}(t,x;t_1)}{\partial t} = \frac{\partial^2 u^{(1)}(t,x;t_1)}{\partial x^2} - 2\varepsilon u^{(0)}(t,x)u^{(1)}(t,x;t_1) + \sigma(t)\delta(t-t_1), \quad (47)$$
$$u^{(1)}(t,0,t_1) = u^{(1)}(t,L;t_1) = 0, u^{(1)}(0,x;t_1) = t.$$

In order to solve the Equations (46) and (47) by the HAM, we choose the initial approximations,

$$u^{(0),0}(t,x) = \varphi(x), u^{(1),0}(t,x;t_1) = t, \quad (48)$$

and the auxiliary linear operators,

$$£_1\left[u^{(0)}(t,x;q)\right] = \frac{\partial u^{(0)}(t,x;q)}{\partial t}, £_2\left[u^{(1)}(t,x;t_1;q)\right] = \frac{\partial u^{(1)}(t,x;t_1;q)}{\partial t}, \quad (49)$$

with the properties,

$$£_1[c_1] = £_2[c_2] = 0, \quad (50)$$

where c_1 and c_2 are the integral constants.

Equations (46) and (47) suggests that we define the nonlinear operators,

$$N\left[u^{(0)}(t,x;q)\right] = \frac{\partial u^{(0)}(t,x;q)}{\partial t} - \frac{\partial^2 u^{(0)}(t,x)}{\partial x^2} + \varepsilon\left[u^{(0)}(t,x)\right]^2 + \varepsilon\int_0^t \left[u^{(1)}(t,x;t_1)\right]^2 dt_1, \quad (51)$$

$$M\left[u^{(1)}(t,x;t_1;q)\right] = \frac{\partial u^{(1)}(t,x;t_1;q)}{\partial t} - \frac{\partial^2 u^{(1)}(t,x;t_1)}{\partial x^2} + 2\varepsilon u^{(0)}(t,x)u^{(1)}(t,x;t_1)$$
$$-(1-\chi_m)\sigma(t)\delta(t-t_1). \quad (52)$$

Using the above definition we construct the zero order deformation equation with $H(t,x) = 1$, we have,

$$(1-q)£_1\left[u^{(0)}(t,x;q) - u^{(0),0}(t,x)\right] = q\hbar N\left[u^{(0)}(t,x;q)\right], \quad (53)$$

$$(1-q)£_2\left[u^{(1)}(t,x;t_1;q) - u^{(1),0}(t,x;t_1)\right] = q\hbar M\left[u^{(1)}(t,x;t_1;q)\right], \quad (54)$$

and the so-called *m*th order deformation equations for $m \geq 1$ are,

$$\pounds_1\left[u^{(0),m}-\chi_m u^{(0),m-1}\right]=\hbar R^{(0),m}\left(\vec{u}^{(0),m-1}\right), \qquad (55)$$

$$\pounds_2\left[u^{(1),m}-\chi_m u^{(1),m-1}\right]=\hbar R^{(1),m}\left(\vec{u}^{(1),m-1}\right), \qquad (56)$$

with the initial conditions,

$$u^{(0),m}(0,x)=1, u^{(1),m}(0,x;t_1)=0, \qquad (57)$$

where,

$$R^{(0),m}\left(\vec{u}^{(0),m-1}\right)=\left(u^{(0),m-1}\right)_t-\left(u^{(0),m-1}\right)_{xx}+\varepsilon\left(u^{(0),m-1}\right)^2+\varepsilon\int_0^t\left[u^{(1),m-1}\right]^2 dt_1, \qquad (58)$$

$$R^{(1),m}\left(\vec{u}^{(1),m-1}\right)=\left(u^{(1),m-1}\right)_t-\left(u^{(1),m-1}\right)_{xx}+2\varepsilon u^{(0),m-1}u^{(1),m-1}-(1-\chi_m)\sigma(t)\delta(t-t_1). \qquad (59)$$

The solutions of the so-called *m*th order deformation equation (55) and (56) are,

$$u^{(0),m}(t,x)=\chi_m u^{(0),m-1}+\hbar\int_0^t R^{(0),m}\left(\vec{u}^{(0),m-1}\right)d\tau+c_1, \qquad (60)$$

$$u^{(1),m}(t,x)=\chi_m u^{(1),m-1}+\hbar\int_0^t R^{(1),m}\left(\vec{u}^{(1),m-1}\right)d\tau+c_2. \qquad (61)$$

We obtain the results for $\sigma(t)=t$ and $\varphi(x)=x$,

$$u^{(0),1}(t,x)=\hbar\left(\frac{t^4}{4}+tx^2\right),\quad u^{(1),1}(t,x)=\hbar tx^2, \qquad (62)$$

and so on.

The proper value of $\hbar$ which ensures that the approximation solution is converge is found from the $\hbar$-curves obtained both from the fifth order HAM WHEP technique approximation of the mean and the variance shown in Figures 7 and 8 respectively. As mentioned S.J. Liao [18], the valid region of $\hbar$ corresponds to the line segments nearly parallel to the horizontal axis.

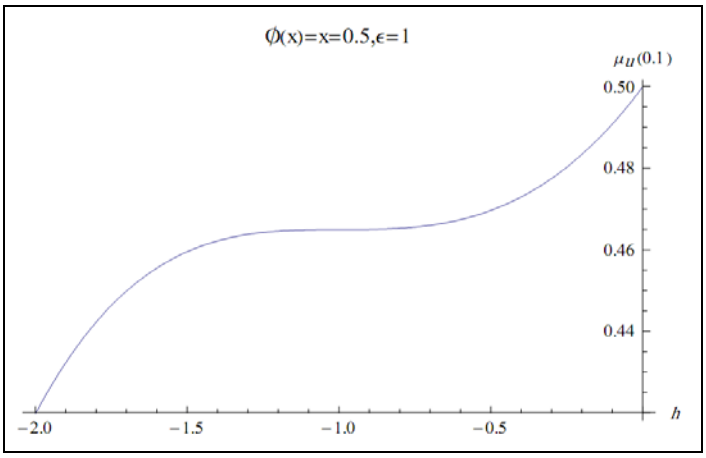

Figure 7**:** The $\hbar$ -curve of the mean obtained from the fifth order HAM WHEP approximation.

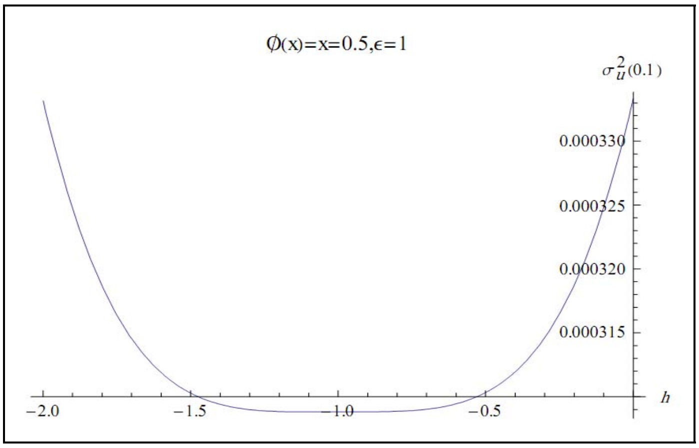

Figure 8**:** The $\hbar$ -curve of the variance obtained from the fifth order HAM WHEP approximation.

Now, considering the case $n=3$, the equation (2) of interest in this paper becomes,

$$\frac{\partial u(t,x;\omega)}{\partial t}=\frac{\partial^2 u(t,x;\omega)}{\partial x^2}-\varepsilon u^3(t,x;\omega)+\sigma(t).n(t;\omega);\quad (t;x)\in(0,\infty)\times(0,L),$$
$$u(t,0)=u(t,L)=0, u(0,x)=\varphi(x), \tag{63}$$

where $\sigma(t).n(t;\omega)$ is the time white noise process scaled by $\sigma(t)$: $\sigma(t)$ is the continuous time part of the random forcing.

Proceeding in the same manner as previously. The proper value of $\hbar$ which ensures that the approximation solution is convergent is found from the $\hbar$-curves obtained both from the fourth order HAM WHEP approximation of the mean and the variance shown in Figures 9 and 10 respectively. The valid region of $\hbar$ corresponds to the line segments nearly parallel to the horizontal axis.

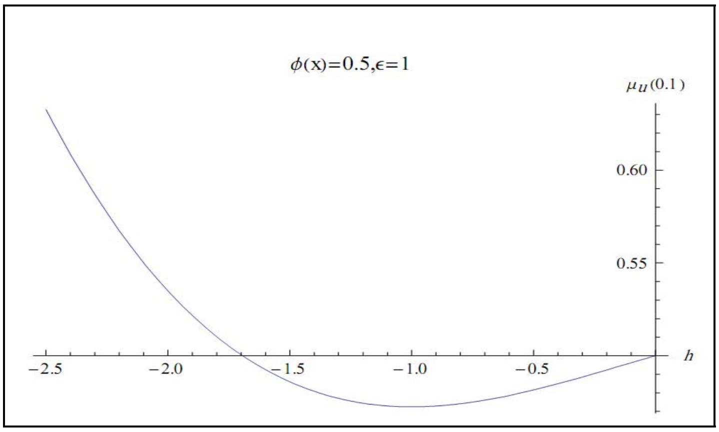

Figure 9**:** The $\hbar$-curve of the mean obtained from the fourth order HAM WHEP approximation

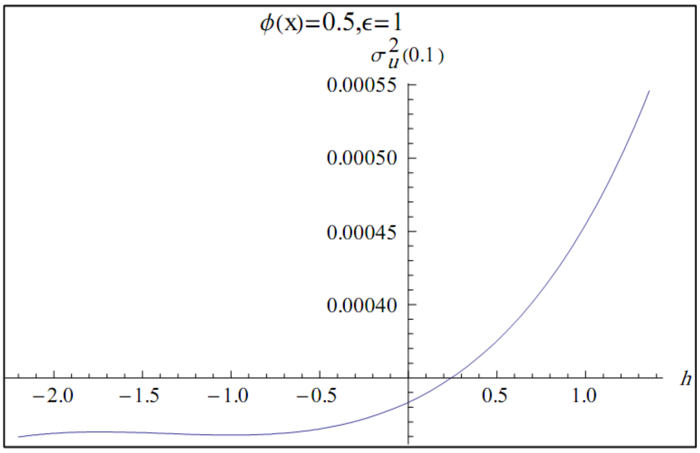

Figure 10**:** The $\hbar$-curve of the variance obtained from the fourth order HAM WHEP approximation.

6 Conclusion

In this paper, the HAM linked to WHEP (HAM WHEP) technique has been applied to solve the generalized stochastic nonlinear diffusion equation with square or cubic nonlinear losses by obtaining the average and variance of the solution process. It has the advantage to overcome the difficulties arising from the Homotopy perturbation method (HPM). In fact, The HPM may lead to divergence because the rate of convergence of the HPM method depends greatly on the initial approximation which is considered as the main disadvantage of the HPM. The HAM WHEP contains the auxiliary parameter $\hbar$, which provides us with a convenient way to adjust and control the convergence region of the series solution. All test problems reveal the accuracy and convergence of the suggested method.

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