Relaxed Two-Coloring of Cubic Graphs

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We show that any graph of maximum degree at most 3 has a two-coloring, such that one color-class is an independent set while the other color induces monochromatic components of order at most 189. On the other hand for any constant C we exhibit a 4-regular graph, such that the deletion of any independent set leaves at least one component of order greater than C. Similar results are obtained for coloring graphs of given maximum degree with $k + \ell$ colors such that k parts form an independent set and ℓ parts span components of order bounded by a constant. A lot of interesting questions remain open.

Keywords: Vertex coloring, bounded size components

1 Introduction

In this paper we consider a relaxation of proper coloring by allowing "errors" of certain controlled kind. We say that a coloring of a graph is C*-relaxed* if all monochromatic components have order at most C. With this definition, 1-relaxed is equivalent to proper coloring. It is easy to see that any graph of maximum degree at most 3 has a 2-relaxed two-coloring. Alon, Ding, Oporowski and Vertigan [1] proved that every graph of maximum degree 4 has a 57-relaxed two-coloring. They also gave a construction of a 6-regular graph for arbitrary C, which does *not* admit a C-relaxed coloring. Haxell, Szabó and Tardos [4] established that even a 6-relaxed two-coloring of graphs of maximum degree 4 is possible and proved that every graph of maximum degree 5 has a C-relaxed two-coloring with some constant C (In fact $C < 20000$).

Earlier work related to relaxed colorings were focusing on special kinds of graphs, like line-graphs of cubic graphs [2, 5]. These works culminated in the result of Thomassen [8], who proved that there exists a two-coloring of the edges of any cubic graph such that not only every monochromatic component is bounded, but is a *path* of length at most five.

2 Two-coloring cubic graphs

In this paper we are concerned about the asymmetric version of the relaxation of proper two-coloring. Namely, we allow larger components in only one of the color classes, the other one has to be an independent set. Obviously, any 2-regular graph has a two-coloring where one of the color-classes is an independent set, and the other induces monochromatic components of order at most 2. Our main theorem claims that a similar statement holds for graphs of maximum degree 3 as well.

Theorem 1. *Let G be a graph of maximum degree at most 3. There exists a partition of the vertex set of G into subsets I and B where I is an independent set and every component of $G[B]$ is of order at most* 189.

We prove Theorem 1 in several steps. Our argument is quite lengthy, here we only give brief synopsis.

2.1 Synopsis of the proof of Theorem 1

After some initial simplification we break the graph G into two pieces: one containing vertices which don't participate in a triangle, the other containing vertices from triangles. We solve our problem separately for each piece, then we finish the proof of Theorem 1 by combining the two-coloring of the two pieces through a series of modifications.

More formally, first we show that Theorem 1 (with a better constant) holds if G is triangle-free.

Theorem 2. *For any triangle-free graph G with $\Delta(G) \leq 3$, there exists a partition of the vertex set into I and B where I is an independent set and no component of $G[B]$ is larger than 6.*

The key point in the proof of this statement is to define an appropriate auxiliary graph and apply the following useful lemma from [4] about *matching transversals*.

Lemma 1. *[4, Corollary 4.3] Let H be a a graph with $\Delta(H) \leq 2$. Suppose that $V(H)$ is partitioned into subsets of size two, $V(H) = W_1 \cup \ldots \cup W_m$, $|W_i| = 2$ for $i = 1, \ldots, m$. Then there exists a "matching transversal", i.e. a subset $T \subseteq V(H)$ of the vertices such that $|W_i \cap T| = 1$ for every $i = 1, \ldots m$ and $\Delta(G[T]) \leq 1$.*

For graphs whose vertex set is the union of vertex disjoint triangles one can also show that Theorem 1 (with a better constant) holds.

Lemma 2. *For any graph G in which every vertex participates in a triangle, and $\Delta(G) \leq 3$, there exists a partition of the vertex set into I and B where I is an independent set and no component of $G[B]$ is larger than 8.*

The proof utilizes the following theorem of Thomassen about certain edge-two-coloring of cubic graphs.

Theorem 3. *[8, Theorem 2.] Let H be a graph of maximum degree at most 3. Then the edge set of H has a red/blue coloring and an orientation of the edges such that*

(i) *each monochromatic component is a directed path of length at most 5, and*

(ii) *each vertex of degree 2 is either an interior vertex of a monochromatic directed path or the endpoint of a monochromatic directed path of length at most 3.*

The third part of the proof of our main theorem, containing the process of combining Theorem 2 and Lemma 2, is quite technical. It starts by taking a "good" two-coloring of the triangle-free part of G (Theorem 2) and the part containing only vertices from triangles. Then we perform a series of small modifications to ensure that each B-component of the "triangle-full" part is joined to at most one B-component of the triangle-free part. In particular we need to use the following strengthened version of Lemma 2, which provides us with the flexibility needed to stick together the two "good" two-colorings. The flexibility is represented by the set X, which can be included in both the "independent" and the "bounded-component" part with some sacrifice in the constant.

Let V_i be the set of vertices of degree i.

Lemma 3. *For any G, which is the vertex disjoint union of triangles, and $\Delta(G) \leq 3$, there exists a partition of the vertex set $V(G)$ into three sets I, B and X, such that*

(i) *$I \subseteq V_3$, $X \subseteq V_2$, $I \cup X$ is an independent set and no component of $G[B \cup X]$ is larger than 21.*

(ii) *every component of $G[B \cup X]$ contains at most three vertices from $B \cap V_2$, all of which are contained in the same triangle. Any component of $G[B \cup X]$ containing exactly one vertex from $B \cap V_2$ is of order at most 8, and any component containing two or three vertices from $B \cap V_2$ is fully contained in a triangle.*

3 4-regular graphs

To complement Theorem 1 we prove that a similar statement cannot hold for 4-regular graphs.

Theorem 4. *For any constant C there exists a 4-regular graph G such that for any independent set $I \subseteq V(G)$, $G[V(G) \setminus I]$ has a component of order larger than C.*

4 More than two colors

We also investigate relaxed colorings of graphs with more than two colors. For this we need the following definition. A graph G is called *C-relaxed (k, ℓ)-colorable* if there exists a C-relaxed $(k+\ell)$-coloring of G such that each of the first k color classes are independent sets. A set of graphs $\mathcal{G}$ is called *(k, ℓ)-colorable* if there exists an absolute constant C, such that every member $G \in \mathcal{G}$ admits a C-relaxed (k, ℓ)-coloring. Obviously, $(k, 0)$-colorability is the same as the usual k-colorability. The main result of [4] could be formulated as the family of 5-regular graphs is $(0, 2)$-colorable. Our main results state that cubic graphs are $(1, 1)$-colorable, while 4-regular graphs are not.

In [4] the maximum degree condition for $(0, k)$-colorability is investigated. We define $\Delta(k, \ell)$ to be the smallest integer Δ such that the family of graphs with maximum degree Δ is *not* (k, ℓ)-colorable. In [4] it is shown that there exists a constant $\delta > 0$, such that for large ℓ, $3 + \delta < \Delta(0, \ell)/\ell < 4$,

Here we give bounds on $\Delta(k, \ell)$ and raise several open questions.

Theorem 5. *Let $\ell > 0$. For any constant C there exists a graph of maximum degree $\Delta = 2(k + 2\ell - 1)$ which is not C-relaxed (k, ℓ)-colorable. That is $\Delta(k, \ell) \leq 2k + 4\ell - 2$.*

Theorem 4 is a special case of Theorem 5 with $k = \ell = 1$. The construction of Alon, Ding, Oporowski and Vertigan [1] is a special case with $k = 0, \ell = 2$.

Theorem 6. *Let k, ℓ be nonnegative integers. The family of graphs of maximum degree at most $k + 3\ell - 1$ is (k, ℓ)-colorable. That is $\Delta(k, \ell) > k + 3\ell - 1$.*

This statement is a consequence of a theorem of [4], a lemma from [7] and our Theorem 1.

5 Open Problems

One would like to know more about the behavior of the function $\Delta(k, \ell)$ in general, or at least tighten the existing asymptotic gap. The following are two important special cases.

Maximum degree condition for $(0, \ell)$-colorability. The main theorem of [4] states that $\Delta(0, 2) = 6$. One of the outstanding questions of the topic is to determine the asymptotics of $\Delta(0, \ell)/\ell$. In [4] it is shown that there exists $\delta > 0$, such that for large ℓ, $3 + \delta < \Delta(0, \ell)/\ell < 4$. It would be of great interest to determine asymptotically $\Delta(0, \ell)$.

Maximum degree condition for $(k,1)$-colorability. Our main result in this paper states that $\Delta(1,1) = 4$. The value of $\Delta(2,1)$ is either 5 or 6. Asymptotically, $\Delta(k,1)$ is between k and $2k$. We conjecture the lower bounds are (closer to) the truth.

Density version. A natural way to weaken the maximum degree condition is by rather bounding the maximum average degree of the graph, which allows a few very large degree vertices.

Let $\mu(G) = \max\{2|E(G[W])|/|W| : W \subseteq V(G)\}$. For non-negative integers k, ℓ what is the supremum value $\alpha(k,\ell)$, such that every graph G with $\mu(G) < \alpha(k,l)$ has a C-relaxed $(k+\ell)$-coloring with some constant C. Obviously $\alpha(k,\ell) \leq \Delta(k,\ell)$. In [4] the determination of $\alpha(0,2)$ was raised. The *wheel* graph shows that $\alpha(0,2) \leq 4$, while Kostochka [6] proved a lower bound of 3. The greedy coloring implies that $\alpha(k,0) = k$, for any k. We would be very much interested interested in the value of $\alpha(1,1)$.

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