Gaussian Interference Channel Capacity to Within One Bit: the Symmetric Case

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Abstract — **The capacity of the two-user Gaussian interference channel has been open for thirty years. The understanding on this problem has been limited. The best known achievable region is due to Han-Kobayashi but its characterization is very complicated. It is also not known how tight the existing outer bounds are. In this work, we show that the existing outer bounds can in fact be arbitrarily loose in some parameter ranges, and by deriving new outer bounds, we show that a simplified Han-Kobayashi type scheme can achieve to within a single bit the capacity for all values of the channel parameters. We also show that the scheme is asymptotically optimal at certain high SNR regimes. Using our results, we provide a natural generalization of the point-to-point classical notion of degrees of freedom to interference-limited scenarios.**

I. Introduction

Interference is a central phenomenon in wireless communication when multiple uncoordinated links share a common communication medium. Most state-of-the-art wireless systems deal with interference in one of two ways:

- orthogonalize the communication links in time or frequency, so that they do not interfere with each other at all;
- allow the communication links to share the same degrees of freedom, but treat each other's interference as adding to the noise floor.

It is clear that both approaches can be sub-optimal. The first approach entails an *a priori* loss of degrees of freedom in both links, no matter how weak the potential interference is. The second approach treats interference as pure noise while it actually carries information and has structure that can potentially be exploited in mitigating its effect.

These considerations lead to the natural question of what is the best performance one can achieve without making any *a priori* assumptions on how the common resource is shared. A basic information theory model to study this question is the two-user Gaussian interference channel, where two point-to-point links with additive white Gaussian noise interfere with each other (Figure 1). The capacity region of this channel is the set of all simultaneously achievable rate pairs (R_1, R_2) in the two interfering links, and characterizes the fundamental tradeoff between the performance achievable in the two links in face of interference. Unfortunately, the problem of characterizing this region has been open for over thirty years. The only case in which the capacity is known is in the *strong* interference case, where each receiver has a better reception of the other user's signal than the intended receiver [1, 2]. The best known strategy for the other cases is due to Han-Kobayashi [1]. This strategy is a natural one and involves splitting the transmitted information of both users into two parts: private information to be decoded only at own receiver and common information that can be decoded at both receivers. By decoding the common information, part of the interference can be cancelled off, with the remaining private information from the other user treated as noise. The Han-Kobayashi strategy allows arbitrary splits of each user's transmit power into the private and common information portions as well as time sharing between multiple such splits. Unfortunately, the optimization among such myriads of possibilities is not well-understood, so while it is clear that it will be no worse than the above-mentioned strategies as it includes them as special cases, it is not very clear how much improvement can be obtained and in which parameter regime would one get significant improvement. More importantly, it is also not clear how close to capacity can such a scheme achieve and whether there will be other strategies that can do significantly better.

We recently made progress on this state of affairs by showing that a very simple Han-Kobayashi type scheme can in fact achieve within 1 bits/s/Hz of the capacity of the channel for *all* values of the channel parameters. That is, for all rate pairs (R_1, R_2) on the boundary of the achievable region, (R_1+1, R_2+1) is not achievable. This result is particularly relevant in the high SNR regime, where the achievable rates are high and in fact grow unbounded as the noise level goes to zero. In fact, in some

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high SNR regimes, we can strengthen our results to show that our scheme is asymptotically optimal. In this conference paper, we will focus on characterizing the symmetric rate of the symmetric interference channel. In a subsequent paper, we will present the general results.

The key feature of the scheme is that the power of the private information of each user should be set such that it is received at the level of the Gaussian noise at the other receiver. In this way, the interference caused by the private information has a small effect on the other link beyond what is already caused by the noise. At the same time, quite a lot of private information can be conveyed in its own link if the direct gain is appreciably larger than the cross gain.

To prove that this scheme is within one bit of optimality, we need good outer bounds on the capacity region of the interference channel. The best-known outer bound is based on giving extra side information to one of the receivers so that it can decode all of the information from the other user (the Z-channel and related bound). It turns out that while this bound is sufficiently tight in some parameter regimes, it can get arbitrarily loose in others. We derive new outer bounds to cover for the other parameters, and show that a very simple Han-Kobayashi type scheme can get within 1 bits/s/Hz of this outer bound for all range of parameters.

The rest of the paper is structured as follows. In Section II, we describe the model. The main results are described in Section III. Using our results, we derive in Section IV a notion of generalized degrees of freedom. We conclude our work in Section V.

II. Model

In this section we describe the model to be used in the rest of this work. We consider a two-user Gaussian interference channel. In this model there are two transmitter-receiver pairs, where each transmitter wants to communicate with its corresponding receiver (cf. Figure 1).

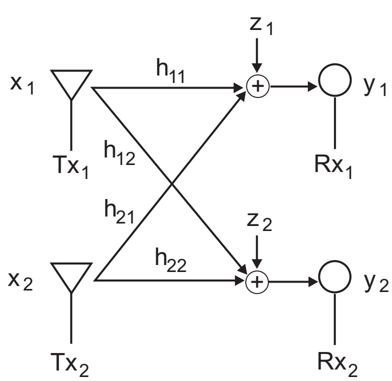

Figure 1: Two-user Gaussian interference channel.

This channel is represented by the equations:

$$\begin{aligned} y_1 &= h_{1,1}x_1 + h_{2,1}x_2 + z_1 \\ y_2 &= h_{1,2}x_1 + h_{2,2}x_2 + z_2 \end{aligned}$$

where for $i = 1, 2$, $x_i \in \mathbb{C}$ is subject to a power constraint P_i, i.e., $E[|x_i|^2] \leq P_i$, and the noise processes $Z_i \sim \mathcal{CN}(0, N_0)$ are i.i.d. over time. For convenience we will denote the power gains of the channels by $g_{i,j} = |h_{i,j}|^2$, $i, j = 1, 2$.

It is easy to see that the capacity region of the interference channel depends only on four parameters: the signal to noise and interference to noise ratios. For $i = 1, 2$, let $\mathsf{SNR}_i = g_{i,i}P_i/N_0$ be the signal to noise ratio of user i, and $\mathsf{INR}_1 = g_{2,1}P_2/N_0$ ($\mathsf{INR}_2 = g_{1,2}P_1/N_0$) be the interference to noise ratio of user 1 (2). As will become apparent from our analysis, this parameterization in terms of SNR and INR is more natural for the interference channel, because it puts in evidence the main factors that determine the channel capacity.

For a given block length n, user i communicates a message $m_i \in \{1, \ldots, 2^{nR_i}\}$ by choosing a codeword from a codebook $\mathcal{C}_{i,n}$, with $|\mathcal{C}_{i,n}| = 2^{nR_i}$. The codewords $\{\mathbf{c}_i(m_i)\}$ of this codebook must satisfy the average power constraint:

$$\frac{1}{n}\sum_{t=1}^{n} |c_i(m_i)[t]|^2 \leq P_i$$

Receiver i observes the channel outputs $\{y_i[t] : t = 1, \ldots, n\}$ and uses a decoding function $f_{i,n} : \mathbb{C}^n \to \mathbb{N}$ to get the estimate $\hat{m}_i$ of the transmitted message m_i. The receiver is in error whenever $\hat{m}_i \neq m_i$. The average probability of error for user i is given by

$$\epsilon_{i,n} = E[P(\hat{m}_i \neq m_i)]$$

where the expectation is taken with respect to the random choice of the transmitted messages m_1 and m_2. Note that due to the interference among users, the probability of error of each user may depend on the codeword transmitted by the other user.

A rate pair (R_1, R_2) is achievable if there exists a family of codebook pairs $\{(\mathcal{C}_{1,n}, \mathcal{C}_{2,n})\}_n$ with codewords satisfying the power constraints P_1 and P_2 respectively, and decoding functions $\{(f_{1,n}(\cdot), f_{1,n}(\cdot)\}_n$, such that the average decoding error probabilities $\epsilon_{1,n}, \epsilon_{2,n}$ go to zero as the block length n goes to infinity.

The capacity region $\mathcal{R}$ of the interference channel is the closure of the set of achievable rate pairs.

III. Symmetric interference channel and symmetric capacity

In order to introduce the main ideas and results in the simplest possible setting, in this paper we focus our analysis of the interference channel capacity region by considering a symmetric interference channel and the symmetric rate point.

In the symmetric interference channel we have $g_{1,1} = g_{2,2} = g_d$, $g_{1,2} = g_{2,1} = g_c$ and $P_1 = P_2 = P$, or equivalently, $\mathsf{SNR}_1 = \mathsf{SNR}_2 = \mathrm{SNR}$ and $\mathsf{INR}_1 = \mathsf{INR}_2 = \mathrm{INR}$. Since the capacity region is known in the strong interference case when $\mathsf{INR}/\mathsf{SNR} \geq 1$, we will focus on the weak interference case where $\mathsf{INR}/\mathsf{SNR} < 1$ (i.e. $g_c/g_d < 1$).

A natural operating point in the symmetric channel is the symmetric rate point (or the symmetric capacity): this is the point in the capacity region that solves the optimization problem:

$$C_{\text{sym}} = \begin{cases} \text{Maximize:} & \min\{R_1, R_2\} \\ \text{Subject to:} & (R_1, R_2) \in \mathcal{R} \end{cases}$$

Due to the convexity and symmetry of the capacity region of the symmetric channel, the symmetric rate point maximizes the sum rate $R_1 + R_2$ and so the symmetric capacity is half the sum capacity of the symmetric channel.

We will use a simple communication scheme that is a special case of the general type of schemes introduced by Han and Kobayashi in [1]. Let us first describe the general Han-Kobayashi setup. For a given block length n user i chooses a private message from codebook $\mathcal{C}^u_{i,n}$ and a common message from codebook $\mathcal{C}^w_{i,n}$. These codebooks satisfy the power constraints P_u and P_w with $P_u + P_w = P$. The sizes of these codebooks are such that $|\mathcal{C}^u_{i,n}| \cdot |\mathcal{C}^w_{i,n}| = 2^{nR_i}$. After selecting the corresponding codewords user i transmits the signal $\mathbf{x}_i = \mathbf{c}^u_i + \mathbf{c}^w_i$ by adding the private and common codewords. The private codewords are meant to be decoded by receiver i, while the common codewords must be decoded by both receivers.

The general Han and Kobayashi scheme allows to generate the codebooks using arbitrary input distributions, and allows to do time sharing between multiple strategies. We will consider a simple scheme where the codebooks are generated by using i.i.d. random samples of a Gaussian $\mathcal{CN}(0, \sigma^2)$ random variable with $\sigma^2 = P_u, P_w$. We also choose P_u such that $g_c P_u = N_0$, i.e. the interference created by the private message has the same power as the Gaussian noise. This is equivalent to choosing $P_u/N_0 = \mathsf{SNR}/\mathsf{INR}$, which is possible only if $\mathsf{INR} > 1$. This we assume for now. In addition, we use a single common-private split, i.e. we do not do time sharing. With this simple Han and Kobayashi scheme we obtain a symmetric rate:

$$\begin{aligned} & R_{HK} \\ &= \min\left\{\frac{1}{2}\log(1+\mathsf{INR}+\mathsf{SNR}) + \frac{1}{2}\log\left(2+\frac{\mathsf{SNR}}{\mathsf{INR}}\right) - 1,\right. \\ &\qquad \left.\log\left(1+\mathsf{INR}+\frac{\mathsf{SNR}}{\mathsf{INR}}\right) - 1\right\} \end{aligned} \tag{1}$$

which can be derived by a straightforward computation.

In order to assess how good our communication scheme performs, we can compare the symmetric rate achieved with an upper bound. We can obtain this upper bound by considering half of any upper bound to the sum capacity of the interference channel. One such upper bound is the sum capacity of the Z-channel [3–5], from which we obtain the following upper bound to the symmetric capacity:

$$R_Z = \frac{1}{2}\log(1+\mathsf{SNR}) + \frac{1}{2}\log\left(1+\frac{\mathsf{SNR}}{1+\mathsf{INR}}\right) \tag{2}$$

Comparing (1) and (2), we can see that the gap between this bound and the first term in (1) is always less than 1 for all values of SNR and INR. Thus, when the first term is the smaller one, our scheme is within 1 bit of the symmetric capacity. On the other hand, there exist values of SNR and INR such that the second term of (1) is arbitrarily smaller than the first term. Indeed, if for example $\mathsf{SNR} \gg 1$ $\mathsf{INR} \gg 1$ and $\mathsf{SNR} \gg \mathsf{INR}^2$, the first term is asymptotically equal to:

$$\log\mathsf{SNR} - \frac{1}{2}\log\mathsf{INR} - 1$$

while the second term is asymptotically equal to:

$$\log\mathsf{SNR} - \log\mathsf{INR} - 1$$

and we see that the gap goes to infinity as INR goes to infinity. In this parameter regime at least, the gap between the achievable rate of our scheme and the Z-channel upper bound is arbitrarily large.

Remark 1. *A slightly better outer bound on the symmetric rate is given in [6]. However, we can show that this bound has a very similar characteristic to the Z-channel bound: it is also within one bit of the achievable rate of our scheme in the first parameter range, and unbounded in the second range.*

This large gap can be due to a very suboptimal scheme, a loose upper bound, or both. It turns out that the large gap is due to the looseness of previous known upper bounds. In order to prove this we need to derive a new upper bound for the sum rate of the interference channel.

Note that we can think of Z-channel as a genie giving one receiver the other transmitter's message in the original interference channel. In order to derive another bound we will make use of the help of another kind of genie. Define

$$\begin{aligned} s_1 &= h_{1,2}x_1 + z_2 \\ s_2 &= h_{2,1}x_2 + z_1 \end{aligned}$$

and consider the genie-aided channel where a genie provides s_1 to receiver 1 and s_2 to receiver 2 (see Figure 2).

In contrast to the Z-channel bound or the bound derived in [6], the information this genie provides does not allow either receiver to completely decode the message of the interfering user. This genie-aided channel is another interference channel with two outputs per user. The usefulness of this genie-aided interference channel is that its capacity can in fact be explicitly computed. This computable capacity then yields an new upper bound on the symmetric rate of the original interference channel, as given in the following theorem.

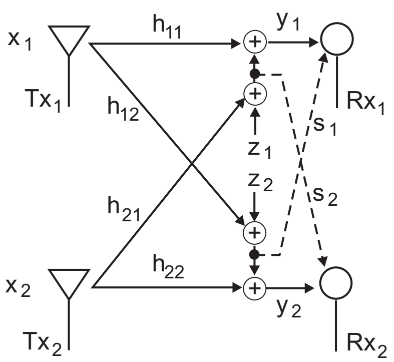

Figure 2: Genie-aided two-user Gaussian interference channel. A genie provides signals s_1 to receiver 1 and s_2 to receiver 2.

Theorem 1. *The symmetric capacity of the symmetric Gaussian interference channel as defined in Section II, equation (1) is upper bounded by*

$$R_{NEW} = \log\left(1 + \mathsf{INR} + \frac{\mathsf{SNR}}{1+\mathsf{INR}}\right) \tag{3}$$

Combine (2) and (3), we have the following upper bound on symmetric rate:

$$R_{UPPER} = \min\left\{\frac{1}{2}\log\left(1+\mathsf{SNR}\right) + \frac{1}{2}\log\left(1+\frac{\mathsf{SNR}}{1+\mathsf{INR}}\right),\right.$$
$$\left.\log\left(1+\mathsf{INR}+\frac{\mathsf{SNR}}{1+\mathsf{INR}}\right)\right\} \tag{4}$$

Comparing (1) and (4), we find that

$$R_{UPPER} - R_{HK} < 1 \tag{5}$$

for all range of parameters SNR and INR < 1.

If $\mathsf{INR} < 1$, we cannot let $P_u/N_0 = \mathsf{SNR}/\mathsf{INR}$. In this case, by giving all power to the private message, i.e., treating interference as noise, we can also get to within one bit of the symmetric capacity. Thus we have the following theorem.

Theorem 2. *We can achieve to within one bit of the symmetric capacity for symmetric Gaussian interference channel by using a simple Han-Kobayashi scheme with* $P_u/N_0 = \min(\mathsf{SNR}/\mathsf{INR}, \mathsf{SNR})$*, i.e., the interference to noise ration of the private message is as close to 1 as possible.*

IV. Generalized Degrees of Freedom

At high SNR, it is well known that the capacity of a point-to-point AWGN link, in bits/s/Hz, is approximately:

$$C_{\text{awgn}} \approx \log \mathsf{SNR} \tag{6}$$

The approximation is in the sense that for $\mathsf{SNR} > 0$dB, the approximation error is within 1 bit. Using our results, we can derive analogous approximations of the interference-channel capacity. The symmetric capacity is approximately:

$$C_{\text{sym}} \approx \begin{cases} \log(\frac{\mathsf{SNR}}{\mathsf{INR}}) & \log \mathsf{INR} < \frac{1}{2}\log \mathsf{SNR} \\ \log \mathsf{INR} & \frac{1}{2}\log \mathsf{SNR} < \log \mathsf{INR} < \frac{2}{3}\log \mathsf{SNR} \\ \log \frac{\mathsf{SNR}}{\sqrt{\mathsf{INR}}} & \frac{2}{3}\log \mathsf{SNR} < \log \mathsf{INR} < \log \mathsf{SNR} \\ \log \sqrt{\mathsf{INR}} & \log \mathsf{SNR} < \log \mathsf{INR} < 2\log \mathsf{SNR} \\ \log \mathsf{SNR} & \log \mathsf{INR} > 2\log \mathsf{SNR} \end{cases} \tag{7}$$

Note that there are five regimes in which the qualitative behaviors of the capacity are different.

The fifth regime is the *very strong interference* regime [2]. Here the interference is so strong that each receiver can decode the other transmitter's information, treating its own signal as noise, before decoding its own information. Thus, interference has no impact on the performance of the other link. The fourth regime is the *strong interference* regime, where the optimal strategy is for both receivers to decode entirely each other's signal, i.e. all the transmitted information is common information [1]. Here, the capacity increases monotonically with INR because increasing INR increases the common information rate.

The capacity in the fourth and fifth regimes follow from previous results. The first three regimes fall into the *weak interference* regime, and the capacity in these regimes is a consequence of the new results we obtain. In these regimes, the interference is not strong enough to be decoded in its entirety. In fact, regime 1 says that if the interference is very weak, then treating interference as noise is optimal. Regime 2 and 3 however say that if the interference is not very weak, decoding it partially can significantly improve performance. Interesting, the capacity is *not* monotonically decreasing with INR in the weak interference regime.

In point-to-point links, the notion of *degrees of freedom* is a fundamental measure of channel resources. It tells us how many signal dimensions are available for communication. In the (scalar) AWGN channel, there is one degree of freedom per second per Hz. When multiple links share the communication medium, one can think of the mutual interference as reducing the available degrees of freedom for useful communication. Our results quantify this reduction. Define

$$\alpha := \frac{\log \mathsf{INR}}{\log \mathsf{SNR}} \tag{8}$$

as the ratio of the interference-to-noise ratio and the signal-to-noise ratio in dB scale, and

$$d_{\text{sym}} := \frac{C_{\text{sym}}}{C_{\text{awgn}}} \tag{9}$$

as the *generalized* degrees of freedom per user, then (7) yields the following characterization:

$$d_{\text{sym}} = \begin{cases} 1-\alpha & 0 \le \alpha \le \frac{1}{2} \\ \alpha & \frac{1}{2} \le \alpha \le \frac{2}{3} \\ 1-\frac{\alpha}{2} & \frac{2}{3} \le \alpha \le 1 \\ \frac{\alpha}{2} & 1 \le \alpha \le 2 \\ 1 & \alpha \ge 2. \end{cases} \tag{10}$$

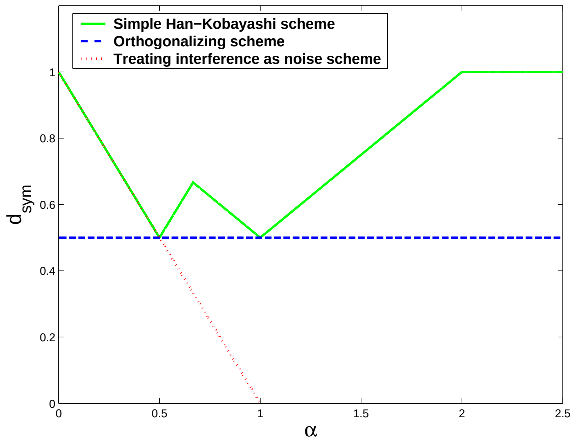

Figure 3: Generalized degrees of freedom per user using different schemes

This is plotted in Figure 3, together with the performance of our baseline strategies of orthogonalizing and treating interference as noise. Note that orthogonalizing between the links, in which each link achieves half the degrees of freedom, is strictly sub-optimal except when $\alpha = \frac{1}{2}$ and $\alpha = 1$. Treating interference as noise, on the other hand, is strictly sub-optimal except for $\alpha \le \frac{1}{2}$. Note also the fundamental importance of comparing the signal-to-noise and the interference-to-noise ratios in dB scale. This is the parameter that significantly impacts performance in the interference channel. The relevance of this in wireless communication is accentuated by the fact that the dB scale is a natural one to measure SNR's and INR's as wireless channel gains have a very large dynamic range.

The approximation $\log \mathsf{SNR}$ of the AWGN capacity is not only within 1 bit, but is also asymptotically tight:

$$\lim_{\mathsf{SNR} \to \infty} [C_{\text{awgn}} - \log \text{SNR}] = 0.$$

We have analogous results in the interference channel case. If we fix α and letting SNR and INR go to infinity, we have the following asymptotic tightness result of the upper bound (4) for some ranges of α.

Theorem 3. *For $0 < \alpha < 1/2$ and $1/2 < \alpha < 2/3$, the upper bound in (4) is asymptotically tight in the sense that the difference between C_{sym} and the upper bound goes to zero as $\mathsf{SNR}, \mathsf{INR}$ go to infinity with α fixed.*

Note that for $\alpha < 1$, the asymptotic gap between the achievable rate of the scheme presented earlier and the upper bound (4) is in fact exactly 1. For $0 < \alpha < 1/2$ and $1/2 < \alpha < 2/3$, we were able to modify the scheme slightly to remove the 1-bit gap asymptotically.

V. Conclusion

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In this paper we present our results on the symmetric capacity of the symmetric Gaussian interference channel. We derive a new outer bound for the symmetric capacity and show that a simple choice of a Han-Kobayashi scheme can get to within one bit of the capacity region. Using these results, we derive simple approximations to the symmetric capacity in the high SNR, INR regime. Using these approximations, we generalize the notion of degrees of freedom from the point-to-point to the interference-limited scenario. We can also show that the approximation is asymptotically tight for some range of the parameters.

References

[1] T. S. Han and K. Kobayashi, *A new achievable rate region for the interference channel,* IEEE Trans. Info. Theory, vol. IT-27, pp. 49 - 60, January 1981.

[2] A. B. Carleial, *A case where interference does not reduce capacity,* (Corresp.), IEEE Trans. Info. Theory, vol. IT-21, pp. 569 - 570, September 1975.

[3] M. H. M. Costa, *On the Gaussian interference channel,* IEEE Trans. Info. Theory, vol. IT-31, pp. 607 - 615, September 1985.

[4] H. Sato, *On degraded Gaussian two-user channels,* (Corresp.), IEEE Trans. Info. Theory, vol. IT-24, pp. 638 - 640, September 1978.

[5] I. Sason, *On achievable rate regions for the Gaussian interference channel,* (Corresp.), IEEE Trans. Info. Theory, vol. IT-50, pp. 1345 - 1356, June 2004.

[6] G. Kramer, *Outer bounds on the capacity of Gaussian interference channels,* (Corresp.), IEEE Trans. Info. Theory, vol. IT-50, pp. 581 - 586, March 2004.