Geometry Teacher's Edition - Common Errors

CK-12 Foundation

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Chapter 1

Geometry TE - Common Errors

1.1 Basics of Geometry

Points, Lines, and Planes

Naming Lines – Students often want to use all the labeled points on a line in its name, especially if there are exactly three points labeled. Tell them they get to pick two, any two, to use in the name. This means there are often many possible correct names for a single line.

Key Exercise: How many different names can be written for a line that has four labeled points?

Answer: 12

Student can get to this answer by listing all the combination of two letters. Recommend that they make the list in an orderly way so they do not leave out any possibilities. This exercise is good practice for counting techniques learned in probability.

Naming Rays – There is so much freedom in naming lines, that students often struggle with the precise way in which rays must be named. They often think that the direction the ray is pointing needs to be taken into consideration. The arrow "hat" always points to the right. The "hat" only indicates that the geometric object is a ray, not the ray's orientation in space. The first letter in the name of the ray is the endpoint; it does not matter if that point comes first or second when reading from left to right on the figure. It is helpful to think of the name of a ray as a starting point and direction. There is only one possible starting point, but often several points that can indicate direction. Any point on the ray other than the endpoint can be the second point in the name.

There is only one point B – English is an ambiguous language. Two people can have the same name; one word can have two separate meanings. Math is also a language, but is different from other languages in that there can be no ambiguity. In a particular figure there can be only one point labeled B.

Key Exercise: Draw a figure in which $\overleftrightarrow{AB}$ intersects $\overline{AC}$.

Answer: There are many different ways this can be drawn. There must be a line with the points A and B, and a segment with one endpoint at A and the other endpoint C could be at any location.

Segments and Distance

Number or Object – The measure of a segment is a number that can be added, subtracted and combine arithmetically with other numbers. The segment itself is an object to which postulates and theorems can be applied. Using the correct notation may not seem important to the students, but is a good habit that will work to their benefit as they progress in their study of mathematics. For example, in calculus whether a variable represents a scalar or a vector is critical. When clear notation is used, the mind is free to think about the mathematics.

Using a Ruler – Many Geometry students need to be taught how to use a ruler. The problems stems from students not truly understanding fractions and decimals. This is a good practical application and an important life skill.

Measuring in centimeters will be learned quickly. Give a brief explanation of how centimeters and millimeters are marked on the ruler. Since a millimeter is a tenth of a centimeter, both fractions and decimals of centimeters are easily written.

Using inches is frequently challenging for students because so many still struggle with fractions. Some may need to be shown how an inch is divided using marks for $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, and $\frac{1}{16}$. These fractions often need to be added and reduced to get a measurement in inches.

Review the Coordinate Plane – Some students will have forgotten how to graph an ordered pair on the coordinate plane, or will get the words vertical and horizontal confused. A reminder that the $x-$coordinate is first, and measures horizontal distance from the origin, and that the $y-$coordinate is second and measures vertical distance from the origin will be helpful. The coordinates are listed in alphabetical order.

Additional Exercises:

1. Points A, B, and C are collinear, with B located between A and C.

$AB = 12$ cm and $AC = 20$ cm. What is BC?

(Hint: Draw and label a picture.)

Answer: 20 cm − 12 cm = 8 cm

Drawing a picture is extremely helpful when solving Geometry problems. It is good to get the students in this habit early. The process of going form a description to a picture also helps them review their vocabulary.

Rays and Angles

Naming Angles with Three Points – Naming, and identifying angles named with three points is often challenging for students when they first learn it. The middle letter of the angle name, the vertex of the angle, is the most important point. Instruct the students to start by identifying this point and working from there. With practice students will become adept at seeing and naming different angles is a complex picture. Review of this concept is also important. Every few months give the students a problem that requires using this important skill.

Using a Protractor – The two sets of numbers on a protractor are convenient for measuring angles oriented in many different directions, but often lead to errors on the part of the students. There is a simple way for students to check their work when measuring an angle with a protractor. Visual inspection of an angle usually can be used to tell if an angle is acute or obtuse. After the measurement is taken, students should notice if their answer matches with the classification.

Additional Exercises:

1. *True or False:* A ray can have a measure

Answer: False. A ray extends infinitely on one direction, so it does not have a length.

2. $< ABC$ has a measure of 100 degrees. Point D is located in the interior of $< ABC$ and $< ABD$ has a measure of 30 degrees. What is the measure of $< DBC$?

(Hint: Draw and label a picture.)

Answer: 100 degrees − 30 degrees = 70 degrees

3. $< XYZ$ has a measure of 45 degrees and $< ZYW$ has a measure of 75 degrees. What is the measure of $< XYW$?

(Hint: Draw and label a picture.)

Answer: 45 degrees + 75 degrees = 120 degrees

Segments and Angles

Congruent or Equal – Frequently students interchange the words congruent and equal. Stress that equal is a word that describes two numbers, and congruent is a word that describes two geometric objects. Equality of measure is often one of the conditions for congruence. If the students have been correctly using the naming conventions for a segment and its measure and an angle and its measure in previous lessons they will be less likely to confuse the words congruent and equal now.

The Number of Tick Marks or Arcs Does Not Give Relative Length – A common misconception is that a pair of segments marked with one tick, are longer than a pair of segments marked with two ticks in the same figure. Clarify that the number of ticks just groups the segments; it does not give any relationship in measure between the groups. An analogous problem occurs for angles.

Midpoint or Bisector – Midpoint is a location, a noun, and bisect is an action, a verb. One geometric object can bisect another by passing through its midpoint. This link to English grammar often helps students differentiate between these similar terms.

Intersects vs. Bisects – Many students replace the word intersects with bisects. Remind the students that if a segment or angle is bisected it is intersected, and it is know that the intersection takes place at the exact middle.

Orientation Does Not Affect Congruence – The only stipulation for segments or angles to be congruent is that they have the same measure. How they are twisted or turned on the page does not matter. This becomes more important when considering congruent polygons later, so it is worth making a point of now.

Labeling a Bisector or Midpoint – Creating a well-labeled picture is an important step in solving many Geometry problems. How to label a midpoint or a bisector is not obvious to many students. It is often best to explicitly explain that in these situations, one marks the congruent segments or angles created by the bisector.

Additional Exercises:

1. Does it make sense for a line to have a midpoint?

Answer: No, a line is infinite in one dimension, so there is not a distinct middle.

Angle Pairs

Complementary or Supplementary – The quantity of vocabulary in Geometry is frequently challenging for students. It is common for students to interchange the words complementary and supplementary. A good mnemonic device for these words is that they, like many math words, go in alphabetical order; the smaller one, complementary, comes first.

Linear Pair and Supplementary – All linear pairs have supplementary angles, but not all supplementary angles form linear pairs. Understanding how Geometry terms are related helps students remember them.

Angles formed by Two Intersection Lines – Students frequently have to determine the measures of the four angles formed by intersecting lines. They can check their results quickly when they realize that there will always be two sets of congruent angles, and that angles that are not congruent must be supplementary. They can also check that all four angles measures have a sum of 360 degrees.

Write on the Picture – In a complex picture that contains many angle measures which need to be found, students should write angle measures on the figure as they find them. Once they know an angle they can use it to find other angles. This may require them to draw or trace the picture on their paper. It is worth taking the time to do this. The act of drawing the picture will help them gain a deeper understanding of the angle relationships.

Proofs – The word proof strikes fear into the heart of many Geometry students. It is important to define what a mathematical proof is, and let the students know what is expected of them regarding each proof.

Definition: A mathematical proof is a mathematical argument that begins with a truth and proceeds by logical steps to a conclusion which then must be true.

The students' responsibilities regarding each proof depend on the proof, the ability level of the students, and where in the course the proof occurs. Some options are (1) The student should understand the logical progression of the steps in the proof. (2) The student should be able to reproduce the proof. (3) The student should be able to create proofs using similar arguments.

Classifying Triangles

Vocabulary Overload – Students frequently interchange the words isosceles and scalene. This would be a good time to make flashcards. Each flashcard should have the definition in words and a marked and labeled figure. Just making the flashcards will help the students organize the material in their brains. The flashcards can also be arranged and grouped physically to help students remember the words and how they are related. For example, have the students separate out all the flashcards that describe angles. The cards could also be arranged in a tree diagram to show subsets, for instance equilateral would go under isosceles, and all the triangle words would go under the triangle card.

Angle or Triangle – Both angles and triangles can be named with three letters. The symbol in front of the letters determines which object is being referred to. Remind the students that the language of Geometry is extremely precise and little changes can make a big difference.

Acute Triangles need all Three – A student may see one acute angle in a triangle and immediately classify it as an acute triangle. Remind the students that unlike the classifications of right and obtuse, for a triangle to be acute all three angles must be acute.

Equilateral Subset of Isosceles – In many instances one term is a subset of another term. A Venn diagram is a good way to illustrate this relationship. Having the students practice with this simple instance of subsets will make it easier for the students to understand the more complex situation when classifying quadrilaterals.

Additional Exercises:

1. Draw and mark an isosceles right and an isosceles obtuse triangle.

Answer: The congruent sides of the triangles must be the sides of the right or obtuse angle.

This exercise lays the groundwork for studying the relationship between the sides and angles of a triangle in later chapters. It is important that students take the time to use a straightedge and mark the picture. Using and reading the tick marks correctly helps the students think more clearly about the concepts.

Classifying Polygons

Vocab, Vocab, Vocab – If the students do not know the vocabulary well, they will have no chance at leaning the concepts and doing the exercises. Remind them that the first step is to memorize the vocabulary. This will take considerable effort and time. The student edition gives a good mnemonic device for remembering the word concave. Ask the students to create tricks to memorize other words and have them share their ideas.

Side or Diagonal – A side of a polygon is formed by a segment connecting consecutive vertices, and a diagonal connects nonconsecutive vertices. This distinction is important when student are working out the pattern between the number of sides and the number of vertices of a polygon.

Squaring in the Distance Formula – After subtracting in the distance formula, students will often need to square a negative number. Remind them that the square of a negative number is a positive number. After the squaring step there should be no negatives or subtraction. If they have a negative in the square root, they have made a mistake.

Additional Exercises:

1. Find the length of each side of the triangle with the following vertices with the distance formula. Then classify each triangle by its sides.

a) $(3,1),(3,5)$, and $(10,3)$

Answer: The triangle is isosceles with side lengths: $4, \sqrt{51}$, and $\sqrt{51}$.

b) $(-3,2),(-8,3)$, and $(-3,-5)$

Answer: This triangle is scalene with side lengths: $7, \sqrt{26}$, and $\sqrt{89}$.

Sometimes students have trouble seeing that they need to take the points two at a time to find the side lengths. By graphing the triangle on graph paper before using the distance formula they can see how to find the side lengths. If they graph the triangle they can also classify it by its angles.

Problem Solving in Geometry

Don't Panic – Problem solving and applications are particularly challenging for many students. Sometimes they just give up. Let the students know that this is difficult. They are probably going to struggle, have to reread the information several time, and will be confused for a while. It is all part of the process. This section will give them strategies to work through the difficulties.

Highlight Important Information – It is nice when students can actually mark up the text of the exercise, but frequently this is not the case. As they read the paragraph have the students take notes or organize the information into a chart. Otherwise the students can just get lost in all the words. Translating from English to math is often the hardest part.

The Last Sentence – When the students are faced with a sizable paragraph of information the most important sentence, the one that asks the question, is usually at the end. Advise the students to read the last sentence first, then as they read the rest of the paragraph they will see how the information they are being given is important.

Does This Make Sense? – It is so hard to get the students to ask themselves this question at the end of a word problem or application. I think they are so happy to have an answer they do not want to know if it is wrong. Keep reminding them. Sometimes it is possible to not accept work with an obviously wrong answer. The paper can be returned to the student so they can look for their mistake. This is a good argument for the importance of showing clear, organized work.

Naming Quadrilaterals – When naming a quadrilateral the letter representing the vertices will be listed in a clockwise or counterclockwise rotation starting from any vertex. Students are accustomed to reading from left to right and will sometimes continue this pattern when naming a quadrilateral.

The Pythagorean Theorem – Most students have learned to use the Pythagorean Theorem before Geometry class and will want to use it instead of the distance formula. They are closely related; the distance formula is derived from the Pythagorean Theorem as will be explained in another chapter. If they are allowed to use the Pythagorean Theorem remind them that it can only be used for right triangles, and that the length of the longest side of the right triangle, the hypotenuse, must be substituted into the "c" variable if it is know. If the hypotenuse is the side of the triangle being found, the "c" stays a variable, and the other two side are substituted for "a" and "b".

1.2 Reasoning and Proof

Inductive Reasoning

The nth Term – Students enjoy using inductive reasoning to find missing terms in a pattern. They are good at finding the next term, or the tenth term, but have trouble finding a generic term or rule for the number sequence. If the sequence is linear (the difference between terms is constant), they can use methods they learned in Algebra for writing the equation of a line.

Key Exercise: Find a rule for the nth term in the following sequence.

$$13, 9, 5, 1, \ldots.$$

Answer: The sequence is linear, each term decreases by 4. The first term is 13, so the point $(1, 13)$ can be used. The second term is 9, so the point $(2, 9)$ can be used. Applying what they know from Algebra I, the slope of the line is -4, and the $y-$intercept is 17, so the rule is $-4n + 17$.

True Means <u>Always</u> True – In mathematics a statement is said to be true if it is always true, no exceptions. Sometimes students will think that a statement only has to hold once, or a few times to be considered true. Explain to them that just one counterexample makes a statement false, even if there are a thousand cases where the statement holds. Truth is a hard criterion to meet.

Sequences – A list of numbers is called a sequence. If the students are doing well with the number of vocabulary words in the class, the term sequence can be introduced.

Additional Exercises:

1. What is the next number in the following number pattern? $1, 1, 2, 3, 5, 8, 13, \ldots$

Answer: This is the famous Fibonacci sequence. The next term in the sequence is the sum of the previous two terms.

$$8 + 13 = 21$$

2. What is the missing number in the following number pattern? $25, 18, ?, 10, 9, \ldots$

Answer: Descending consecutive odd integers are being subtracted from each term, so the missing number is 13.

Conditional Statements

The Advantages and Disadvantages of Non-Math Examples – When first working with conditional statements, using examples outside of mathematics can be very helpful for the students. Statements about the students' daily lives can be easily broken down into parts and evaluated for veracity. This gives the students a chance to work with the logic, without having to use any mathematical knowledge. The problem is that there is almost always some crazy exception or grey area that students will love to point out. This is a good time to remind students of how much more precise math is compared to our daily language. Ask the students to look for the idea of what you are saying in the non-math examples, and use their powerful minds to critically evaluate the math examples that will follow.

Converse and Contrapositive – The most important variations of a conditional statement are the converse and the contrapositive. Unfortunately, these two sound similar, and students often confuse them. Emphasize the converse and contrapositive in this lesson. Ask the students to compare and contrast them.

Converse and Biconditional – The converse of a true statement is not necessarily true! The important concept of implication is prevalent in Geometry and all of mathematics. It takes some time for students to completely understand the direction of the implication. Daily life examples where the converse is obviously not true is a good place to start. The students will spend considerable time deciding what theorems have true converses (are bicondtional) in subsequent lessons.

Key Exercise: What is the converse of the following statement? Is the converse of this true statement also true?

If it is raining, there are clouds in the sky.

Answer: The converse is: If there are clouds in the sky, it is raining. This statement is obviously false.

Practice, Practice, Practice – Students are going to need a lot of practice working with conditional statements. It is fun to have the students write and share conditional statements that meet certain conditions. For example, have them write a statement that is true, but that has an inverse that is false. There will be some creative, funny answers that will help all the members of the class remember the material.

Deductive Reasoning

Inductive or Deductive Reasoning – Students frequently struggle with the uses of inductive and deductive reasoning. With a little work and practice they can memorize the definition and see which form of reasoning is being used in a particular example. It is harder for them to see the strengths and weaknesses of each type of thinking, and understand how inductive and deductive reasoning work together to form conclusions.

Recognizing Reasoning in Action – Use situations that the students are familiar with where either inductive or deductive reasoning is being used to familiarize them with the different types of logic. The side by side comparison of the two types of thinking will cement the students' understanding of the concepts. It would also be beneficial to have the students write their own examples.

Key Exercise: Is inductive or deductive reasoning being used in the following paragraph? Why did you come to this conclusion?

1. The rules of Checkers state that a piece will be crowned when it reaches the last row of the opponent's side of the board. Susan jumped Tony's piece and landed in the last row, so Tony put a crown on her piece.

Answer: This is an example of detachment, a form of deductive reasoning. The conclusion follows from an agreed upon rule.

2. For the last three days a boy has walked by Ana's house at 5 pm with a cute puppy. Today Ana decides to take her little sister outside at 5 pm to show her the dog.

Answer: Ana used inductive reasoning. She is assuming that the pattern she observed will continue.

Which is Better? – Students quickly conclude that inductive reasoning is much easier, but often miss that deductive reasoning is more sure and frequently provides some insight into the answer of that important question, "Why?".

Additional Exercises:

1. What went wrong in this example of inductive reasoning?

Teresa learned in class that John Glenn (the first American to orbit Earth) had to eat out of squeeze tubes, and her mom says the food served in airplanes is not very good. She just had a yummy pizza for lunch. She sees a pattern. Food gets better as one approaches the center of the earth. Therefore the food in a submarine must be delicious!

Answer: She carried the pattern too far.

Algebraic Properties

Commutative or Associate – Students sometimes have trouble distinguishing between the commutative and associate properties. It may help to put these properties into words. The associate property is about the order in which multiple operations are done. The commutative is about the first and second operand having different roles in the operation. In subtraction the first operand is the starting amount and the second is the amount of change. Often student will just look for parenthesis; if the statement has parenthesis they will choose associate, and they will usually be correct. Expose them to an exercise like the one below to help break them of this habit.

Key Exercise: What property of addition is demonstrated in the following statement?

$$(x + y) + z = z + (x + y)$$

Answer: It is the commutative property that ensures these two quantities are equal. On the left-hand side of the equation the first operand is the sum of x and y, and on the right-hand side of the equation the sum if x and y is the second operand.

Transitive or Substitution – The transitive property is actually a special case of the substitution property. The transitive property has the additional requirement that the first statement ends with the same number or object with which the second statement begins. Acknowledging this to the students helps avoid confusion, and will help them see how the properties fit together.

Key Exercise: The following statement is true due to the substitution property of equality. How can the statement be changed so that the transitive property of equality would also ensure the statement's validity?

If $ab = cd$, and $ab = f$, then $cd = f$.

Answer: The equality $ab = cd$ can be changed to $cd = ab$ due to the symmetric property of equality. Then the statement would read:

If $cd = ab$, and $ab = f$, then $cd = f$.

This is justified by the transitive property of equality.

Diagrams

Keeping It All Straight – At this point in the class the students have been introduced to an incredible amount of material that they will need to use in proofs. Laying out a logic argument in proof form is, at first, a hard task. Searching their memories for terms at the same time makes it near impossible for many students. A notebook that serves as a "tool cabinet" full of the definitions, properties, postulates, and later theorems that they will need, will free the students' minds to concentrate on the logic of the proof. After the students have gained some experience, they will no longer need to refer to their notebook. The act of making the book itself will help the students collect and organize the material in their heads. It is their collection; every time they learn something new, they can add to it.

All Those Symbols – In the back of many math books there is a page that lists all of the symbols and their meanings. The use of symbols is not always consistent between texts and instructors. Students should know this in case they refer to other materials. It is a good idea for students to keep a page in their notebooks where they list symbols, and their agreed upon meanings, as they learn them in class. Some of the symbols they should know at this point in are the ones for equal, congruent, angle, triangle, perpendicular, and parallel.

Don't Assume Congruence! – When looking at a figure students have a hard time adjusting to the idea that even if two segments or angles look congruent they cannot be assumed to be congruent unless they are marked. A triangle is not isosceles unless at least two of the sides are marked congruent, no matter how much it looks like an isosceles triangle. Maybe one side is a millimeter longer, but the picture is too small to show the difference. Congruent means exactly the same. It is helpful to remind the students that they are learning a new, extremely precise language. In geometry congruence must be communicated with the proper marks if it is known to exist.

Communicate with Figures – A good way to have the students practice communicating by drawing and marking figures is with a small group activity. One person in a group of two or three draws and marks a figure, and then the other members of the group tell the artist what if anything is congruent, perpendicular, parallel, intersecting, and so on. They take turns drawing and interpreting. Have them use as much vocabulary as possible in their descriptions of the figures.

Two-Column Proofs

Diagram and Plan – Students frequently want to skip over the diagramming and planning stage of writing a proof. They think it is a waste of time because it is not part of the end result. Diagramming and marking the given information enables the writer of the proof to think and plan. It is analogous to making an outline before writing an essay. It is possible that the student will be able to muddle through without a diagram, but in the end it will probably have taken longer, and the proof will not be written as clearly or beautifully as it could have been if a diagram and some thinking time had been used. Inform students that as proofs get more complicated, mathematicians pride themselves in writing simple, clear, and elegant proofs. They want to make an argument that undeniably true.

Teacher Encouragement – When talking about proofs and demonstrating the writing of proofs in class, take time to make a well-drawn, well-marked diagram. After the diagram is complete, pause, pretend like you are considering the situation, and ask students for ideas of how they would go about writing this proof.

Assign exercises where students only have to draw and mark a diagram. Use a proof that is beyond their ability at this point in the class and just make the diagram the assignment.

When grading proofs, use a rubric that assigns a certain number of points to the diagram. The diagram should be almost as important as the proof itself.

Start with "Given", but Don't End With "Prove" – After a student divides the statement to be proved into a given and prove statements he or she will enjoy writing the givens into the proof. It is like a free start. Sometimes they get a little carried away with this and when they get to the end of the proof write "prove" for the last reason. Remind them that the last step has to have a definition, postulate, property, or theorem to show why it follows from the previous steps.

Scaffolding – Proofs are challenging for many students. Many students have a hard time reading proofs. They are just not used to this kind of writing; it is very specialized, like a poem. One strategy for making students accustom to the form of the proof is to give them incomplete proofs and have them fill in the missing statements and reason. There should be a progression where each proof has less already written in, and before they know it, they will be writing proofs by themselves.

Segment and Angle Congruence Theorems

Number or Geometric Object – The difference between equality of numbers and congruence of geometric objects was addressed earlier in the class. Before starting this lesson, a short review of this distinction to remind students is worthwhile. If the difference between equality and congruence is not clear in students' heads, the proofs in this section will seem pointless to them.

Follow the Pattern – Congruence proofs are a good place for the new proof writer to begin because they are fairly formulaic. Students who are struggling with proofs can get some practice with this style of writing while already knowing the structure of the proof.

1^{st} State the "if" side in congruence form.

2^{nd} Change the congruence of segments into equality of numbers.

3^{rd} Apply the analogous property of equality.

4^{th} Change the equality of numbers back to congruence of segments.

Theorems – The concept of a theorem and how it differs from a postulate has been briefly addressed several times in the course, but this is the first time theorems have been the focus of the section. Now would be a good time for students to start a theorem section in their notebook. As they prove, or read a proof of each theorem it can be added to the notebook to be used in other proofs.

Additional Exercises:

1. Prove the following statement.

If $AB = AC$, triangle ABC is isosceles.

Answer:

Table 1.1

Statement	Reason
$AB = AC$	Given
$\overline{AB} \cong \overline{AC}$	Definition of congruent segments.
Triangle ABC is isosceles.	Definition of isosceles triangle.

Proofs About Angle Pairs

Mark-Up That Picture – Angles are sometimes hard to see in a complex picture because they are not really written on the page; they are the amount of rotation between two rays that are directly written on the page. It is helpful for students to copy diagram onto their papers and mark all the angles of interest. They can use highlighters and different colored pens and pencils. Each pair of vertical angles or linear pairs can be marked in a different color. Using colors is fun, and gives the students the opportunity to really analyze the angle relationships.

Add New Information to the Diagram – It is common in geometry to have multiple questions about the same diagram. The questions build on each other leading the student though a difficult exercise. As new information is found it should be added to the diagram so that it is readily available to use in answering the next question.

Try a Numerical Example – Sometimes students have trouble understanding a theorem because they get lost in all the symbols and abstraction. When this happens, advise the students to assign a plausible number to the measures of the angles in question and work form there to understand the relationships. Make sure the student understands that this does not prove anything. When numbers are assigned, they are looking at an example, using inductive reasoning to get a better understanding of the situation. The abstract reasoning of deductive reasoning must be used to write a proof.

Inductive vs. Deductive Again – The last six sections have given the students a good amount of practice drawing diagrams, using deductive reasoning, and writing proofs, skills which are closely related. Before moving on to Chapter Three, take some time to review the first two sections of this chapter. It is quite possible that students have forgotten all about inductive reasoning. Now that they have had practice with deductive reasoning they can compare it to inductive reasoning and gain a deeper understanding of both. They should understand that inductive reasoning often helps a mathematician decide what should be attempted to be proved, and deductive reasoning proves it.

Review – The second section of chapter two contains information about conditional statements that will be used in the more complex proofs in later chapters. Since the students did not get to use most of it with these first simple proofs, it would be a good idea to draw their attention to it again and talk briefly about the more complex proof that will be coming.

1.3 Parallel and Perpendicular Lines

Lines and Angles

Marking the Diagram – Sometimes students confuse the marks for parallel and congruent. When introduction them to the arrows that represent parallel lines, review the ticks that represent congruent segments. Seeing the two at the same time helps avoid confusion.

When given the information that two lines or segments are perpendicular, students don't always immediately see how to mark the diagram accordingly. They need to use the definition of perpendicular and mark one of the right angles created by the lines with a box.

Symbol Update – Students should be keeping a list of symbols and how they will be used in this class in their notebooks. Remind them to update this page with the symbols for parallel and perpendicular.

Construction – The parallel and perpendicular line postulates are used in construction. Constructing parallel and perpendicular lines with a compass and straightedge is a good way to give students kinesthetic experience with these concepts. Construction can also be done with computer software. To construct a parallel or perpendicular line the student will select the line they want the new line to be parallel or perpendicular to, and the point they want the new line to pass through, and chose construct. The way the programs have the students select the line and then the point reinforces the postulates.

Additional Exercises:

1. Write a two-column proof of the following conditional statement.

If $\overline{AB}$ is perpendicular to $\overline{BC}$, triangle ABC is a right triangle.

Answer:

Table 1.2

Statement	Reason
$\overline{AB}$ is perpendicular to $\overline{BC}$	Given.
$< B$ is right	Definition of perpendicular.
triangle ABC is a right triangle	Definition of right triangle.

Parallel Lines and Transversals

The Parallel Hypothesis – So far seven different pairs of angles that may be supplementary or congruent have been introduced. All seven of these pairs are used in the situation where two lines are being crossed by a transversal forming eight angles. Some of these pairs require the two lines to be parallel and some do not. Students sometimes get these confuse on when they need parallel lines to apply a postulate or theorem, and if a specific pair is congruent or supplementary. A chart like the one below will help them sort it out.

Table 1.3

	Type of Angle Pair	Relationship
Do Not Require Parallel Lines	Linear Pairs	Supplementary
	Vertical Angles	Congruent
Parallel Lines Required	Corresponding Angles	Congruent
	Alternate Interior Angles	Congruent
	Alternate Exterior Angles	Congruent
	Consecutive Interior Angles	Supplementary
	Consecutive Exterior Angles	Supplementary

Patty Paper Activity – When two lines are intersected by a transversal eight angles are formed in two sets of four. When the lines are parallel, the two sets of four angles are exactly the same. To help students see this relationship, have them darken a set of parallel lines on their binder paper a few inches apart and draw a transversal through the parallel lines. Now they should trace one set of four angles on some thin paper (tracing paper or patty paper). When they slide the set of four angles along the transversal they will coincide with the other set of four angles. Have them try the same thing with a set of lines that are not parallel. This will help students find missing angle measures quickly and remember when they can transfer numbers down the transversal. It does not help them learn the names of the different pairs of angles which in important for communicating with others about mathematical concepts and for writing proofs.

Additional Exercises:

1. One angle of a linear pair has a measure twice as large as the other angle. What are the two angle measures?

Answer:

$$x + 2x = 180$$
$$x = 60$$

The angles measure 60 degrees and 120 degrees

Proving Lines Parallel

When to Use the Converse – It takes some experience before most students truly understand the difference between a statement and its converse. They will be able to write and recognize the converse of a statement, but then will have a hard time deciding which one applies in a specific situation. Tell them when you know the lines are parallel and are looking for angles, you are using the original statements; when you are trying to decide if the lines are parallel or not, you are using the converse.

Additional Exercise:

1. Prove the Converse of the Alternate Exterior Angle Theorem.

Answer: Refer to the image used to prove the Converse of the Alternate Interior Angle Theorem in the text.

Table 1.4

Statement	Reason
$\angle ABC \cong \angle HFE$	Given
$\angle HFE \cong \angle GFB$	Vertical Angles Theorem
$\angle ABC \cong \angle GFB$	Transitive Property of Angle Congruence
$\overleftrightarrow{AD}$ is parallel to $\overleftrightarrow{GE}$	Converse of the Corresponding Angles Postulate.

The Converse of the Alternate Exterior Angle Theorem could also be proved using the Converse of the Alternate Interior Angle Theorem. This would demonstrate to the students that once a theorem has been proved it, can be used in the proof of other theorems. It demonstrates the building block nature of math.

Table 1.5

Statement	Reason
$\angle ABC \cong \angle HFE$	Given

Table 1.5: (continued)

Statement	Reason
$\angle HFE \cong \angle GFB$ $\angle DBF \cong \angle GFB$	Vertical Angles Theorem
$\angle DBF \cong \angle GFB$	Transitive Property of Angle Congruence
$\overleftrightarrow{AD}$ is parallel to $\overleftrightarrow{GE}$	Converse of the Alternate Interior Angles Theorem.

Proving the theorem in several ways gives students a chance to practice with the concepts and their proof writing skills. Similar proofs can be assigned for the other theorems in this section.

Slopes

Order of Subtraction – When calculating the slope of a line using two points it is important to keep straight which point was made point one and which one was point two. It does not matter how these labels are assigned, but the order of subtraction has to stay the say in the numerator and the denominator of the slope ratio. If students switch the order they will get the opposite of the correct answer. If they have a graph of the line, ask them to compare the sign of the slope to the direction of the line. Is the line increasing or decreasing? Does that match the slope?

Graphing Lines with Integer Slopes – The slope of a line is the ratio of two numbers. When students are asked to graph a line with an integer slope they often fail to realize what and where the second number is. Frequently they will make the "run" of the line zero and graph a vertical line. It is helpful to have them write the integer that is the slope, as a ratio over one, before then do any graphing. Really, they only need to do this a few times on paper before they are able to graph the lines correctly. They will begin to see the ratio correctly in their heads.

Zero or Undefined – Students need to make these associations:

Zero in numerator – slope is zero – line is horizontal

Zero in denominator – slope is undefended – line is vertical

They frequently switch these around. After the relationships are explained in class, remind them frequently, maybe have a poster up in the room or write the relationship on a corner of the board that does not get erased.

Use Graph Paper – Making a connection between the numbers that describe a line and the line itself is an important skill. Requiring that the students use graph paper encourages them to make nice, thoughtful graphs, and helps them make this connection.

Additional Exercises:

1. Find the slope of the line that is perpendicular to the line passing through the points $(5, -7)$, and $(-2, -3)$.

Answer: The slope of the line passing through the given points is $\frac{-4}{7}$, so the line perpendicular to this line has the slope $\frac{7}{4}$.

Equations of Lines

The $y-$axis is Vertical – When using the slope-intercept form to graph a line or write an equation, it is common for students to use the $x-$intercept instead of the $y-$intercept. Remind them that they want to use the vertical axis, $y-$intercept, to begin the graph. Requiring that the $y-$intercept be written as a point, say $(0, 3)$ instead of just 3, helps to alleviate this problem.

Where's the Slope – Students are quickly able to identify the slope as the coefficient of the $x-$variable when a line is in slope-intercept form, unfortunately they sometimes extend this to standard form. Remind the students that if the equation of a line is in standard form, or any other form, they must first algebraically convert it to slope-intercept form before they can easily read off the slope.

Key Exercises:

1. Write the equation $3x + 5y = 10$ in slope-intercept form.

Answer: $y = -\frac{3}{5}x + 2$

2. What is the slope of the line $2x - 3y = 7$?

Answer: $\frac{3}{2}$

3. Are the lines below parallel, perpendicular, or neither?

$$6x + 4y = 7$$
$$6x - 4y = 7$$

Answer: These lines are neither parallel nor perpendicular.

Why Use Standard Form – The slope-intercept form of the line holds so much valuable information about the graph of a line, that students probably won't understand why any other form would ever be used. Mention to them that standard form is convenient when putting equations into matrices, something they will be doing in their second year of algebra, to motivate them to learn and remember the standard form.

Perpendicular Lines

Complementary, Supplementary, or Congruent – When finding angle measures students generally need to decide between three possible relationships: complementary, supplementary, and congruent. A good way for them to practice with these and review their equation solving skills, is to assign variable expressions to angle measures, state the relationship of the angles, and have the students use this information to write an equation that when solved will lead to a numerical measurement for the angle.

Key Exercise:

1. Two vertical angles have measures $2x - 30^\circ$ and $x + 60^\circ$.

Set-up and solve an equation to find x. Then find the measures of the angles.

Answer:

$$2x - 30^\circ = x + 60^\circ$$
$$x = 90^\circ$$

Both angles have a measure of 150 degrees.

2. The outer rays of two adjacent angle with measures $4x + 10°$ and $5x - 10°$ are perpendicular. Find the measures of each angle.

Answer:

$$\begin{aligned} 5x - 10 + 4x + 10 &= 90 \\ x &= 10 \end{aligned}$$

The angles have measures of 50 degrees and 40 degrees.

3. The angles of a linear pair have measures $3x + 45°$ and $2x + 35°$. Find the measure of each angle.

Answer:

$$\begin{aligned} 2x + 35 + 3x + 45 &= 180 \\ x &= 20 \end{aligned}$$

The angles have measures of 105 degrees and 75 degrees.

Encourage students to take the time to write out and solve the equation neatly. This process helps them avoid errors. Many times students will find the value of x, and then stop without plugging in the value to the expression for the angle measures. Have the students verify that their final answers are angle measures that have the desired relationship.

Additional Exercise:

1. Perpendicular lines form an angle with measure $8x + 10°$. What is the value of x?

Answer:

$$\begin{aligned} 8x + 10° &= 90° \\ x &= 10° \end{aligned}$$

Perpendicular Transversals

The Perpendicular Distance – In theory, measuring along a perpendicular line makes sense to the students, but in practice, when lining up the ruler or deciding which points to put in the distance formula, there are many distractions. Students can evaluate their decision by taking a second look to see if the path they chose was the shortest one possible.

Multi-Step Procedures – When working on an exercise that requires many different steps, like the last problem in this section, students sometimes become lost in the process or overwhelmed before they begin. A good way to ground students, and help them move through the problem, is to create, or have them create, a To-do list. Writing out the steps that need to be completed will help them understand the process, give them a sense of satisfaction as the check off parts they have completed, and help them organize their work. Creating the list could be a good group activity.

Where to Measure? – Now that the students know to measure along a line that is perpendicular to both parallel lines, they might wonder where along the lines to measure. When working on a coordinate plane it is best to start with a point that has integer coordinates, just to keep the problem simple and accurate. They will get the same distance no matter where they measure though. An alternate definition of parallel lines is two lines that are a constant distance apart.

Addition Exercises:

1. Prove the Converse of the Perpendicular Transversal Theorem.

Answer: Refer to the figure at the top of page 178, at the beginning of this lesson.

Table 1.6

Statement	Reason
$\overleftrightarrow{KN}$ is perpendicular to $\overleftrightarrow{QT}$	Given
$\overleftrightarrow{OR}$ is perpendicular to $\overleftrightarrow{QT}$	Given
$\angle QPO$ is right	Definition of Perpendicular Lines
$\angle PST$ is right	Definition of Perpendicular Lines
$\angle QPO \cong \angle PST$	Right Angle Theorem
$\overleftrightarrow{OR}$ is parallel to $\overleftrightarrow{KN}$	Converse of Corresponding Angle Postulate

Non-Euclidean Geometry

Separate Worlds – The geometry presented in this section is completely separate from the geometry in the rest of the text. The study of non-Euclidean geometry is excellent for developing critical thinking skills. It also demonstrates to the students what an influential role postulates play and how important it is to carefully evaluate them before accepting them as true. This section is best used for enrichment and should be treated differently from the other sections. If the students attempt to memorize the postulates in this section it may compromise their ability to recall analogous postulates of Euclidean Geometry. Exploring taxicab geometry is a wonderful way to spend a day in class, but it is not something that has to be included on tests. This is a decision that the instruction can make based on the ability of the students in a particular class.

Projects – This section opens the door to many possible projects that students can complete as part of the class or for extra credit. More advanced students in particular will have the ability and interest to explore the topic of Non-Euclidean geometry independently. Topics can include further exploration of taxicab geometry, other types of Non-Euclidean geometry, like spherical geometry, or research into the mathematician who developed these fields. This may make a good group project, where each group presents its findings to the class.

Encourage Creativity - Have students write their own problems involving taxicab geometry. This type of geometry lends itself to application and story problems. Students can be creative and funny. They will enjoy sharing problems with their classmates and solving each other's challenges. Writing word application helps students solve similar exercises. When formulating their question and deciding what information to give and how to give it, they become more aware of the structure of a word problem. If the students are enjoying this line of study and there is time, they may create their own type of geometry by setting up a system of postulates.

Abstraction and Modeling– This section briefly addresses the fact that mathematics is an abstraction and that it usually needs to be modified before it can be helpful in applications to the world in which we live. This is an important concept applicable to all areas of mathematics that is easily seen while studying geometry. This knowledge will help students understand why math is useful and how they will benefit from what they are learning in this class.

1.4 Congruent Triangles

Triangle Sums

Interior vs. Exterior Angles – Students frequently have trouble keeping interior and exterior angles straight. They may fail to identify to which category a specific angle belongs and include an exterior angle in a sum with two interior. They also sometimes use the wrong total, 360 degrees verses 180 degrees. Encourage the students to draw the figure on their papers and color code it. They can highlight or use a specific color of pencil to label all the exterior angle measures and another color for the interior angle measures. Then it is easy to do some checks on their work. Each interior/exterior pair should have a sum of 180 degrees, all of the interior angles should add to 180 degrees, and the measures of the exterior angles total 360 degrees.

Find All the Angles You Can – When a student is asked to find a specific angle in a complex figure and they do not immediately see how they can do it, they can become stuck, and don't know how to proceed. A good strategy is to find any angle they can, even if it is not the one they are after. Finding other angles keeps their brains active and working, they practice using angle relationships, and the new information will often help they find the target angle. Many exercises are not designed to do in one step. It is important that the students know this.

Congruent Angles in a Triangle – In later sections students will study different ways of determining if two or more angles in a triangle are congruent, and will then have to use this information to find missing angles in a triangle. To start them on this process it is good to have them work with triangles in which two angles are stated to be congruent.

Key Exercises:

1. An acute triangle has two congruent angles each measuring 70 degrees. What is the measure of the third angle?

Answer: $180 - 2 * 70 = 40$ degrees

2. An obtuse triangle has two congruent angles. One angle of the triangle measures 130 degrees. What are the measures of the other two angles?

Answer: The two remaining angles must be congruent since a triangle can not have more than one obtuse angle.

$(180 - 130) \div 2 = 25$ degrees

Congruent Figures

Rotation Difficulties – When congruent triangles are shown with different orientations, many students find it difficult to rotate the figures in their head to align corresponding sides and angles. One recommendation is to redraw the figures on paper so that they have the same orientation. It may be necessary for students to physically rotate the paper at first. After students have had some time to practice this skill, most will be able to skip this step.

Stress the Definition – The definition of congruent triangles requires six congruencies, three pairs of angles and three pairs of sides. If students understand what a large requirement this is, they will be more motivated to develop the congruence shortcuts in subsequent lessons.

The Language of Math – Many students fail to see that math is a language, a form of communication, which is extremely dense. Just a few symbols hold great amounts of information. The congruence statements for example, not only tell the reader which triangles are congruent, but which parts of the triangle correspond. When put in terms of communication students have an easier time understanding why they must put the corresponding vertices in the same order when writing the congruence statement.

Third Angle Theorem by Proof – In the text an example is given to demonstrate the Third Angle Theorem, this is inductive reasoning. A deeper understanding of the theorem, and different types of reasoning, can be gained by using deductive reasoning to write a proof. It will also reinforce the idea that theorems must be proved, and shows how inductive and deductive reasoning work together.

Key Exercise: Prove the Third Angle Theorem.

Answer: Refer to the figures on the top of page 213, where the example of the Third Angle Theorem is given.

Table 1.7

Statement	Reason
$\angle W \cong \angle C$	Given
$\angle V \cong \angle A$	Given
$m\angle V + m\angle W + m\angle X = 180$	Triangle Sum Theorem
$m\angle C + m\angle A + m\angle T = 180$	Triangle Sum Theorem
$m\angle V + m\angle W + m\angle X = m\angle C + m\angle A + m\angle T$	Substitution Property of Equality
$m\angle C + m\angle A + m\angle X = m\angle C + m\angle A + m\angle T$	Subtraction Property of Equality
$m\angle X = m\angle T$	Substitution Property of Equality

Triangle Congruence Using SSS

One Triangle or Two – In previous chapters, students learned to classify a single triangle by its sides. Now students are comparing two triangles by looking for corresponding pairs of congruent sides. Evaluating the same triangle in both of these ways helps the students remember the difference, and is a good way to review previous material. For instance, students could be asked to draw a pair of isosceles triangles that are not congruent, and a pair of scalene triangles that can be shown to be congruent with the SSS postulate.

Correct Congruence Statements – Determining which vertices of congruent triangles correspond is more difficult when no congruent angles are marked. Once the students have determined that the triangles are in fact congruent using the SSS Congruence postulate, it is advisable for them to mark congruent angles before writing the congruence statement. Corresponding congruent angles are found by matching up side markings. The angle made by the sides marked with one and two tick marks corresponds to the angle made by the corresponding sides in the other triangle, and so on.

Translation Rotation – Translating a triangle on a coordinate plane in order to see if it fits exactly over another triangle is a good way to demonstrate that two triangles are congruent. The notation used to describe these translations can sometimes be confusing. The text writes out the movement in words "D is 7 units to the right and 8 units below A". If students use other materials for reference, they may see this same translation as $(7, -8)$. This could be confused with the point located at $(7, -8)$. It may be helpful to alert students to this difference.

Additional Exercises:

1. Use the congruence statement and given information to find the indicated measurements.

$$\triangle ABC \cong \triangle ZYX$$
$$m\angle A = 52^\circ$$
$$m\angle Y = 85^\circ$$
$$AC = 12 \text{ cm}$$

Find XY and $m\angle X$.

Answer: $XY = 12$ cm and $m\angle X = 43$ degrees.

Triangle Congruence Using ASA and AAS

An Important Distinction – At first students may not see why it is important to identify whether ASA or AAS is the correct tool to use for a specific set of triangles. They both lead to congruent triangles, right? Yes, but this will not always be the case, as they will see in the next lesson. Sometimes the configuration of the corresponding congruent sides and angles in the triangles determines if the triangles can be proved to be congruent or not. Knowing this will motivate students to study the difference between ASA and AAS.

Flowchart Proofs – Flowchart proofs do a much better job of showing implication than two-column proofs. In a two-column proof one statement following another does not necessarily mean that the previous statement implies the next. Sometimes all the given information is listed at the beginning or another parallel argument needs to be developed before the implication is made. This can be confusing for students without much experience with proofs, or who have trouble understanding the argument. In a flowchart proof the implications are clearly indicated with arrows, and when parallel arguments are being developed, they are arranged vertically. The flowchart holds much more information.

Different Folks – People think and learn in different ways. When teaching, it is best to provide a few different explanations and have a variety of ways to present content. Some students, the linear thinkers, will understand two-column proof perfectly, and others, the special thinkers, will find flowchart proofs clearer. It is best to use both so that all students understand and develop their reasoning skills. One option to introduce the flowchart format is to have the students go back to key two-column proofs provided in the text and convert them to flowchart proofs.

Patterns and Structure – All of the shortcuts to triangle congruence require three pieces of information, therefore the box of the flowchart proof that states that two triangles are congruent will have three boxes leading into it. These kinds of structural relationships help students write and understand flowchart proofs and should be noted. It is also helpful to give students incomplete flowchart proofs and have them fill in the missing information. Subsequent proofs can be given with less information provided each time, until the boxes are all empty, and then with no help at all. The only problem is that sometimes there is more than one way to write a proof and a different chart may be required for the proof that the student wants to write. In that case, students can start from scratch if they like.

Proof Using SAS and HL

AAA – Students sometimes have to think for a bit to realize that AAA does not prove triangle congruence. Ask them to think back to the definition of triangle. Congruent triangles have the same size and shape. Most students intuitively see that AAA guarantees that the triangles will have the same shape. To see that triangles can have AAA and be different sizes ask them to consider a triangle they are familiar with, the equiangular triangle. They can draw an equiangular triangle on their paper, and you can draw an equiangular triangle on the board. The triangles have AAA, but are definitely different in size. This is a counterexample to AAA congruence. Have the students note that the triangles are the same shape; this relationship is called similar and will be studied in later chapters.

SSA – Student will have a hard time seeing the two possible triangles with SSA. The best way to describe it when the congruent parts are set up, is to tell them to take that the last congruent side can bend in, so that the third side is short to make one triangle, and bent out, so that the third side is long, to make the other triangle. Some students will see it right away and others will really have to play around with their triangles for awhile in order to understand.

Why Not LL? – Some students may wonder why there is not a LL shortcut for the congruence of right triangles. It also leads to SSS when the Pythagorean theorem is applied. Have the students explore the situation with a drawing. They can draw out two congruent right triangles and mark sides so that the triangles have LL. There is already a congruence guarantee for this, SAS. What would the non-right triangle congruence be for HL? Is this a guarantee? (It would be SSA, and no, this does not work in triangles that are right.)

Importance of Right Triangles – When using math to model situations that occur in the world around us the right triangle is used frequently. Have the students think of right angles that they see every day: walls with the ceilings and the floors, widows, desks, and many more constructed objects. Right triangles are also important in trigonometry which they will be studying soon. Stressing the usefulness of right triangles will motivate them to think about why HL guarantees triangle congruence but SSA, in general, does not.

Using Congruent Triangles

The Process – When students first start examining pairs of triangles to determine congruence it is difficult for them to sort out all the sides and angles.

The first step is for them to copy the figure onto their paper. It is helpful to color code the sides and angles, congruent sides marked in one color and the congruent angles in another. Some congruent parts will not be marked in the original figure that is given to the students in the text. For example, there could be an overlapping side that is congruent to itself, due to the reflexive property; mark it as well. Then they should do a final check to ensure that the congruent parts do correspond.

The next step is for them to count how many pairs of congruent corresponding sides and how many pairs of congruent corresponding angles there are. With this information they can eliminate some possibilities from the list of way to prove triangles congruent. If there is no right angle they can eliminate HL, or if they only have one set of corresponding congruent angles, they can eliminate both ASA and AAS.

If at this point there is still more than one possibility, they are going to need to decide if an angle is between two sides or if a side is between two angles. Remind them that both ASA and AAS can be used to guarantee triangle congruence, and that SAS works, but that SSA can not be used to prove two triangles are congruent.

If all postulates and theorems have been eliminated, then it is not possible to determine if the triangles are congruent.

AAS or SAA – Sometimes students try to list the congruent sides and angles in a circle as they move around the triangle. This could result in AAS or SAA when there are two pairs of congruent angles and one pair of congruent sides that is not between the angles. They know AAS proves congruence and want to know if SAA does as well. When this occurs it is best to redirect their thinking process. With two sets of angles and one set of sides there are only two possibilities, the side is between the angles or it is another side. When it is between the angles we have ASA, if it is either of the other two sides we use SAA. This same situation occurs with SSA, but is even more important since SSA is not a test for congruence. A good way for the students to remember this is that when the order of SSA is reversed it makes an inappropriate word. This word should not be used in class or in proofs, even if it is spelled backwards.

Isosceles and Equilateral Triangles

The Useful Definition of Congruent Triangles – The arguments used in the proof of the Base Angle Theorem apply what the students have learned about triangles and congruent figures in this chapter, and what they learned about reasoning and implication in the second chapter. It is a lot of information to bring together and students may need to review before they can fully understand the proof.

They have been practicing with proofs throughout the chapter, so they should be adept with the logic at this point. If they are having trouble, a review of the Deductive Reasoning section in Chapter Two: Reasoning and Proof will help. It could be assigned as reading the night before the current lesson will be done in class.

This is a good point to summarize what the students have learned in this chapter about congruent triangles and demonstrate how it can be put to use. To understand this proof, students need to remember that the definition of congruent triangles requires three pairs of congruent sides and three pairs of congruent angles, but realize that not all six pieces of information need to be verified before it is certain that the triangles are congruent. There are shortcuts. The proof of the Base Angle Theorem uses one of these shortcuts and jumps to congruence which implies that the base angles, a pair of corresponding angles of congruent triangles, are congruent.

To a student new to geometry this argument is not as straightforward as it may seem to an instructor experienced in mathematical proofs. Plan to take some time explaining this important proof.

A Proved Theorem Can Be Used – Now that the students have the proof of the Base Angle Theorem they can use it as opportunities present themselves. They should be on the lookout for isosceles triangles in the proofs of other theorems, in complex figures, and in all other situations. When they spot them, they need to immediately apply the Base Angle Theorem and mark those base angles congruent. This is true for the converse as well. When they spot a triangle with congruent angles, they should mark the appropriate sides congruent. Students sometimes do not realize what a powerful tool this theorem is and that they will be using it extensively throughout this class, and in math classes they will take in the future.

Additional Exercise:

1. What are the measures of the angles of an equilateral triangle? What postulates or theorems did you use to obtain your answer?

Answer: 60 degrees, Base Angle Theorem, Triangle Sum Theorem

Congruence Transformations

Reflection or Rotation – When looking a two triangle, where one is a transformation of the other, students sometimes have trouble distinguishing between a reflection and a rotation. This is particularly true when the triangles are almost equilateral. When demonstrating these transformations, it is best to use an obviously scalene triangle. Good use of labels is also helpful. The prime notation clearly indicates the new location of each vertex under the transformation. A rotation preserves the order of the vertices, and a reflection reverses the order of the vertices. If the students are unsure of what transformation has been applied to the figure, have them choose one vertex and then move counterclockwise around the polygon listing off the vertices as they occur. If they start with the image of that first vertex in the new figure, and again move counterclockwise, they will get the images of the vertices in the same order for a rotation, and in reverse order for a reflection.

Don't Just Memorize, Reason – This section contains many ordered pair rules for different transformation. Students will try to memorize them without really thinking about them or looking for patterns. This is challenging, if not impossible for most students. Have the students discuss similarities and differences between the rules. Ask them if the rule surprises them, or seems logical. Why? If they really get stuck, they can do a test. Graph a scalene triangle and apply different rules to it until the desired transformation occurs. Students will be motivated to use reason to shorten the guess and check process. In geometry problems are written so that students will have to think about them for awhile, and figure out an answer. Once students realize that they are not supposed to know the answer immediately, they are much more willing to spend time thinking about an exercise.

Additional Exercise:

1. Graph a scalene triangle in the first quadrant of the coordinate axis. Reflect the triangle over the $x-$axis. Take this new triangle and reflect it over the $y-$axis. What single transformation would take the first triangle to the final triangle? How can this be predicted by the ordered pair rules?

Answer: A single rotation of 180 degrees about the origin would result in the final triangle. The reflection over the $x-$axis takes the opposite of the $y-$coordinate and the reflection over the $y-$axis takes the opposite of the $x-$coordinate. If opposite of both coordinates are taken, the result is a rotation of 180 degrees about the origin.

1.5 Relationships Within Triangles

Midsegments of a Triangle

Don't Forget the $\frac{1}{2}$ - In this section there are two types of relationships that the students need to keep in mind when writing equations with variable expressions. The first involves the midpoint. When the expressions represent the two parts of a segment separated by the midpoint they just have to set the expressions equal to each other. The second is when comparing the length of a side of the triangle with the midsegment parallel to it. In this case they need to multiply the expression representing the side of the triangle by $\frac{1}{2}$, and then set it equal to the expression representing the midsegment. They may forget the $\frac{1}{2}$ or forget to use parenthesis and distribute. Remind them that they need to multiply the entire expression by $\frac{1}{2}$, not just the first term.

Additional Exercises:

1. The proof given in the text of the Midsegment Theorem is a paragraph proof. Write the second part of the proof as a two-column proof.

Answer: Refer to the triangle used in the text proof on the top of page 267.

Table 1.8

Statement	Reason
$\overline{AB}, \overline{CB}$, and $\overline{AC}$ are midsegments of $\triangle XYZ$	Given
$\overline{AB}$ is parallel to $\overline{XY}$	Midsegment Theorem (1)
$\angle BAC \cong \angle XCA$	Alternate Interior Angle Theorem
$\overline{CB}$ is parallel to $\overline{XZ}$	Midsegment Theorem (1)
$\angle XAC \cong \angle BCA$	Alternate Interior Angle Theorem
$\overline{AC} \cong \overline{AC}$	Reflexive Property of Congruence
$\triangle AXC \cong \triangle CBA$	ASA Postulate
$\overline{AB} \cong \overline{XC}$	Definition of Congruent Triangles
C is the midpoint of $\overline{XY}$	Definition of Midsegment
$XC = CY$	Definition of Midpoint
$XY = XC + CY$	Segment Addition Postulate
$XY = XC + XC$	Substitution Property of Equality

Table 1.8: (continued)

Statement	Reason
$AB = XC$	Definition of Congruent Segments
$XY = AB + AB = 2AB$	Substitution Property of Equality
$AB = \frac{1}{2}XY$	Division Property of Equality

Perpendicular Bisectors in Triangles

Construction Frustrations – Using a compass and straightedge to make clean, accurate constructions takes a bit of practice. Some students will pick up the skill quickly and others will struggle. What is nice about doing construction in the classroom is that it is often the students that typically struggle with mathematics, the more artistically minded students that excel and learn from constructing figures.

Recommend that students invest in a decent compass. They can get a quality, metal compass with some weight behind it for less that $20. The $2 variety often does not hold the pencil steady.They slip, and are extremely frustrating.

Here are some other tips for good construction: (1) Hold the compass at an angle. (2) Try rotating the paper while holding the compass steady. (3) Work on a stack of a few papers so that the needle of the compass can really dig into the paper and will not slip.

Perpendicular Bisector Quirks – There are two key ways in which the perpendicular bisector of a triangle is different form the other segments in the triangle that the students will learn about in subsequent sections. Since they are learning about the perpendicular bisector first these differences do not become apparent until the end of the chapter.

The perpendicular bisector of the side of a triangle does not have to pass through a vertex. Have the students explore in what situations the perpendicular bisector does pass through the vertex. They should discover that this is true for equilateral triangles and for the vertex angle of isosceles triangles.

The point of concurrency of the three perpendiculars of a triangle, the circumcenter, can be located outside the triangle. This is true for obtuse triangles. The circumcenter will be on the hypotenuse of a right triangle. This is also true for the orthocenter, the point of concurrency of the altitudes.

Same Construction for Midpoint and Perpendicular Bisector – The Perpendicular Bisector Theorem is used to construct the perpendicular bisector of a segment and to find the midpoint of a segment. When finding the midpoint, the students should make the arcs, one from each endpoint with the same compass setting, to find two equidistant points, but instead of drawing in the perpendicular bisector, they can just line up their ruler and mark the midpoint. This will keep the drawing from getting overcrowded and confusing.

Angle Bisectors in Triangles

Check with a Third – When constructing the point of concurrency of the perpendicular bisectors or angle bisectors of a triangle, it is strictly necessary to construct only two of the three segments. The theorems proved in the texts ensure that all three segments meet in one point. It is advisable to construct the third segment as a check of accuracy. Sometimes the compass will slip a bit while the student is doing the construction. If the three segments form a little triangle, instead of meeting at a single point, the student will know that their drawing is not accurate and can go back and check their marks.

Inscribed Circles – For a circle to be inscribed in a triangle, all three sides of the triangle must be tangent to the circle. A tangent to a circle intersects the circle in exactly one point. After accurately finding the incenter, students may have a difficult time finding the correct compass setting that will construct the inscribed circle. The best method is to place the center of the compass at the incenter, choose one side, and adjust the compass setting until the compass brushes by that side of the triangle, without passing through it. The word tangent does not have to be introduced at this point if the students already have enough vocabulary to learn. When the incenter is correctly placed, the compass should also hit the other two sides of the triangle once, creating the inscribed circle.

Additional Exercises:

1. Construct an equilateral triangle. Now construct the perpendicular bisector of one of the sides. Construct the angle bisector from the angle opposite of the side with the perpendicular bisector. What do you notice about these two segments? Will this be true of a scalene triangle?

Answer: The segments should coincide on the equilateral triangle, but not on the scalene triangle.

2. Construct an equilateral triangle. Now construct one of the angle bisectors. This will create two right triangles. Label the measures of the angles of the right triangles. With your compass compare the lengths of the shorter leg to the hypotenuse of either right triangle. What do you notice?

Answer: The hypotenuse should be twice the length of the shorter leg.

Medians in Triangles

Vocabulary Overload – So far this chapter has introduced to a large number of vocabulary words, and there will be more to come. This is a good time to stop and review the new words before the students become overwhelmed. Have them make flashcards, or play a vocabulary game in class.

Label the Picture – When using the Concurrency of Medians Theorem to find the measure of segments, it is helpful for the students to copy the figure onto their paper and write the given measures by the appropriate segments. When they see the number in place, it allows them to concentrate on the relationships between the lengths since they no longer have to work on remembering the specific numbers.

Median or Perpendicular Bisector – Students sometimes confuse the median and the perpendicular bisector since they both involve the midpoint of a side of the triangle. The difference is that the perpendicular bisector must be perpendicular to the side of the triangle, and the median must end at the opposite vertex.

Key Exercises:

1. In what type(s) of triangles are the medians also perpendicular bisectors? Is this true of all three medians?

Answer: This is true of all three medians of an equilateral triangle, and the median that intersects and vertex angle of an isosceles triangle.

Applications – Students are much more willing to spend time and effort learning about topics when they know of their applications. Questions like the ones below improve student motivation.

Key Exercises:

In the following situations would it be best to find the circumcenter, incenter, or centroid?

1. The drama club is building a triangular stage. They have supports on all three corners and want to put one in the middle of the triangle.

Answer: Centroid, because it is the center of mass or the balancing point of the triangle

2. A designer wants to fit the largest circular sink possible into a triangular countertop.

Answer: Incenter, because it is equidistant from the sides of the triangle.

Altitudes in Triangles

Extending the Side – Many students have trouble knowing when and how to extend the sides of a triangle when drawing in an altitude. First, this only needs to be done with obtuse triangles when drawing the altitude that intersects the vertex of one of the acute angles. It is the sides of the triangle that form the obtuse angle that need to be extended. The students should rotate their paper so that the vertex of the acute angle they want to start an altitude from is above the other two, and the segment opposite of this vertex is horizontal. Now they just need to extend the horizontal side until it passes underneath the raised vertex.

The Altitude and Distance – The distance between a point and a line is defined to be the shortest segment with one endpoint on the point and the other on the line. It has been shown that the shortest segment is the one that is perpendicular to the line. So, the altitude is the segment along which the distance between a vertex and the opposite side is measured. Seeing this connection will help students remember and understand why the length of the altitude is the height of a triangle when calculating the triangle's area using the formula $A = \frac{1}{2}\text{bh}$.

Explorations – When students discover a property or relationship themselves it will be much more meaningful. They will have an easier time remembering the fact because they remember the process that resulted in it. They will also have a better understanding of why it is true now that they have experience with the situation. Unfortunately, students sometimes become frustrated with explorations. They may not understand the instruction, or they may not be carefully enough and the results are unclear. Some of the difficulties can be alleviated by have the students work in groups. They can work together to understand the directions and interpret the results. Students strong in one area, like construction, can take on that part of the task and help the others with their technique.

Some guidelines for successful group work.

- Groups of three work best.
- The instructor should choose the groups before class.
- Students should work with new groups as often as possible.
- Desks or tables should be arranged so that the members of the group are physically facing each other.
- The first task of the group is to assign jobs: person one reads the directions, person two performs the construction, person three records the results. Students should regularly trade tasks.

Inequalities in Triangles

The Opposite Side/Angle – At first it may be difficult for students to recognize what side is opposite a given angle or what angle is opposite a given side. If it is not obvious to them from the picture, obtuse, scalene triangles can be confusing, they should use the names. For $\triangle ABC$, the letters are divided up by the opposite relationship, the angle with vertex A is opposite the side with endpoints B and C. Being able to determine these relationships without a figure is important when studying trigonometry.

Small, Medium, and Large - When working with the relationship between the sides and angles of a triangle, students will summarize the theorem to "largest side is opposite largest angle". They sometimes forget that this comparison only works within one triangle. There can be a small obtuse triangle in the same figure as a large acute triangle. Just because the obtuse angle is the largest in the figure, does not mean the side opposite of it is the longest among all the segments in the figure, just that it is the longest in that obtuse triangle. If the triangles are connected or information is given about the sides of both triangles, a comparison between triangles could be made. See exercise #9 is the text.

Add the Two Smallest – The triangle inequality says that the sum of the lengths of any two sides of a triangle is greater than the length of the third side. In practice it is enough to check that the sum of the lengths of the smaller two sides is larger than the length of the longest side. When given the three sides lengths for a triangle, students who do not fully understand the theorem will add the first two numbers instead of the smallest two. When writing exercises it is easy to always put the numbers in ascending order without thinking much about it. Have the students try to draw a picture of the triangle. After making a few sketches they will understand what they are doing, instead of just blindly following a pattern.

Indirect Proof – Students will not really understand the method of indirect proof the first time they see it. Let them know that this is just the first introduction, and that in subsequent lessons they will be given more examples and opportunities to learn this new method of proof. If students think they are supposed to understand something perfectly the first time they see it, and they don't, they will become frustrated with themselves and mathematics. Let them know that the brain needs time, and multiple exposures to master these challenging concepts.

Inequalities in Two Triangles

Use Color – The figures is this section now have two triangles instead of just one and are therefore more complex. The students may need some help sorting out the shapes. A good way for them to begin this process is to draw the figure on their paper and use highlighters to color code the information.

Both of the theorems presented in this section require two pairs of congruent sides. The first step is for student to highlight these four sides in a common color, let's say yellow. Once they have identified the two pairs of congruent sides, they know the hypothesis of the theorem has been filled and they can apply the conclusion.

The conclusions of these theorems involve the third side and the angle between the two congruent sides. These parts of the triangles can be highlighted in a different color, let's say pink.

Now the students need to determine if they need to use the SAS Inequality theorem or the SSS inequality theorem. If they know one of the pink angles is bigger than the other, than they will use the SAS Inequality theorem and write an inequality involving the pink sides. If they know that one of the pink sides is bigger than the other, they will apply the SSS Inequality theorem, and write an inequality involving the two pink angles.

Having a step-by-step process is good scaffolding for students as they begin working with new types of problems. After the students have gained some experience, they will no longer need to go through all the steps.

Solving Inequalities – Students learned to solve inequalities in algebra, but a short review may be in order. Solving inequalities involves the same process as solving equations except the equal sign is replaced with an inequality, and there is the added rule that if both sides of the inequality are multiplied or divided by a negative number the direction of the inequality changes. Students frequently want to change the direction of the inequality when it is not required. They might mistakenly change the inequality if they subtract from both sides, or if result of multiplication or division is a negative even if the number used to change the inequality was not negative. In geometry it is most common to be working with all positive numbers, but depending on how the students apply the Properties of Inequalities, they may create some negative values.

Indirect Proof

Why Learn Indirect Proof – For a statement to be mathematically true it must always be true, no exceptions. This frequently makes it easier to prove that a statement is false than to prove it is true.

Indirect proof gives mathematician the choice between proving a statement true or proving a statement false and can therefore greatly simplify some proofs. Letting the students know that indirect proof can be a potential shortcut will motivate them to learn to use this type of logic.

Review the Contrapositive - Proving a statement using indirect proof is equivalent to proving the contrapositive of the statement. If students are having trouble setting up indirect proofs, or even if they are not, it is a good idea to have them review conditional statements and the contrapositive. The second section of Chapter Two: Reasoning and Proof is about conditional statements. Have the students reread this section before working on indirect proof in class. The first step to writing an indirect proof, can be to have them write out the contrapositive of the statement they want to prove. This will reduce confusion about what statement to start with, and what statement concludes the proof.

Does This Really Prove Anything? – Even after students have become adept with the mechanics of indirect proof, they may not be convinced that what they are doing really proves the original statement. This is the same as asking if the contrapositive is equivalent to the original statement. Using examples outside the field of mathematics can help students concentrate on the logic.

Start with the equivalence of the contrapositive. Does statement (1) have the same meaning as statement (2)?

1. If you attend St. Peter Academy, you must wear a blue uniform.

2. If you don't wear a blue uniform, you don't attend St. Peter Academy.

Let the students discuss the logic, and have them create and share their own examples.

If a good class discussion ensues, and the students provide many statements on a single topic, it may be possible to write some indirect proofs of statements not concerned with mathematics. This could be a good bonus assignment or project that when presented to the class will make the logic of indirect proof clearer for other students.

1.6 Quadrilaterals

Interior Angles

All Those Polygons – Although they have probably been taught it before, not all students will remember the names of the different polygons. There are not very many opportunities in life to use the word heptagon. Add these words to their vocabulary list.

This is most likely the first time they have been introduced to a polygon with a variable number of sides, an $n-$gon. This notation can be used when referring to a polygon that does not have a special name in common use, like a $19-$gon. It can also be used when the number of sides of the polygon is unknown.

Key Exercise:

1. What is the measure of each interior angle of a regular $n-$gon if the sum of the interior angles is 1080 degrees?

Answer: 135 degrees

First the number of sides needs to be found:

$$1080 = 180(n-2)$$
$$n = 8$$

Now the total of 1080 degrees needs to be divided into 8 congruent angles.

$$1080 \div 8 = 135$$

Sketchpad Alternatives – Many students become particularly engaged in a topic when they are able to investigate it while playing around with the computer. Here are a couple of ways to use Geometers' Sketchpad in the classroom as an alternative or supplement to direct instruction.

Angle Sum Conjecture – Have student draw different convex polygons and measure the sum of their interior angles.

1. The students should observe that for each type of polygon, no matter how many were drawn, they all have the same interior angle sum.
2. The students should drag a vertex of each polygon toward the center to create a concave polygon, and notice if the sum stays the same. (It won't.)
3. Put the sums in order on the board: $180, 360, 540, \ldots$ Ask the students to find the pattern in this sequence of numbers. Lead them to discovering the Angle Sum formula from the pattern.

Exterior Angles

Clockwise or Counterclockwise But Not Both – At this point in the class, student are usually good at recognizing vertical angles. They will understand that the exterior angles made by extending the sides of the polygon in a clockwise rotation are congruent, at each vertex, to the exterior angle formed by extending the sides counterclockwise. What they will sometimes do is include both of these angles when using the Exterior Angle Sum theorem. Reinforce that the number of exterior angles is the same as the number of interior angles and sides, one at each vertex.

Interior or Exterior – Interior and exterior angles come in linear pairs. If one of these angles is known at a particular vertex, it is simple to find the other. When finding missing angles in a polygon, students need to decide from the beginning if they are going to use the interior or exterior sum. Most likely, if the majority of the known angle measures are from interior angles they will use the interior sum. They need to convert the exterior angle measures to interior angle measures before including them in the sum. If there are more exterior angle measures given, they can convert the interior angle measures and use the sum of 360 degrees. It is important that they make a clear choice. They may mix the two types of angles in one summation if they are not careful.

Do a Double Check – Students often do not take the time to think about their answers. Going over the arithmetic and logic is one way to check work, but it is common to not recognize the error the second time either. A better strategy is to use other relationships to do the checking. In this lesson if the exterior sum was used, the work can be checked with the interior sum.

What's the Interior Sum of a Nonagon Again? – If students do not remember the interior sum for a specific polygon, and do not remember the formula, they can always convert to the exterior angle measures using the linear pair relationship. The sum of the exterior angles is always 360 degrees. This strategy will work just as well as using the interior sum. Remind the students to be creative. When taking a test, they may not know an answer directly, but many times they can figure out the answer in an alternative way.

Sketchpad Alternative – The activities designed for students to explore interior angles in the pervious section can be easily adapted for exterior angles, and be used with this section. Using Sketchpad to extend the sides of the polygon helps students gain an understanding of where the exterior angle is in relation to the polygon.

Classifying Quadrilaterals

Tree Diagram – Most students will need practice working with the classification of quadrilaterals before they completely understand and remember all of the relationships. The Venn diagram is an important mathematical tool and should definitely be used to display the relationships among the different types of quadrilaterals. A tree diagram will also make an informative visual. Using both methods will reinforce the students' understanding of quadrilaterals, and their ability to make good diagrams.

Parallel Line Properties – In the second section of Chapter Three: Parallel and Perpendicular Lines, the students learn about the relationships between the measures of the angles formed by parallel lines and a transversal. Many of the quadrilaterals studied in this section have parallel sides. The students can apply what they learned in chapter three to the quadrilaterals in this chapter. They may have trouble seeing the relationships because instead of lines the quadrilaterals are made of segments. Recommend that the students draw the figures on their papers and extend the sides of the quadrilaterals so they can see all four angles made by the intersection of the lines. These angles will be useful when looking for specific information about the quadrilateral.

Show Clear, Organized Work – When using the distance or slope formula to verify information about a quadrilateral on the coordinate plane, students will often do messy scratch work as if they are the only ones that will need to read it. In this situation, the work is a major part of the answer. They need to communicate their thoughts on the situation. They should write as if they are trying to convince the reader that they are correct. As students progress in their study of mathematics, this is more often the case than the need for a single numerical answer. They should start developing good habits now.

Symmetry – Most students have already studied symmetry at some point in their education. A review here may be in order. When studying quadrilaterals, symmetry is a good property to consider. Symmetry is also important when discussing the graphs of key functions that the students will be studying in the next few years. It will serve the students well to be adept in recognizing different types of symmetry.

Using Parallelograms

Proofs Using Congruent Triangles – The majority of the proofs in this section use congruent triangles. The quadrilateral of interest is somehow divided into triangles that can be proved congruent with the theorems and postulates of the previous chapters. Once the triangles are known to be congruent, the definition of congruent triangles ensures that certain parts of the quadrilateral are also congruent. Students should be made aware of this pattern if they are having difficulty writing or understanding the proofs of the properties of various quadrilaterals. If they are still struggling they should spend some time reviewing sections two through six of Chapter Four: Congruent Triangles.

The Diagonals of Parallelograms – The properties concerning the sides and angles of parallelograms are fairly intuitive, and students pick them up quickly. More emphasis should be placed on what is known, and not known about the diagonals. Students frequently try to use the incorrect fact that the diagonally of a parallelogram are congruent. Rectangles are the focus of an upcoming lesson, but demonstrating to students that the diagonals of a quadrilateral are only congruent in the special case where all the angles of the parallelogram are congruent. For a general parallelogram, the measures of the two pieces of the same diagonal separated by the other diagonal, can be set equal to each other, but no comparison can be made between diagonals.

Additional Exercises:

1. Quadrilateral $ABCD$ is a parallelogram.

$AB = 2x + 5, BC = x - 3$, and $DC = 3x - 10$

Find the measures of all four sides of the quadrilateral.

(Hint: Draw and label a picture. Remember the name of a polygon lists the vertices in a circular order.)

Answer:

$$2x + 5 = 3x - 10 \qquad AB = CD = 35 \text{ and } BC = AD = 12$$
$$x = 15$$

2. JACK is a parallelogram.

$m\angle A - 10x - 60^\circ$ and $m\angle C = 2x + 45^\circ$

Find the measures of all four angles.

Answer:

$$10x - 60 + 2x + 45 = 180 \qquad m\angle C = m\angle J = 77.5^\circ$$
$$x = 16\ \frac{1}{4} \qquad m\angle A = m\angle K = 102.5^\circ$$

Proving Quadrilaterals are Parallelograms

Proof Practice − The proofs in this section may seem a bit repetitive, but students will benefit from practicing these proofs since they review important concepts learned earlier in the course. To avoid loosing the students' attention, find different ways of presenting the proofs. One idea is to divide the students into groups, and have each group demonstrate a different proof to the class.

Parallel or Congruent − When looking at a marked figure students will sometimes see the arrows that designate parallel segments and take that the segments to be congruent. This could be due to the misreading of the marks, or mistakenly thinking parallel always implies congruence. Warn students not to make this error. The last method of proof in this section which utilizes that one pair of sides are both congruent and parallel, along with an example of a trapezoid where the parallel sides are not congruent, will help students remember the difference.

Additional Exercises:

1. KATE is a parallelogram with a perimeter of 40 cm.

$KA = 3x + 8$, and $AT = x + 4$

Find the length of each side.

(Hint: Draw and label a picture. Remember the name of a polygon lists the vertices in a circular order.)

Answer:

$$2(3x + 8) + 2(x + 4) = 40 \qquad KA = ET = 14 \text{ cm}$$
$$x = 2 \qquad AT = KE = 6 \text{ cm}$$

2. SAMY is a parallelogram with diagonals intersecting at point X.

$SX = x + 5$, $XM = 2x - 7, AX = 12x$

Find the length of each diagonal.

Answer:

$$x + 5 = 2x - 7 \qquad SM = 34 \text{ cm}$$
$$x = 12 \qquad AY = 288 \text{ cm}$$

3. JEDI is a parallelogram.

$m\angle J = 2x + 60$, and $m\angle D = 3x + 45$

Find the measures of the four angles of the parallelogram.

Does this parallelogram have a more specific categorization?

Answer:

$$2x + 60 = 3x + 45$$
$$x = 15$$

All four angles measure 90 degrees.
JEDI is a rectangle.

Rhombi, Rectangles, and Squares

The Power of the Square – Students should know by the classification of quadrilaterals that all the theorems for parallelograms, rectangles, and rhombi, also apply to squares. It is a good idea to talk about this in class though in case they have not put it together on their own. A combination of these theorems and the definition of a square can be combined to from some interesting exercises.

Key Exercises:

1. SQUR is a square.

$m\angle XUR = 3x - 9$, and $SU = x$

Find the length of both diagonals.

(Hint: Draw and label a picture. Remember the name of a polygon lists the vertices in a circular order.)

Answer:

$$3x - 9 = 45 \qquad SU = QR = 18$$
$$x = 18$$

Information Overload – Quite a few theorems are presented in this chapter. Remembering them all and which quadrilaterals they apply to can be a challenge for students. If they are unsure, and cannot check reference material, a test case can be drawn. For example: Do the diagonals of a parallelogram bisect the interior angles of that parallelogram? First they need to draw a parallelogram that clearly does not fit into any subcategory. It should be long and skinny, so no rhombi properties are mistakenly attributed to it. It should also be well slanted over, so as not to be mistaken for a rectangle. Now they can draw in the diagonals. It will be obvious that the diagonals are not bisecting the interior angles. They could also try to recreate the proof, but that will probably be more time consuming and it requires a bit of skill.

Additional Exercises:

1. DAVE is a rhombus with diagonals that intersect at point X.

$DX = 3$ cm, and $AX = 4$ cm

How long is each side of the rhombus?

Answer:

$$\begin{aligned} 32 + 42 &= DA^2 && \text{since } \triangle DXA \text{ is right} \\ DA &= 5 \\ DA = AV = VE = ED &= 5 \text{ cm} \end{aligned}$$

Trapezoids

Average for the Median – Students who have trouble memorizing formulas may be intimidated by the formula for the length of the median of a trapezoid. Inform them that they already know this formula; it is just the average. The application of the formula makes since, the location of the median is directly between the two bases, and the length of the median is exactly between the lengths of the bases. They will have no problem finding values involving the median.

Where Are We? – Most students have five other classes and a demanding social and family life. It is easy for them to forget how what they are learning today relates to the chapter and to the class. Use the Venn diagram of the classification of quadrilaterals to orient them in the chapter. They are no longer learning about parallelograms, but have moved over to the separate trapezoid area. When student are able to organize their new knowledge, they are better able to retain and apply it.

Does it Have to be Isosceles? – Students may have trouble remembering which theorems in this section apply only to isosceles trapezoids. Note that base angle, and diagonal congruence apply only to isosceles trapezoids, but the relationship of the length of the median to the bases is the same for all trapezoids.

Additional Exercises:

1. TRAP is a trapezoid. The median has length 4 cm, and one of the bases has length 7 cm. What is the length of the other base?

Answer: 1 cm

Seven is three more than four, so the other base must be three less than four.

OR solve the equation $4 = (7 + x) \div 2$

2. $WXYZ$ is a trapezoid. The length of one base is twice the length of the other base, and the median is 9 cm. How long is each base?

Answer:

$$(x + 2x) \div 2 = 9 \qquad \text{The bases are 6 cm and 12 cm.}$$
$$x = 6$$

Kites

Only One Congruent Set – It is important to note that in a kite, only one set of interior angles are congruent, and only one of the diagonals is bisected. Sometimes students struggle with identifying where these properties hold. It is the nonvertex angles that are congruent, and the diagonal connecting the nonvertex angles that is bisected. The single line of symmetry of a kite shows both these relationships.

Break it Up – When working with a kite, it is sometimes easier to think of it as two isosceles triangles, or four right triangles, instead of one quadrilateral.

At this point in the class, students have had extensive experience working with isosceles triangles, and can easily apply the Base Angle theorem to see that the nonvertex angles of the kite are congruent. They have also seen that the segment from the vertex angle creates many symmetries in the triangle, and it will make since to them that the diagonal connecting the nonvertex angles is bisected.

They can also think of a kite as four right triangles. This will help them remember that the diagonals are perpendicular, and remind them that the Pythagorean theorem can be used to find missing segment measures. Noticing that the right triangles are in two congruent sets will help them identify congruent segments and angles.

Additional Exercises:

1. Refer to the kite used to prove the diagonal properties on page 396.

Prove that $\triangle AYR$ is congruent to $\triangle TYR$.

Answer:

Table 1.9

Statement	Reason
$\overline{AR} \cong \overline{TR}$	Given
$\overline{AT} \perp \overline{PR}$	Kite Diagonal Theorem
$\angle AYR$ is right	Definition of Perpendicular
$\angle TYR$ is right	Definition of Perpendicular
$\angle TYR \cong \angle AYR$	Right Angle Theorem
$\triangle AYR \cong \triangle TYR$	HL

1.7 Similarity

Ratio and Proportion

Keep it in Order – When writing a ratio, the order of the numbers is important. When the ratio is written in fraction form the amount mentioned first goes in the numerator, and the second number goes in the denominator. Remind the students it is important to keep the values straight, especially if they are looking at the male to female student ratio at their top three college choices.

To Reduce or Not to Reduce – When a ratio is written in fraction form it can be reduced like any other fraction. This will often make the arithmetic simpler and is frequently required by instructors for fractions in general. But when reducing a ratio, useful information can be lost. If the ratio of girls to boys in a classroom is 16 to 14, it may be best to use the fraction $\frac{16}{14}$ because it gives the total number of students in the class where the reduced ratio $\frac{8}{7}$ does not.

Consistent Proportions – A proportion can be correctly written in many ways. As long as the student sets up the ratios in a consistent, orderly fashion, they will most likely have written a correct proportion. There should be a common tie between the two numerators, the two denominators, the numbers in the first ratio, and the numbers in the second ratio. They should think about what the numbers represent, and not just use them in the order given in the exercise, although the numbers are usually given in the correct order.

Key Example:

1. Junior got a new hybrid. He went 525 miles on the first five gallons that came with the car. He just put 12 gallons in the tank. How far can he expect to go on that amount of gas?

Answer:

$$\frac{525}{5} = \frac{x}{12}$$
$$x = 1260$$

He can expect to go 1,260 miles.

Note: Students will be tempted to put the 12 in the numerator of the second ratio because it was the third number given in the exercise, but it should go in the denominator with the other amount of gas.

The Fraction Bar is a Grouping Symbol – Students know that parenthesis are a grouping symbol and that they need to distribute when multiplying a number with a sum or difference. A fraction bar is a more subtle grouping symbol that students frequently overlook, causing them to forget to distribute. To help them remember have them put parenthesis around sums and differences in proportions before they cross-multiply.

Example: $\frac{x+3}{5} = \frac{x-8}{7}$ becomes $\frac{(x+3)}{5} = \frac{(x-8)}{7}$

Properties of Proportions

Everybody Loves to Cross-Multiply – There is something satisfying about cross-multiplying and students are prone to overusing this method. Remind them that cross-multiplication can only be used in proportions, when two rations are equal to each other. It is not appropriate to cross-multiply when two fractions are being added or subtracted.

Examples:

Table 1.10

Cross Multiply	Don't Cross Multiply
$\frac{3}{4} = \frac{10}{x}$ $\frac{x+3}{4} = \frac{10}{x}$	$\frac{3}{4} + \frac{10}{x}$ $\frac{x+3}{4} - \frac{10}{x}$

Only Cancel Common Factors – When reducing a fraction or putting a ratio in simplest terms, students often try to cancel over an addition or subtraction sign. This problem occurs most frequently when students work with fractions that contain variable expressions. To combat this error, go back to numerical examples. Students will see that what they are doing does not make sense when the variables are removed. Then go back to example with variables. Hopefully the students will be able to carry over the concept.

Examples:

Table 1.11

Can be Reduced	Can't be Reduced
$\frac{3 \cdot 2}{5 \cdot 2}$ $\frac{3(x-4)}{3 \cdot 2}$	$\frac{3+2}{5+2}$ $\frac{x-4}{4}$

Color-Code the Proofs – The proof in this section requires many substitutions of similar looking expressions. It is difficult to see where everything is coming from and moving to. When presenting the proof in class use colors so the variables will be easier to follow. Another option is to have the students do the color-coding. Once they understand the mechanics of the proof in the lesson, they will be able to do the similar proof in the exercises.

Similar Polygons

A Common Vocabulary Error – Students frequently interchange the words proportional and similar. Remind them that proportional describes a relationship between numbers, and similar describes a relationship between figures, like equal and congruent.

Compare and Contrast Similar with Congruent – If your students have already learned about congruent figures, now would be a good time to review. The definitions of congruent and similar are very close. Ask the students if they can identity the difference; it's only one word. You can also point out that congruent is a subset of similar like square is a subset of rectangle, or mother is a subset of women. Understanding the differences between congruent and similar will be important in upcoming lessons when proving triangles similar.

Use that Similarity Statement – In some figures, which sides of similar polygons correspond is obvious, but when the polygons are almost congruent, or oriented differently, the figure can be misleading. Students usually begin by using the figure and then forget to use the similarity statement when necessary. Remind them about this information as they start working on more complicated problems. The similarity statement is particularly useful for students that have a hard time with visual-spatial processing.

Who's in the Numerator – When writing a proportion students sometimes carelessly switch which polygon's measurements are in the numerator. To combat this I tell the students to choose right from the beginning and BE CONNSISTENT throughout the problem. When it comes to writing proportions if the students focus on being orderly and consistent, they will usually come up with a correct setup.

Bigger or Smaller – After completing a problem it is always a good idea to take a minute to decide if the answer makes sense. This is hard to get students to do. When using a scale factor, a good way to check that the correct ratio was used is to notice if the number got bigger or smaller. Is that what we expected to happen?

Update the List of Symbols – In previous lessons it has been recommended that students create a reference page in their note books that contains a list of all the symbols and how they are being used in this class. Students should add the symbol for similar to the list, and take few minutes to compare it to the symbols they already know. Sometimes students will read the similarity symbol as "approximately equal". It is standard to use two wavy lines for approximately equal and one wavy line for similar, but this is not always the case.

Similarity by AA

Definition of Similar Triangles vs. AA Shortcut – Let the students know what a deal they are getting with the AA Triangle Similarity Postulate. The definition of similar polygons requires that all three corresponding pairs of angles be congruent, and that all three pairs of corresponding sides are proportional. This is a significant amount of information to verify, especially when writing a proof. The AA postulate is a significant shortcut; only two piece of information need to be verified and all the rest comes for free. When students see how much this reduces the work, they will be motivated to understand the proof and will enjoy using the postulate. Everybody likes to use a tricky shortcut.

Get Some Sun – It is always a good idea to create some variety in the class. It will keep students' minds active. Although it is time consuming, get some yard sticks and take the students outside to measure a tree or a flagpole using their shadows and similar triangles. Have them evaluate their accuracy. They will have to measure carefully if they are to get a reasonable numbers. This will give them some practice using a rule and converting units. The experience will also help them put what they are learning about similar triangles into their long term memory.

Trigonometry – Let the students know that the next chapter is a about trigonometry, and that the AA Triangle Similarity Postulate is what make trigonometry possible. If the students know what an important postulate this is, they will be motivated to understand and learn how to apply it. Mentioning what is to come will start to prepare their minds and make learning the material in the next chapter that much easier. Here are some problems that involve similar right triangles to accustom the students to this new branch of mathematics.

Key Exercises:

1. $\triangle ABC$ is a right triangle with right angle C, and $\triangle ABC \sim \triangle XYZ$.

Which angle in $\triangle XYZ$ is the right angle?

Answer: $\angle Z$

2. $\triangle CAT \sim \triangle DOG$, $\angle A$ is a right angle

$CA = 5 \text{ cm}, CT = 13 \text{ cm}$

What is DG?

Answer: $DG = 13$ cm,

Students must use the Pythagorean theorem and the definition of similar polygons.

Similarity by SSS and SAS

The "S" of a Triangle Similarity Postulate – At this point in the class, students have shown that a significant number of triangles are congruent. They have learned the process well. When teaching them to show that triangles are similar, it is helpful to build on what they have learned. The similarity postulates have S's and A's just like the congruence postulates and theorems. The A's are treated exactly the same in similarity postulates as they were in congruence theorems. Each "A" in a similarity shortcut stands for one pair of congruent corresponding angles in the triangles.

The S's represent a different requirement in similarity postulates then they did in congruence postulates and theorems. Congruent triangles have congruent sides, but similar triangles have proportional sides. Each "S" is a similarity postulate represents a ratio of corresponding sides. Once the ratios (two for SAS and three for SSS) are written, equality of the ratios must be verified. If the ratios are equal, the sides in question are proportional, and the postulate can be applied.

It is sometimes hard for student to adjust to this new side requirement. They have done so much work with congruent triangles that it is easy for them to slip back into congruent mode. Warn them not to fall into the old way of thinking.

Table 1.12

Triangle Congruence Postulates and Theorems	Triangle Similarity Postulates
"S" $\leftrightarrow$ congruent sides	"S" $\leftrightarrow$ proportional sides
SSS	AA
SAS	SSS
ASA	SAS
AAS	
HL	

Only Three Similarity Postulates – Students will sometimes try to use ASA, or other congruence theorems to show that two triangles are similar. Bring it to their attention that there are only three postulates for similarity, and that they do not all have the same side and angle combinations as congruence postulates or theorems.

Proportionality Relationships

Similar Triangles Formed by an Interior Parallel Segment – Students frequently are presented with a triangle that contains a segment that is parallel to one side of the triangle and intersects the other two sides. This segment creates a smaller triangle in the tip of the original triangle. There are two ways to consider this situation. The two triangles can be considered separately, or the Triangle Proportionality Theorem can be applied.

(1) Consider the two triangles separately.

The original triangle and the smaller triangle created by the parallel segment are similar as seen in the proof of the Triangle Proportionality theorem. One way students can tackle this situation is to draw the triangles separately and use proportions to solve for missing sides. The strength of this method is that it can be used for all three sides of the triangles. Students need to be careful when labeling the sides of the larger triangle; often the lengths will be labeled as two separate segments and the students will have to add to get the total length.

(2) Use the Triangle Proportionality theorem.

When using this theorem it is much easier to setup the proportions, but there is the limitation that the theorem can not be used to find the lengths of the parallel segments.

Ideally student will be able to identify the situations where each method is the most efficient, and apply it. This may not happen until the students have had some experience with these types of problem. It is best to have students use method (1) at first, then after they have worked a few exercises on their own, they can use (2) as a shortcut in the appropriate situations.

Additional Exercises:

1. $\triangle ABC$ has point E on $\overline{AB}$, and F on $\overline{BC}$ such that $\overline{EF}$ is parallel to $\overline{AC}$.

$AE = 5 \text{ cm}, EB = 3 \text{ cm}, BF = 4 \text{ cm}, AC = 10 \text{ cm}$

Find EF and FC.

(Hint: Draw and label a picture, then draw another figure where the two triangles are shown separately.)

Answer:

$$\frac{3}{8} = \frac{EF}{10} \qquad \frac{3}{5} = \frac{4}{FC} \quad \text{or} \quad \frac{3}{8} = \frac{4}{FC+4}$$

$$EF = 3\frac{3}{4} \text{ cm} \qquad FC = 6\frac{2}{3} \text{ cm}$$

Similarity Transformations

Scale Factor Compared to Segment and Area Ratios – When a polygon is dilated using scale factor k, the ratio of the image of the segment to the original segment is k. This is true for the sides of the polygon, all the special segments of triangles studied in chapter five, and the perimeter of the polygon. The relationship holds for any linear measurement. Area is not a linear measurement and has a different scale factor. The ratio of the area of the image to the area of the original polygons is k^2. Student frequently forget to square the scale factor when working with the ratios of a figure and its image. This is an important concept that is frequently used on the SAT and on other standardized tests.

Key Exercises:

$\triangle ABC$ has coordinates at the following vertices. $A(1,13), B(6,1)$, and $C(1,1)$.

1. Graph $\triangle ABC$.

2. Use the distance formula to find the length of each side of $\triangle ABC$.

3. Calculate the perimeter of $\triangle ABC$.

4. Calculate the area of $\triangle ABC$.

$\triangle A'B'C'$ is the image of $\triangle ABC$ under a dilation centered at the origin with scale factor 3.

5. Graph $\triangle A'B'C'$.

6. Use the distance formula to find the length of each side of $\triangle A'B'C'$.

7. Calculate the perimeter of $\triangle A'B'C'$.

8. Calculate the area of $\triangle A'B'C'$.

Compare $\triangle A'B'C'$ to $\triangle ABC$.

9. What is the ratio of each set of corresponding side lengths, the perimeters, and the areas? What do you notice when these ratios are compared to the scale factor.

Answers:

1.

2. $AC = 12, BC = 5, AB = 13$

3. 30

4. 30

5.

6. $A'C' = 36, B'C' = 15, A'B' = 39$

7. 90

8. 270

9. The ratios of the side lengths and the perimeter are 3 : 1 the same as the scale factor. The ratio of the areas is 9 : 1, the square of the scale factor.

Self-Similarity (Fractals)

More Complex Fractals – Students need to begin learning about fractals with the simple examples given in the text. Once they have taken some time to work with, and understand the self-similar relationship, it is amazing to see how complex and beautiful fractal can become. Numerous examples of exquisite fractals can be found on-line. If you are lucky enough to have access to computers and a projector, have the students search for fractals and choose their favorite to share with the class. Student will begin to realize the importance of what there are learning when they see what a huge ocean they are dipping their toe into.

Applications – Many students need to know how a subject is useful before they are motivated to spend time and energy learning about it. Throughout the text there have been references to modeling and how mathematical concepts often need to be adjusted to fit the world around us. Fractals are used to model many aspects of nature including tree branches, shells, and the coast line. Knowing of the applications of fractals motivates students. If time permits give a more in-depth explanation, or use this topic to assign research projects.

Video Time – Self-similarity and fractals make up an extremely complex visual topic. There are many videos in common use that can give a much more exciting and attention grabbing explanation than most teachers can deliver while standing in front of the classroom. These videos are not hard to come by, and they give an excellent explanation of the material. It is a nice change of pace for the students, and it gives the instructor some precious time to catch-up on paperwork. It is the best approach for all.

Create Your Own Fractal – Having the students create their own fractal outside of class is a fun, creative project. This gives the more artistically minded students an opportunity to shine in the class, and the products make beautiful wall decorations. Here are some guidelines for the assignment.

1. The fractal should fill the top half of a piece of $8\frac{1}{2} \times 11$ inch plain white paper turned vertically. To give them more space, provide them with legal size paper. Be aware that each student will probably require more than one piece before they create their final product.

2. The fractal should be boldly colored to accentuate the self-similarity.

3. The students should be encouraged to be creative and original in their design.

4. The bottom half of the paper will have a paragraph explaining the self-similarity in the fractal. They should explain why their design is a fractal.

5. Create a rubric to give to the students at the time the project is assigned so that they will feel like they are being graded fairly. It is hard to evaluate artwork in a way that everyone feels is objective.

1.8 Right Triangle Trigonometry

The Pythagorean Theorem

Presenting the Proof – The proof of the Pythagorean theorem given in this lesson provides a wonderful review of, and use for, what the students just learned about similar triangles. Sometimes it is difficult for students to see the three right triangles contained in the figure and how the sides correspond. It is helpful to make an additional drawing of the three triangles so that they are separate, and oriented in the same direction. Using both figures for reference students can more easily verify the proportions used in this proof.

Skipping Around – Not all texts present material in the same order, and many instructors have a preferred way to develop concepts that is not always the same as the one used in the text. The Pythagorean theorem is frequently moved from place to place. If the students have not done similar figures yet, or if area has already been covered, the proof of the Pythagorean theorem given in the exercises may be the better place to focus the students' attention. Proofs are hard for most students to understand. It is important to choose one that the students can feel good about. Don't limit the possibilities to these two, research other methods, and pick the one that is most appropriate for your class. Or better yet, pick the best two or three. Different proofs will appeal to different students.

The Height Must be Measured Along a Segment That is Perpendicular to the Base – When given an isosceles triangle where the altitude is not explicitly shown, student will frequently try to use the length of one of the sides of the triangle for the height. The will do this repeatedly, even after you tell them that they must find the length of the altitude that is perpendicular to the segment that's length is being used for the base in the formula $A = \frac{1}{2}\text{bh}$. Sometimes they do not know what to do, and are just trying something, which is, in a way, admirable. The more common explanation though is that they forget. The students have been using this formula for years, they think they know this material, so they just plug and chug, not realizing that the given information has changed. Remind the students that now that they are in Geometry class, there is an extra step. The new challenge is to find the height, and then they can do the easy part and plug it into the formula.

Derive the Distance Formula – After doing an example with numbers to show how the distance formula is basically just the Pythagorean theorem, use variables to derive the distance formula. Most students will understand the proof if they have seen a number example first. Point out to the students that the number example was inductive reasoning, and the proof was deductive reasoning. Taking the time to do this is a good review of logic and algebra as well as great proof practice.

Converse of The Pythagorean Theorem

Mnemonic Devise for Acute and Obtuse Triangles – Many students have trouble remembering that the inequality with the greater tan is true when the triangle is acute, and that the equation with the less than is true for obtuse triangles. It seams backwards to them. One way to present this relationship is to compare the longest side and the angle opposite of it. In a right triangle, the equation has an equal sing; the hypotenuse is the perfect size. When the longest side of the triangle is shorter than what it would be in a right triangle, the angle opposite that side is also smaller, and the triangle is acute. When the longest side of the triangle is longer than what is would be in a right triangle, the angle opposite that side is also larger, and the triangle is obtuse.

Review Operations with Square Roots – Some of the exercises in this section require students to do operations with square roots. This is an essential skill for working with special right triangle which is an important topic that is also covered in this chapter. Many students struggle with using roots in algebra, and they have probably not thought about this topic for a year. Depending on the level of the class, it may be wise to take a day, or half a day, to review operations with square roots. Here are some sample problems of the basic operations with square roots that student will have to know how to do in order to be successful in this chapter.

Simplify:

1. $\sqrt{9} =$

2. $\sqrt{50} =$

3. $5\sqrt{96} =$

Multiply:

4. $\sqrt{2} * \sqrt{5} =$

5. $9\sqrt{6} * 4\sqrt{7} =$

6. $\sqrt{10} * \sqrt{14} =$

Square:

7. $(\sqrt{7})^2 =$

8. $(3\sqrt{2})^2 =$

Add or Subtract:

9. $\sqrt{3} + 7\sqrt{3} =$

10. $3\sqrt{5} - \sqrt{20} =$

Answers:

1. 3

2. $5\sqrt{2}$

3. $20\sqrt{6}$

4. $\sqrt{10}$

5. $36\sqrt{42}$

6. $\sqrt{140} = 2\sqrt{35}$

7. 7

8. $9 * 2 = 18$

9. $8\sqrt{3}$

10. $3\sqrt{5} - 2\sqrt{5} = \sqrt{5}$

Using Similar Right Triangles

Separate the Three Triangles – The altitude from the right angle of a triangle divides the triangle into two smaller right triangles that are similar to each other, and to the original triangle. All the relationships among the segments in this figure are based on the similarity of the three triangles. Many students have trouble rotating shapes in their minds, or seeing individual polygons when they are overlapping. It is helpful for these students to draw the triangles separately and oriented in the same direction. After going through the process of turning and redrawing the triangles a few times, they will remember how the triangles fit together, and this step will no longer be necessary.

Color-Coded Flashcards – It is difficult to describe in words which segments to use in the geometric mean to find the desired segment. Labeling the figure with variables and using a formula is the standard method. The relationship is easier to remember if the labeling of the triangles is kept the same every time the figure is drawn. What the students need to remember, is the location of the segments relative to each other. Making color-coded pictures or flashcards will be helpful. For each relationship the figure should be drawn on both sides of the card. The segment whose measure is to be found should be highlighted in one color on the front, and on the back, the two segments that need to be used in the geometric mean should be highlighted with two different colors. Using two colors on the back is important because the segments often overlap. Making these cards will be helpful even if the students never use them. Those that have trouble remembering the relationship will use these cards frequently as a reference.

Add a Step and Find the Areas – The exercises in this section have the students find the base or height of triangles. They have all the information that they need to also calculate the areas of these triangles. Students need practice with multi-step problems. Having them find the area will help them think through a more complex problem, and give them practice laying out organized work for calculations that are more complex. Chose to extend the assignment or not based on how well the students are doing with the material, and how much time there is to work on this section.

Additional Exercise:

1. Refer to the figure used to give the relationship of the altitude as the geometric mean of the lengths of the two segments of the hypotenuse on page 478 of the text.

Let $f = 3$ cm and $c = 10$ cm. What are the values of d and e?

Answer:

$$\begin{aligned}
3 &= \sqrt{e} * (10 - e) \qquad\qquad e = 1 \text{ or } 9 \text{ cm, so } d = 9 \text{ or1 cm such that the sum is } 10\\
9 &= 10e - e^2\\
0 &= e^2 - 10e + 9
\end{aligned}$$

Special Right Triangles

Memorize These Ratios – There are some prevalent relationships and formulas in mathematics that need to be committed to long term memory, and the ratios made by the sides of these two special right triangles are definitely among them. Students will use these relationships not only in the rest of this class, but also in trigonometry, and in other future math classes. Students are expected to know these relationships, so the sooner learn to use them and commit them to memory, the better off they will be.

Two is Greater Than the Square Root of Three – One way that students can remember the ratios of the sides of these special right triangles, is to use the fact that in a triangle, the longest side is opposite the largest angle, and the shortest side is opposite the smallest angle. At this point in the class, students know that the hypotenuse is the longest side in a right triangle. What sometimes confuses them is that in the $30 - 60 - 90$ triangle, the ratio of the sides is $1 : 2 : \sqrt{3}$, and if they do not really think about it, they sometimes put the $\sqrt{3}$ as the hypotenuse because it might seem bigger than 2. Using the opposite relationship is a good method to use when working with these triangles. Just bring to the students' attention that $2 > \sqrt{3}$.

Table 1.13

$45-45-90$ Triangle	$30-60-90$ Triangle
$x^2+x^2=c^2$ $2x^2=c^2$ $x\sqrt{2}=c$	$x^2+b^2=(2x)^2$ $x^2+b^2=4x^2$ $b^2=3x^2$ $b=x\sqrt{3}$

Derive with Variables – The beginning of the last chapter offers students a good amount of experience with ratios. If they did well on those sections, it would benefit them to see the derivation of the ratios done with variable expressions. It would give them practice with a rigorous derivation, review and apply the algebra they have learned, and help them see how the triangles can change in size.

Exact vs. Decimal Approximation – Many students do not realize that when they enter $\sqrt{2}$ into a calculator and get 1.414213562, that this decimal is only an approximation of $\sqrt{2}$. They also do not realize that when arithmetic is done with an approximation, that the error usually grown. If 3.2 is rounded to 3, the error is only 0.2, but if the three is now multiplied by five, the result is 15, instead of the 16 it would have been if original the original number had not been rounded. The error has grown to 1.0. Most students find it more difficult to do operations with radical expressions than to put the numbers into their calculator. Making them aware of error magnification will motivate them to learn how to do operations with radicals. In the last step, it may be nice to have a decimal approximation so that the number can be easily compared with other numbers. It is always good to have an exact form for the answer so that the person using your work can round the number to the desired degree of accuracy. Less accuracy is needed for building a deck than sending a robot to Mars?

Tangent Ratio

Trig Thinking – Students sometimes have a difficult time understanding trigonometry when they are first introduced to this new branch of mathematics. It is quite a different way of thinking when compared to algebra or even geometry. Let them know that as they begin their study of trigonometry in the next few sections the calculations won't be difficult, the challenge will be to understand what is being asked. Sometimes students have trouble because they think it must be more difficult than it appears to be. Most students find they like trigonometry once they get the feel of it.

Ratios for a Right Angle – Students will sometimes try to take the sine, cosine or tangent of the right angle in a right triangle. They should soon see that something is amiss since the opposite leg is the hypotenuse. Let them know that there are other methods of finding the tangent of angles 90 degrees or more. The triangle based definitions of the trigonometric functions that the students are learning in this chapter only apply to angles in the interval 0 degrees $< m <$ 90 degrees.

The Ratios of an Angle – The sine, cosine, and tangent are ratios that are associated with a specific angle. Emphasize that there is a pairing between an acute angle measure, and a ratio of side lengths. Sine, cosine, and tangent is best described as functions. If the students' grasp of functions is such that introducing the concept will only confuse matters, the one-to-one correspondence between acute angle and ratio can be taught without getting into the full function definition. When students understand this, they will have an easier time using the notation and understanding that the sine, cosine, and tangent for a specific angle are the same, no matter what right triangle it is being used because all right triangles with that angle will be similar.

Use Similar Triangles – Many students have trouble understanding that the sine, cosine, and tangent of a specific angle measure do not depend on the size of the right triangle used to take the ratio. Take some time to go back and explain why this is true using what the students know about similar triangle. It will be a great review and application.

Sketchpad Activity:

1. Students can construct similar right triangles using dilation from the transformation menu.
2. After choosing a specific angle they should measure the corresponding angle in all the triangles. Each of these measurements should be equal.
3. The legs of all the right triangles can be measured.
4. Then the tangents can be calculated.
5. Student should observe that all of ratios are the same.

Remind the students that if the right triangles have one set of congruent acute angles, then they are similar by the AA Triangle Similarly Postulate. Once the triangles are known to be similar it follows that their sides are proportional. The ratios are written using two sides of one triangle and compared to the ratios of the corresponding sides in the other triangle. This is different but equivalent to the ratios students probably used to find missing sides of similar triangles in previous sections.

Sine and Cosine Ratio

Trig Errors are Hard to Catch – The math of trigonometry is, at the point, not difficult. Not much computation is necessary to chose two number and put them in a ratios. What students need to be aware of is how easy it is to make a little mistake and not realize that there is an error. When solving an equation the answer can be substituted back into the original equation to be checked. The sine and cosine for acute angles do not have a wide range. It is extremely easy to mistakenly use the sine instead of the cosine in an application and. The difference often is small enough to seem reasonable, but still definitely wrong. Ask the student to focus on accuracy as they work with these new concepts. Remind them to be slow and careful.

Something to Consider – Ask the students to combine their knowledge of side-angle relationships in a triangle with the definition of sine. How does the length of the hypotenuse compare to the lengths of the legs of a right triangle? What does that mean about the types of numbers that can be sine ratios? With leading questions like these students should be able to see that the sine ratio for an acute angle will always be less than one. This type of analysis will prepare them for future math classes and increase their analytical thinking skills. It will also be a good review of previous material and help them check there work when they first start writing sine and cosine ratios.

Rationalizing the Denominator – Sometimes student will not recognize that $\frac{1}{\sqrt{2}}$ and $\frac{\sqrt{2}}{2}$ are equivalent. Most likely, they learned how to rationalize denominators in algebra, but it is nice to do a short review before using these types of ratios in trigonometry. Student will have to be able to easily switch between the two forms of the number when working with the unit circle in later classes.

Two-Step Problems – Having the students write sine, cosine, and tangent ratios as part of two-step problems will help them connect the new material that they have learned to other geometry they know. They will remember it longer, and be better able to see where it can be applied.

Key Exercise:

1. $\triangle ABC$ is a right triangle with the right angle at vertex C.

$AC = 3$ cm and $BC = 4$ cm

What is the sign of $\angle A$?

Answer: $AB = 5$ cm by the Pythagorean theorem, therefore $\sin A = \frac{4}{5}$.

Note: The sine of an angle does not have units. The units will cancel out in the ratio.

Inverse Trigonometric Ratios

Regular or Arc – Students will sometimes be confused about when to use the regular trigonometric function and when to use the inverse. They understand to concepts, but do not want to go through the entire thought process each time they must make the decision. I give them this short rule of thumb to help them remember: When looking for a ratio or side length, use regular and when looking for an angle use arc. They can associate "angle" and "arc" in their minds. Use the alliteration.

Which Trig Ratio – A common mistake students make when using the inverse trigonometric functions to find angles in right triangles is to use the wrong function. They may use arcsine instead of arccosine for example. There is a process that students can use to reduce the number of these kinds of errors.

1. First, the students should mark the angle whose measure is to be found. With the angle in question highlighted, it is easier for the students to see the relationship the sides have to that angle. It is fun for the students to use colored pencils, pens, or highlighters.

2. Next, the students should look at the sides with known side measures and determine their relationship to the angle. They can make notes on the triangle, labeling the hypotenuse, the adjacent leg and the opposite let. If they are having trouble with this I have them look for the hypotenuse first and always highlight it green, then they and decide between opposite and adjacent for the remaining to sides.

3. Now, they need to look at the two sides they have chosen, and decide if they need to use sine, cosine, or tangent. It might help to have a mnemonic device to help them remember the definitions of the trigonometric functions. A common one is *soh-cah-toa*. The student can write this abbreviation on the top of every paper and refer to it when necessary. For example, in an exercise, if they decide it is the adjacent leg to the angle, and the hypotenuse that they have measures for, that is the *"ah"* portion of *cah*. They will know to use cosine, and be reminded that the length of the adjacent leg will e in the numerator of the ratio.

Make a Graph – Sometimes student will have a hard time seeing a pattern in a list of numbers. One way to help them remember the general trends in the trigonometric ratios is two have them make a graph. They can put the angle measure on the horizontal axis and the ratio, in decimal form, on the vertical axis. They will have to use different scales, of course. Now they can use their calculators to find the trig values of different angles at every five or ten degrees between zero and ninety and plot points on their graph. The comparison would be most meaningful if they put all three on the same set of axes with different colors. The process of making the graph and the visual representation of the pattern will form an impression in the students' minds that will be useful and lasting.

Acute and Obtuse Triangles

Law of Sines or Law of Cosines – At first, it may be difficult for student to determine if they need to use the Law of Sines or the Law of Cosines to find a measure in a particular situation. Here is a good thought process for them to use.

1. First have them look for the two, fairly easy to recognize, Law of Cosines situations. They have all three sides and are looking for an angle, or have two angles and the included side and are looking for the third side.
2. If it is not one of these, then they need to try to set-up a Law of Sines proportion.

The Third Angle – Remind students that the three angles of a triangle have a sum of 180 degrees, and that this fact is often helpful when applying the Law of Sines or Cosines. Sometimes they may not be able to fine the angle they want directly, but if they find the third angle, they can use the Triangle Sum Theorem to get the measure they need.

Two Exercises in One - Sometimes the students will have to use both the Law of Sines and the Law of Cosines to find a measure in a specific triangle. For instance, let's say they have two sides and the included angle of a triangle, but they do not want to find the third side, they want to find the other angles. It will be necessary to use the Law of Cosines to get the third side, and then use that third side to get the ratio for the triangle so the Law of Sines could be used. Remind students to be creative when solving exercises. They should use all of their mathematical knowledge to figure out the solution.

Triangle Labeling – Stress the labeling convention of using a capital letter for a vertex and the same letter in lower case for the opposite side. It is especially important when using the Law of Cosines to fine an angle. Students need to verify that they start the Law of Cosines with the side opposite of the angle they are finding.

Multiplication Before Addition – At this level, students usually faithfully follow this application of the order of operations except, when using the Law of Cosines to fine an angle. The last term of the Law of Cosines is $-2\text{abcos}C$. This is four values being multiplied together. The cosine function is new to students. They will not see it as the representation of a number and will separate it from the other terms. Frequently, they will subtract the $-2ab$ from the $a^2 + b^2$ or add it to the c^2 on the other side of the equation. Use a big multiplication symbol when writing and using the formula. It can be written, $-2ab * \cos C$, to remind students to use the proper order of operations.

1.9 Circles

About Circles

Circle Vocabulary – This section has quite a few vocabulary words. Some the students will already know, like radius, and some, like secant, will be new. Encourage the students to make flashcards or a vocabulary list. They should know the word definition and have pictures drawn and labeled. It is also important for students to know the relationships between the words. The radius is half the length of the diameter and the diameter is the longest chard in a circle. Make knowing the vocabulary a specific assignment, otherwise many students will forget to take the time to learn the vocabulary well.

Circle or Disk – The phrase "a point on the circle" is commonly used. This will confuses the students that do not realize that the circle is the set of points *exactly* some set distance from the center, and not the points less than or equal to that distance from the center or the circle. What is happening is that they are confusing the definition of a disk and a circle. Emphasize to the students that a circle is one dimensional; it only contains the points on the edge. Another option is to give them the definition of a disc along with that of a circle, so that they can compare and contrast the two definitions.

Inscribed or Circumscribed – An inscribed circle can also be described as a circumscribed polygon. The different ways that these vocabulary words can be used can make learning the relationships complicated. As a guide, tell the students that the object inscribed is on the inside. Starting with that, they can work out the rest. For practice, ask the students to draw different figures that are described in words, like a circumscribed hexagon, or a circle inscribed in an octagon.

Square the Radius – When working with the equation of a circle, students frequently forget that the radius is squared in the equation, especially when the radius is an irrational number. Explaining the equation of the circle in terms of the Pythagorean theorem will help the students remember and understand how to graph this conic section.

Completing the Square – Completing the square to put the equation of a conic section in standard form is a nice little math trick. It exemplifies the kinds of moves mathematicians use to manipulate expressions and equations. Students find it difficult to do especially when fractions are involved and they have trouble retaining the process for more than a few days. Give them many opportunities to practice.

Tangent Lines

Bringing It All Together – This section makes use of many concepts students have previously learned in the class. It will help students to start to prepare for the final, or for an end of the quarter cumulative test. Students will need time to go back and review the topics used in this section as well as the normal time allotted to learn the new material. Below is a list of subjects the students must be competent at to be successful with this section. A day spent reviewing these will help avoid frustration.

Review Topics:

1. The converse of a conditional statement and proof by contradiction
2. Proofs that employ congruent triangles and there corresponding parts
3. The Pythagorean theorem and its converse
4. Equations of lines and circles, including slopes of perpendicular lines
5. The proportionality of the sides of similar triangles
6. Polygons: the sum of interior angles and regular polygons

All the Radii of a Circle Are Congruent – It may seem obvious, but frequently students forget to use the fact that all the radii of a circle are congruent. This follows directly from the definition of a circle. Remind students to use this fact when setting up equations and assigning variables to different radii in the same circle.

Congruent Tangents – In this section the Tangent Segment Theorems is proved and applied. Remind student that this is only true for tangents and does not extend to secants. Sometimes student will see a secant enter a circle and think the distance from the exterior point to where the secant intersects the circle is the same as a tangent or another secant from that same point.

Hidden Tangent Segments – Sometimes it is difficult for students to recognize tangent segments because they are imbedded in a more complex figure, or the tangent segment is extended in some way. A common situation where this occurs is when there is an inscribed circle. Tell the students to be on the lookout for tangent segments. They should look at segments individually and as part of the whole. Sometimes it is helpful to use a small sticky note to cover parts of the figure so they do not distract from the area of focus.

Common Tangents and Tangent Circles

Using Trigonometry to Find Side Measurers in Right Triangles – Using the definitions of sine, cosine, or tangent to find the measures of sides in a right triangle is a common application of trigonometry that is put to use in this section. Students will need a bit of practice and perhaps a step-by-step process when learning this skill. With some experience though, this will become an easy, enjoyable task.

Step-by-Step Process:

1. Highlight the side of the right triangle that's measure is to be found. Place a variable, say x, by that side.
2. Chose one of the acute angles of the right triangle whose measure is known to work from. Highlight this angle in another color.
3. Chose another sides of the triangle whose measure is known. Highlight that side in the same color as the other side.
4. Decide what relationships (opposite leg, adjacent leg, or hypotenuse) the highlighted sides have to the highlighted angle.
5. Decide which of the three trigonometric ratios utilize those side relationships.
6. Write out the definition of that trigonometric ratio.
7. Substitute in the highlighted values.
8. Solve the equation by either multiplying or dividing. It is best to not round the decimal approximation of the trigonometric ratio taken from the calculator. Round after the multiplication or division has taken place.

Key Exercises:

1. $\triangle ABC$ is a right triangle with the right angle at vertex C.

$m\angle A = 52^\circ$ and $AC = 10$ cm. Find BC.

Answer:

$$\tan 52^\circ = \frac{BC}{10} \qquad BC \approx 12.8 \text{ cm}$$

2. $\triangle DEF$ is a right triangle with the right angle at vertex F.

$m\angle D = 22^\circ$ and $DF = 1143$ ft. Find DE.

Answer:

$$\cos 22^\circ = \frac{1143}{DE} \qquad DE \approx 1200 \text{ ft}$$

Arc Measure

Naming Major Arcs and Semicircles – When naming and reading the names of major arcs and semicircles, the three letter system is sometimes confusing for students. When naming an angle with three letters, the first place to look is to the middle letter, the vertex. It is just the opposite for a three letter arc name. First, the students should locate the endpoints of the arc at the ends of the name. For a major arc they have two arcs to choose from. The major arc uses three letters and is the long way around. Any of the other points on the major arc can be used to designate that the long path is being taken. A semicircle divides the circle into two congruent arcs. A third letter is needed to designate which half of the circle is being named.

Look For Diameters – When working exercises that call for students to find the measures of arcs by adding and subtracting arc and angle measures in a circle, students often forget that a diameter divides the circle in half, or into two 180 degree arcs. Remind the students to be on the lookout for diameters when finding arc measures.

Using Trigonometry to Find Angle Measures in Right Triangles – A similar process is needed for finding angles in right triangles as for finding sides in right triangles in the previous lesson. Students need some scaffolding when they first learn to use this method.

1. Highlight the angle whose measure is to be found.
2. Two sides of the right triangle must be known. Highlight these two sides.
3. Decide what relationship the highlighted sides have to the angle in question.
4. Decide which trigonometric ratio used those side relationships.
5. Write and solve an equation. Remember to use the inverse of the trigonometric ratio on the calculator since it is the angle that needs to be found.

Key Example:

1. $\triangle ABC$ is a right triangle with the right angle at vertex C.

$AC = 12$ cm, and $AB = 17$ cm. Find $m\angle B$.

Answer:

$$\sin B = \frac{12}{17} \qquad\qquad m\angle B \approx 45^\circ$$

Chords

Update the Theorem List – Students should be keeping a notebook full of all the theorems they have learned in geometry class. These theorems are like tools that can be used to work exercises and write proofs. This section has quite a few different theorems about the relationships or chords and angles that need to be included in their notebook. Each entry should have the name of the theorem, the written statement of the theorem, and a picture to illustrate the relationship. Not only will this be good reference material, making the notebook will help the students to remember the material.

Algebra Review – Students may need a bit of a review before correctly squaring algebraic expressions and solving quadratic equations in geometric applications.

1. In example three the equation of a line is substituted into the equation of a circle so points of intersection can be found. When the binomial is substituted for the $y-$variable in the circle equation, it must be squared. Students frequently try to "distribute" the square instead of using the FOIL method. Make a point of writing out the binomial twice, and multiplying. Students should know and be able to use the pattern for a perfect square binomial, but they will understand why they have to use the pattern when they see the long way written out once and awhile, and will be more likely to remember.

2. In the same example the quadratic formula is used to solve for the two possible values of the $x-$variable. Students will benefit from a brief explanation of how quadratic equations are solved. First, when the student realized that it is a second degree equation they need to solve for zero. Then the equation can be factored or the quadratic formula can be applied. The students should remember the process quickly when they see it. This is an important topic of algebra, and it is always good to review to eliminate misconceptions.

Tips and Suggestions – There are a few strategies that students should keep in mind when working on the exercises in this section.

1. Draw in segments to create right triangles, central angles, and any other useful geometric objects.

2. Remember to split the length of the chord in half if only half of it is used in a right triangle. Don't just use the numbers that are given. The theorems must be applied to get the correct number, and multiple steps will usually be necessary.

3. Use trigonometry of right triangles to find the angles and segment lengths needed to complete the exercise.

4. Don't forget that all radii are congruent. If you have the length of one radius, you have them all, including the ones you add to the figure.

5. Employ the Pythagorean theorem and any other tool you have from previous lessons that might be useful.

Inscribed Angles

Inscribed Angle or Central Angle – When students spot an arc/angle pair to use in solving a complex circle exercise, the first step is to identify the angle as a central angle, an inscribed angle, or possibly neither. If necessary, they can trace the sides of the angle back from the arc to see where the vertex is located. If the vertex is at the center of the circle, it is a central angle, and the measure of the arc and the angle are equal. If the vertex is on the circle, it is an inscribed angle, and the students must remember to double the angle measure. A good mnemonic device is to think of the arc of an inscribed angle being farther away from the vertex than the arc of a central angle. Therefore the measure of the arc will be larger. If the vertex is at neither the exact center or on the circle, no arc/angle relationship can be determined with only one arc.

What to Look For – Students can be overwhelmed by the number of different relationships that need to be used to solve these circle exercises. Sometimes they can just get paralyzed and not know where to start. In small groups, or as a class, have them create a list of possible tools that are commonly used in these types of situations.

Does the figure contain?

1. A triangle with a sum of 180 degrees

2. A convex quadrilateral with a sum of 360 degrees

3. A right triangle formed with a tangent
4. An isosceles triangle formed with two radii
5. A diameter creating a 180 degree semicircle
6. Arcs covering the entire 360 degrees of the circle
7. Central or Inscribed angles
8. Tangents that form right angles
9. Similar triangle with proportional sides
10. Congruent triangles with congruent corresponding parts

Any New Information is Good – If students can not immediately see how to find the measure they are after, advise them to find any measure they can. This keeps their mind active and working. Frequently, they will be able to use the new information to find other measures, and will eventually work their way around to the desired answer. This might not br the most efficient method, but the students' technique will improve with practice.

Angles of Chords, Secants, and Tangents

Where's the Vertex? – When determining the relationships between angles and arcs in a circle the location of the vertex of the angle is the determining factor. There are four possibilities.

1. The vertex of the angle is at the center of the circle, it is a central angle, and the arc and angle have the same measure.
2. The vertex of the angle is on the circle. The angle could be made by two cords, an inscribed angle, or by a chord and a tangent. In either situation, the measure of the arc is twice that of the angle.
3. The vertex of the angle is inside the circle, but not at the center. In this case two arcs are necessary, and the angle measure is the average of the measures of the arcs cut off by the chords that form the vertical angles.
4. The vertex of the angle is outside the circle. Then the two intersected arcs have to be subtracted and the difference divided by two. Note the similarity to an average.

Students often need help organizing information in this way. It is best to do this with them, as a class activity so that in the future they will be able to do it for themselves.

Use the Arcs – It is typical to have more than one angle intercepting a specific arc. In this case a measure can be moved to an arc and then back out to another angle. Another situation students should look for is when a circle is divided into two arcs. One arc can be represented as $360-$ (an expression for the other arc). Students sometimes miss these kinds of moves. It may be beneficial to have students share with the class the different strategies and patterns they see when working on these exercises.

Additional Exercises:

1. Two tangent segments with a common endpoint intercept a circle dividing it into two arcs, one of which is twice as big as the other. What is the measure of the angle formed by the by the two tangents?

Answer:

$$x + 2x = 360 \qquad \text{angle measure} = (240 - 120) \div 2 = 60 \text{ degrees}$$
$$x = 120$$

2. Two intersecting chords intercept congruent arcs. What kind of angles do the chords form?

Answer: central angles

Segments of Chords, Secants, and Tangents

Chapter Study Sheet – This chapter contains many relationships for students to remember. It would be helpful for them to summarize all of these relationships on a single sheet of paper to use when studying. Some instructors allow students to use these sheets on the exam in order to encourage students to make the sheets. The value of a study sheet is in its making. Students should know this and make them regardless of whether they can be used on the exams. Sometimes if students know that they will be able to use the study sheet, they will not work to remember all of the relationships, and their ability to learn the material is compromised. It is a hard issue to work around and each instructor needs to deal with it as he or she feels best with their particular classes.

When to Add – When writing proportions involving secants, students will have a difficult time remembering to add the two segments together to form the second factor. A careful study of the proof will help them remember this detail. When they see secants, have them picture the similar triangles that could be drawn. Remind them, and give them ample opportunity to practice.

Have Them Subtract – One way to give students more practice with the lengths of secants in circles is to give them exercises where the entire length of the secant is given, and they have to setup an expression using subtraction to use in the proportion.

Key Examples:

1. A secant and a tangent segment have a common exterior endpoint. The secant has a total length of 12 cm and the tangent has length 7 cm. What is the measure of the both segments of the secant?

Answer:

Let one segment of the secant be x, so the other can be represented by $20 - x$.

$$7^2 = (12 - x) * 12$$

The secant is composed of two segments
with approximate lengths of 4.1 cm and 7.9 cm

$$x \approx 7.9$$

2. Two secant segments have a common endpoint outside of a circle. One has interior and exterior segments of lengths 10 ft and 12 ft respectively and the other has a total measure of 18 ft. What is the measure of the two segments composing the other secant?

Answer:

$12(10+12) = (18-x)*18$ The secant is composed of two segments with lengths $3\frac{1}{3}$ ft and $14\frac{2}{3}$ ft

$x = 3\ \frac{1}{3}$

1.10 Perimeter and Area

Triangles and Parallelograms

The Importance of Units – Students will give answers that do not include the proper units, unless it is required by the instructor. When stating an area, square units should be included, and when referring to a length, linear units should be used. Using proper units helps reinforce the basic concepts. With these first simple area problems including the units seems like a small detail, but as the students move to more complex situations combining length, area, and volume, units can be a helpful guide. In physics and chemistry dimensional analysis is an important tool.

The Power of Labeling – When doing an exercise where a figure needs to be broken into polynomials with known area formulas, it is important for the student to draw on and label the figure well. Each polygon, so far only parallelograms and triangles, should have their base and height labeled and the individual area should be in the center of each. By solving these exercises in a neat, orderly way student will avoid errors like using the wrong values in the formulas, overlapping polygons, or leaving out some of the total area.

Subtracting Areas – Another way of finding the area of a figure that is not a standard polygon is to calculate a larger known area and then subtracting off the areas of polygons that are not included in the target area. This can often result in fewer calculations than adding areas. Different minds work in different ways, and this method might appeal to some students. It is nice to give them as many options as possible so they feel they have the freedom to be creative.

The Height Must Be Perpendicular to the Base – Students will frequently take the numbers from a polygon and plug them into the area formula without really thinking about what the numbers represent. In geometry there will frequently be more steps. The students will have to use what they have learned to find the correct base and height and then use those numbers in an area formula. Remind students that they already know how to use a formula; many exercises in this class will require more conceptual work.

Write Out the Formula – When using an area formula, it is a good idea to have the students first write out the formula they are using, substitute numbers in the next step, and then solve the resulting equation. Writing the formula helps them memorize it and also reduces error when substituting and solving. It is especially important when the area is given and the student is solving for a length measurement in the polygon. Students will be able to do these calculations in their heads for parallelograms, and maybe triangles as well, but it is important to start good habits for the more complex polygons to come.

Trapezoids, Rhombi, and Kites

It's Arts and Crafts Time – Student have trouble remembering how to derive the area formulas. At this level it is required that they understand the nature of the formulas and why the formulas work so they can modify and apply them in less straightforward situations. An activity where student follow the explanation by illustrating it with shapes that they cut out and manipulate is much more powerful then just listening and taking notes. It will engage the students, keep their attention, and make them remember the lesson longer.

Trapezoid

1. Have student use the parallel lines on binder paper to draw a trapezoid. They should draw in the height and label it h. They should also label the two bases b_1 and b_2.
2. Now they can trace and cut out a second congruent trapezoid and label it as they did the first.
3. The two trapezoids can be arranged into a parallelogram and glued down to another piece of paper.
4. Identify the base and height of the parallelogram in terms of the trapezoid variables. Then substitute these expressions into the area formula of a parallelogram to derive the area formula for a trapezoid.
5. Remember that two congruent trapezoids were used in the parallelogram, and the formula should only find the area of one trapezoid.

Kite

1. Have the students draw a kite. They should start by making perpendicular diagonals, one of which is bisecting the other. Then they can connect the vertices to form a kite.
2. Now they can draw in the rectangle around the kite.
3. Identify the base and height of the parallelogram in terms of d_1 and d_2, and then substitute into the parallelogram area formula to derive the kite area formula.
4. Now have the students cut off the four triangles that are not part of the kite and arrange them over the congruent triangle in the kite to demonstrate that the area of the kite is half the area of the rectangle.

Rhombus

The area of a rhombus can be found using either the kite or parallelogram area formulas. Use this as an opportunity to review subsets and what they mean in terms of applying formulas and theorems.

Areas of Similar Polygons

Adjust the Scale Factor - It is difficult for students to remember to square and cube the scale factor when writing proportions involving area and volume. Writing and solving a proportion is a skill they know well and have used frequently. Once the process is started, it is hard to remember to add that extra step of checking and adjusting the scale factor in the middle of the process. Here are some ways to reinforce this step in the students' minds.

1. Inform students that this material is frequently used on the SAT and other standardized tests in some of the more difficult problems.
2. Play with graph paper. Have students draw similar shape on graph paper. They can estimate the area by counting squares, and then compare the ratio of the areas to the ratio of the side lengths. Creating the shapes on graph paper will give the students a good visual impression of the areas.
3. Write out steps, or have the students write out the process they will use to tackle these problems. (1) Write a ratio comparing the two polygons. (2) Identify the type of ratio: linear, area, or volume. (3) Adjust the ratio using powers or roots to get the desired ratio. (4) Write and solve a proportion.

4. Mix-up the exercises so that students will have to square the ratio in one problem and not in the next. Keep them on the lookout. Make them analyze the situation instead of falling into a habit.

Additional Exercises:

1. The ratio of the lengths of the sides of two squares is 1 : 2. What is the ratio of their areas?

Answer: 1 : 4

2. The area of a small triangle is 15 cm^2, and has a height of 5 cm. A larger similar triangle has an area of 60 cm^2. What is the corresponding height of the larger triangle?

Answer:

area ratio 15 : 60 or 1 : 4 height linear ratio 1 : 2 = 10 cm

Height of larger triangle is twice the of the smaller triangle. $5 * 2$

3. The ratio of the lengths of two similar rectangles is 2 : 3. The larger rectangle has a width of 18 cm. What is the width of the smaller rectangle?

Answer:

$$\frac{2}{3} = \frac{x}{18}$$
$$x = 12$$

The width of the smaller rectangle is 12 cm.

Circumference and Arc Length

Pi is an Irrational Number – Many students can give the definition of an irrational number. They know that an irrational number has an infinite decimal that has no pattern, but they have not really internalized what this means. Infinity is a difficult concept. A fun way to help the students develop this concept is to have a pi contest. The students can chose to compete by memorizing digits of pi. They can be given points, possible extra credit, for ever ten digits or so, and the winner gets a pie of their choice. The students can also research records for memorizing digits of pi. The competition can be done on March 14^{th}, pi day. When the contest is introduced, there is always a student who asks "How many points do I get if I memorize it all?" It is a fun way to reinforce the concept of irrational numbers and generate a little excitement in math class.

How is Pi Calculated? - Students frequently ask how mathematicians calculate pi and how far they have gotten. One method is by approximating the circumference of a circle with inscribed or circumscribed polygons. Inspired students can try writing the code themselves, and possibly sharing it with the class. There are many other more commonly used methods, but they involve calculus or other mathematics that is beyond geometry students.

There Are Two Values That Describe an Arc – The measure of an arc describes how curved the arc is, and the length describes the size of the arc. Whenever possible, have the students give both values with units so that they will remember that there are two different numerical descriptions of an arc. Often student will give the measure of an arc when asked to calculate its length.

Arc Length Fractions – Fractions are a difficult concept for many students even when they have come as far as geometry. For many of them putting the arc measure over 360 does not obviously give the part of the circumference included in the arc. It is best to start with easy fractions. Use a semi-circle and show how 180/360 reduces to $\frac{1}{2}$, then a ninety degree arc, and then a 120 degree arc. After some practice with fractions they can easily visualize, the students will be able to work with that eighty degree arc.

Exact or Approximate – When dealing with the circumference of a circle there are often two ways to express the answer. The students can give exact answers, such as 2π cm or the decimal approximation 6.28 cm. Explain the strengths and weaknesses of both types of answers. It is hard to visualize 13π feet, but that is the only way to accurately express the circumference of a circle with diameter 13 ft. The decimal approximations $41, 40.8, 40.84$, can be calculated to any degree of accuracy, are easy to understand in terms of length, but are always slightly wrong. Let the students know if they should give one, the other, or both forms of the answer.

Circles and Sectors

Reinforce – This section on area of a circle and the area of a sector is analogous to the previous section about circumference of a circle and arc length. This gives students another chance to go back over the arguments and logic to better understand, remember, and apply them. Focus on the same key points and methods in this section, and compare it to the previous section. Mix-up exercises so students will see the similarities and learn each more thoroughly.

Don't Forget the Units – Remind students that when they calculate an area the units are squared. When an answer contains the pi symbol, students are more likely to leave off the units. In the answer 7π cm^2, the π is part of the number and the cm^2 are units of area.

Draw a Picture – When applying geometry to the world around us, it is helpful to draw, label, and work with a picture. Visually organize information is a powerful tool. Remind students to take the time for this step when calculating the areas of the irregular shapes that surround us.

Additional Exercises:

1. What is the area between two concentric circles with radii 5 cm and 12 cm?

(Hint: Don't subtract the radii.)

Answer: $144\pi - 25\pi = 119\pi$ cm^2

2. The area of a sector of a circle with radius 6 m, is 12π cm^2. What is the measure of the central angle that defines the sector?

Answer:

$$12\pi = \frac{x}{360} * \pi * 6^2$$
$$x = 120$$

The central angle measures 120 degrees.

3. A square with side length $5\sqrt{2}$ cm is inscribed in a circle. What is the area of the region between the square and the circle?

Answer: $\pi * 5^2 - (5\sqrt{2})^2 = 25\pi - 50 \approx 28.5$ cm^2

Regular Polygons

The Regular Hypothesis – Make it clear to students that these formulas only work for regular polygons, that is, polygons will all congruent sides and angles. The regular restriction is part of the hypothesis. Many times the hypotheses of important theorems in mathematics are quite restrictive, but that does not necessarily limit the value of the theorem. If a polygon is approximately regular, then the formula can be used to get an approximate area. Also, the method of breaking the polygon into triangles can be applied to non-regular polygons, but each triangle may be different and therefore each area computed separately. It is important to understand how the theorem or formula was derived so it can be adapted to other situations. Knowing this will motivate students to work to understand the formulas.

Numerous Variables and Relationships – Polygons come with an entire new set of variable. Students need to learn what these knew variable represent, how they are related to the triangles that makeup the polygon, and how they are related to each other. This will take some time and practice. If students do not have time to memorize what the variables represent, they will not understand how they are being put together in the various formulas. Find convenient breaking points and give students time, examples, or activities to help them become familiar with the material. If it all comes too fast, student will get lost and frustrated.

What is n? - Students will frequently be given the value of n, but will not realize it because it is not given in the form they are expecting. An exercise may state, "Each side of a regular hexagon is nine inches long." The students will see the nine and assign it to the variable s, but not notice that they are also being given the value of n; a hexagon has six sides.

Let the Radius be One – The simplification of only considering polygons inscribed in a unit circle by letting the radius be equal to one, may seem a bit odd to students. Let them know that this is being done in preparation for more advanced trigonometry. In trigonometry ratios of side lengths of similar triangles are considered, and the size of the triangles in not important. Letting them know that this simplification becomes useful in the future will reassure them that the course of their mathematics education is well designed.

Additional Exercises:

1. The area of a regular hexagon is $24\sqrt{3}\ \text{cm}^2$. What is the length of each side?

Answer:

$$24\sqrt{3} = \frac{1}{2} * s * \frac{1}{2} s\sqrt{3} * 6$$
$$s = 4$$

Each side has a length of 4 cm.

Geometric Probability

Count Carefully – Counting is the most challenging aspect of probability for students. It is easy to make an error when thinking of all the possible outcomes and determining how many of them are favorable. The best way to guard against errors is to make logical, orderly lists. The goal is for the students to see a pattern so that eventually they will be able to get the count without listing all of the possibilities.

Table 1.14

Outcome	Favorable?	Outcome	Favorable?
(N_1, N_2)	No	(D, N_1)	No
(N_1, D)	No	(D, N_2)	No
(N_1, Q)	Yes	(D, Q)	Yes
(N_2, N_1)	No	(Q, N_1)	Yes
(N_2, D)	No	(Q, N_2)	Yes
(N_2, Q)	Yes	(Q, D)	Yes

Does Order Matter? – One of the hardest decisions for a new probability student to make when analyzing a situation is to determine if different permutations should be counted separately or not. Take Example Two from this section. Is there a first coin and a second coin? Does it matter if Charmane sees the quarter of the dime first? In this case, it does not because the two coins are taken at the same time. To compare have the students consider the situation where first one coin is drawn and then a second. In this case there are more possible outcomes. The probability remains the same, 50%, but now it is calculated as $\frac{6}{12}$ instead of $\frac{3}{6}$. It is not always the case that considering order results in the same probability as when order is not considered. Note that this example can be reduced to the simpler question of whether a quarter is one of the two coins drawn.

Additional Exercises:

1. A man throws a dart at a circular target with radius 6 inches. He is equally likely to hit anywhere in the target. What is the probability that he is within 2 inches of the center of the target?

Answer:

$\frac{\pi 2^2}{\pi 6^2} = \frac{1}{9} \approx .11$ There is an approximately 11% chance that he will hit within 2 inches of the target.

2. There is a 110 mile stretch of road between the centers of two cities. A hospital is located 30 miles from the center of one city. If an accident is equally likely to occur anywhere between the two cities, what is the probability it is within ten miles of the hospital?

Answer:

$\frac{20}{110} \approx .18$ There is an approximately 18% chance that the accident will be within 10 miles of the hospital.

1.11 Surface Area and Volume

The Polyhedron

Polygon or Polyhedron – A polyhedron is defined using polygons, so in the beginning students will understand the difference. After some time has passes though, students tend to get these similar sounding words confused. Remind them that polygons are two-dimensional and polyhedrons are three-dimensional. The extra letters in polyhedron represents it spreading out into three-dimensions.

The Limitations of Two Dimensions – It is difficult for students to see the two-dimensional representations of three-dimensional figures provided in books and on computer screens. A set of geometric solids is easily obtained through teacher supply companies, and are extremely helpful for students as they familiarize themselves with three-dimensional figures. When first counting faces, edges, and vertices most students need to hold the solid in their hands, turn it around, and see how it is put together. After they have some experience with these objects, student will be better able to read the figures drawn in the text to represent three-dimensional objects.

Assemble Solids – A valuable exercise for students as they learn about polyhedrons is to make their own. Students can cut out polygons from light cardboard and assemble them into polyhedrons. Patterns are readily available. This hands-on experience with how three-dimensional shapes are put together will help them develop the visualization skills required to count faces, edges, and vertices of polyhedrons described to them.

Computer Representations – When shopping on-line it is possible to "grab" and turn merchandise so that they can be seen from different perspectives. The same can be done with polyhedrons. With a little poking around students can find sights that will let them virtually manipulate a three-dimensional shape. This is another possible option to develop the students' ability to visualize the solids they will be working with for the remainder of this chapter.

Using the Contrapositive – If students have already learned about conditional statements, point out to them that Example Six in this section makes use of the contrapositive. Euler's formula states that if a solid is a polyhedron, then $v + f = e + 2$. The contrapositive is that if $v + f \neq e + 2$, then the solid is not a polyhedron. Students need periodic review of important concepts in order to transfer them to their long-term memory. For more review of conditional statements, see the second section of the second chapter of this text.

Representing Solids

Each Representation Has Its Use – Each of the methods for making two-dimensional representations of three-dimensional figures was developed for a specific reason and different representation are most appropriate depending on what aspect of the geometric solid is of interest.

Perspective – used in art, and when one wants to make the representation look realistic

Isometric View – used when finding volume

Orthographic View – used when finding surface area

Cross Section – used when finding volume and the study of conic sections (circles, ellipses, parabolas, and hyperbolas) is based on the cross sections of a cone

Nets – used when finding surface area or assembling solids

Ask the students to think of other uses for these representations.

When students know and fully understand the options, they will be able to choose the best tool for each task they undertake.

Isometric Dot Paper – If students are having trouble making isometric drawings, they might benefit from the use of isometric dot paper. The spacing of the dots allows students to make consistent lengths and angles on their polyhedrons. After some practice with the dot paper, they should be able to make decent drawings on any paper. A good drawing will be helpful when calculating volumes and surface areas.

Practice – Most students will need to make quite a few drawings before the result is good enough to be helpful when making calculations. The process of making these representations provides the student with an opportunity to contemplate three-dimensional polyhedrons. The better their concept of these solids, the easier it will be for them to calculate surface areas and volumes in the sections to come.

Additional Exercises:

1. In the next week look around you for polyhedrons. Some example may be a cereal box or a door stop. Make a two-dimensional representation of the object. Choose four objects and use a different method of representation for each.

These can make nice decorations for classroom walls and the assignment makes students look for way to apply what they will learn in this chapter about surface area and volume.

Prisms

The Proper Units – Students will frequently use volume units when reporting a surface area. Because the number describes a three-dimensional figure, the use of cubic units seems appropriate. This shows a lack of understanding of what exactly it is that they are calculating. Provide students with some familiar applications of surface area like wrapping a present or painting a room, to improve their understanding of the concept. Insist on the use of correct units so the student will have to consider what exactly is being calculated in each exercise.

Review Area Formulas – Calculating the surface area and volume of polyhedrons requires the students to find the areas of different polygons. Before starting in on the new material, take some time to review the area formulas for the polygons that will be used in the lesson. When students are comfortable with the basic area calculations, they can focus their attention on the new skill of working with three-dimensional solids.

A Prism Does Will Not Always Be Sitting On Its Base – When identifying prisms, calculating volumes, or using the perimeter method for calculating surface area, it is necessary to locate the bases. Students sometimes have trouble with this when the polyhedron in question is not sitting on its base. Remind students that the mathematical definition of the bases of a prism is two parallel congruent polygons, not the common language definition of a base, which is something an object sits on. Once students think they have identified the bases, they can check that any cross-section taken parallel to the bases is congruent to the bases. Thinking about the cross-sections will also help them understand why the volume formula works.

Additional Exercises:

1. A prism has a base with area 15 cm^2 and a height of 10 cm. What is the volume of the prism?

Answer: $V = 15 * 10 = 150 \text{ cm}^3$

2. A triangular prism has a height of 7 cm. Its base is an equilateral triangle with side length 4 cm. What is the volume of the prism?

Answer: $V = Bh = \frac{1}{2} * 4 * 2\sqrt{3} * 7 = 28\sqrt{3} \text{ cm}^3 \approx 48.5 \text{ cm}^3$

3. The volume of a cube is 27 cm^3. What is the cube's surface area?

Answer: $SA = 3 * 3 * 6 = 36 \text{ cm}^2$

Cylinders

Understand the Formula – Many times students think it is enough to remember and know how to apply a formula. They do no see why it is necessary to understand how and why it works. The benefit of fully understanding what the formula is doing is versatility. Substituting and simplifying works wonderfully for standard cylinders, but what if the surface area of a composite solid needs to be found.

Key Exercise:

1. Find the surface area of the piece of pipe illustrated in the composite solid part of this section.

Answer: $A = 2B + L = 2 * (\pi 3^2 + \pi 2^2) + 2\pi * 3 * 5 = 56\pi \text{ cm}^2 \approx 176.9 \text{ cm}^2$

Make and Take Apart a Cylinder – Students have a difficult time understanding that the length of the rectangle that composes the lateral area of a cylinder has length equal to the circumference of the circular base. First, review the definition of circumference with the students. A good way to describe the circumference is to talk about an ant walking around the circle. Next, let them play with some paper cylinders. Have them cut out circular bases, and then fit a rectangle to the circles to make the lateral surface.

After some time spent trying to tape the rectangle to the circle, they will understand that the length of the rectangle matches up with the outside of the circle, and therefore, must be the same as the circumference of the circle.

The Volume Base – In the past, when students used formulas, they just needed to identify the correct number to substitute in for each variable. Calculating a volume requires more steps. To find the correct value to substitute into the B in the formula $V = Bh$, usually requires doing a calculation with an area formula. Students will often forget this step, and use the length of the base of the polygon that is the base of the prism for the B. Emphasize the difference between b, the linear measurement of the length of a side of a polygon, and B, the area of the two-dimensional polygon that is the base of the prism. Students can use dimensional analysis to check their work. Volume is measured in cubic units, so three linear measurements, or a linear unit and a squared unit must be fed into the formula.

Additional Exercises:

1. The volume of a 4 in tall coffee cup is approximately 50 in^3. What is the radius of the base of the cup?

Answer:

$$50 = \pi r^2 * 4$$
$$r \approx 2$$

The cup has a radius of approximately 2 inches.

Pyramids

Don't Forget the $\frac{1}{3}$ – The most common mistake students make when calculating the volume of a pyramid is to forget to divide by three. They also might mistakenly divide by three when trying to find the volume of a prim. The first step students should take when beginning a volume calculation, is to make the decision if the solid is a prism or a pyramid. Once they have chosen, they should immediately write down the correct volume formula.

Prism or Pyramid – Some students have trouble deciding if a solid is a prism or a pyramid. Most try to make the determination by looking for the bases. This is especially tricky if the figure is not sitting on its base. Another method for differentiating between these solids is to look at the lateral faces. If there are a large number of parallelograms, the figure is probably a prism. If there are more triangles, the figure is most likely a pyramid. Once the student has located the lateral faces, then they can make a more detailed inspection of the base or bases.

Height, Slant Height, or Edge – A pyramid contains a number of segments with endpoints at the vertex of the pyramid. There is the altitude which is located inside right pyramids, the slant height of the pyramid is the height of the triangular lateral faces, and there are lateral edges, where two lateral faces intersect. Students frequently get these segments confused. To improve their understanding, give them the opportunity to explore with three-dimensional pyramids. Have the students build pyramids out of paper or light cardboard. The slant height of the pyramid should be highlighted along each lateral face in one color, and the edges where the lateral faces come together in another color. A string can be hung form the vertex to represent the altitude of the pyramid. The lengths of all of these segments should be carefully measured and compared. They should make detailed observations before and after the pyramid is assembled. Once the students have gained some familiarity with pyramids and these different segments, it will make intuitive sense to them to use the height when calculating volume, and the slant height for surface area.

Additional Exercises:

1. A square pyramid is placed on top of a cube. The cube has side length 3 cm. The slant height of the triangular lateral faces of the pyramid is 2 cm. What is the surface area of this composite solid?

Answer:

$$
\begin{aligned}
A &= 5 * \text{area of the square} + 4 * \text{area of the triangle} \\
&= 5 * (3 * 3) + 4 * \left(\frac{1}{2} * 3 * 2\right) \\
&= 57 \text{ cm}^2
\end{aligned}
$$

Note: The top face of the cube is covered by the base of the pyramid so neither square is included in the surface area of the composite figure.

Cones

Mix'em Up – Students have just learned to calculate the surface area and volume of prisms, cylinders, and cones. Most students do quite well when focused on one type of solid. They remember the formulas and how to apply them. It is a bit more difficult when students have to choose between the formulas for all four solids. Take a review day here. Have the students work in small groups during class on a worksheet or group quiz that has a mixture of volume and surface area exercises for these four solids. The extra day will greatly help to solidify the material learned in the last few lessons.

Additional Exercises:

1. A cone of height 9 cm sits on top of a cylinder of height 12 cm. Both cone and cylinder have radius with length 4 cm. Find the volume and surface area of the composite solid.

Answer:

$$V = \pi r^2 h_1 + \frac{1}{3}\pi r^2 h_2 = \pi * 4^2 * 12 + \frac{1}{3} * \pi * 4^2 * 9 = 240\pi \text{ cm}^3 \approx 750 \text{ cm}^3$$

$$A = \pi r^2 + 2\pi rh + \pi rl \qquad \text{and} \qquad \begin{aligned} r^2 + h_2 &= l^2 \\ 4^2 + 9^2 &= l^2 \\ l &\approx 9.8 \end{aligned}$$

$$
\begin{aligned}
A &\approx \pi 4^2 + 2\pi * 4 * 12 + \pi * 4 * 9.8 \\
A &\approx 475 \text{ cm}^2
\end{aligned}
$$

2. A cone of radius 7 cm is carved out of a square prism of the same height. The square base of the prism has area 225 cm^2 and height 30 cm. What is the volume of this composite solid?

Answer:

$$
\begin{aligned}
V &= \text{Volume of Prism–Volume of Cone} \\
V &= 225 * 30 – \frac{1}{3} * \pi 7^2 * 30 = 6750 – 490\pi \text{ cm}^3 \approx 5210 \text{ cm}^3
\end{aligned}
$$

Spheres

Expand on Circles – Students learned about circles earlier in the course. Review and expand on this knowledge as they learn about spheres. Ask the students what they know about circles. Being able to demonstrate their knowledge will build their confidence and activate their minds. Now modify the definitions that the students have provided to fit the three-dimensional sphere. Students will learn the new material quickly and will remember it because it is now neatly filed away with their knowledge of circles.

Explore Cross-Sections – One of the goals of this chapter is to develop the students' ability to think about three-dimensional objects. Most students will need a significant amount of practice before becoming competent at this skill. Take some time and ask the students to think about what the cross-sections of a sphere and a plane will look like. Explore trends. What happens to the cross-section as the plane moves farther away from the center of the circle? A cross-section that passes through the center of the sphere makes the largest possible circle, or the great circle of the sphere.

Cylinder to Sphere – It would be a good exercise for students to take the formula for the surface area of a cylinder and derive the formula for the surface area of a sphere. It is just a matter of switching a few variables, but it would be a good exercise for them. During the lesson, ask them to do it in their notes, wait a few minutes and then do it on the board or ask one of them to put their work on the board. It should look something like this:

$$\begin{aligned}
\text{A}_{\text{cylinder}} &= \text{bases} + \text{lateral area} \\
\text{A}_{\text{cylinder}} &= 2 * \pi r^2 + 2\pi r h \\
\text{A}_{\text{sphere}} &= 2 * \pi r^2 + 2\pi r * r && \text{substitute in } h = r \\
\text{A}_{\text{sphere}} &= 2 * \pi r^2 + 2\pi\ r^2 \\
\text{A}_{\text{sphere}} &= 4\pi r^2
\end{aligned}$$

Note to students that in the last line the like terms where combine. This can only be done because both terms had the factor πr^2. The coefficients could have been different, but to combine terms using the distributive property they must have the exact same variable combination. Here the π is being treated as a variable even though it represents a number. This is a more complex application of like terms than students are used to seeing.

Limits – The formula for the volume of a sphere is developed using the idea of a limit. Explain this to the students or the logic might seem fuzzy to them. The limit is a fundamental concept to all of calculus. It is worthwhile to give it some attention here.

Similar Solids

Surface Area is Squared – Surface area is a two-dimensional measurement taken of a three-dimensional object. Students are often distracted by the solid and use cubed units when calculating surface area or mistakenly cube the ratio of linear measurements of similar solids when trying to find the ratio of the surface areas. Remind them, and give them many opportunities to practice with exercises where surface area and volume are both used.

Don't Forget to Adjust the Ratio – There are three distinct ratios that describe the relationship between similar solids. When the different ratios and their uses are the subject of the lesson, students usually remember to use the correct ratio for the given situation. In a few weeks when it comes to the chapter test or on the final at the end of the year, students will frequently forget that the area ratio is different from the volume ratio and the linear ratio. They enjoy writing proportions and when they recognize that a proportion will be used, they get right to it without analyzing the ratios. One way to remind them is to have them use units when writing proportions. The units on both sides of the equal sign have to match before they can cross-multiply. Give them opportunities to consider the relationship between the different ratios with questions like the one below.

Key Exercise:

1. If a fully reduced ratio is raised to a power, will the resulting ratio be fully reduced? Explain your reasoning.

Answer: Yes, two numbers make a fully reduced ratio if they have no common factors. Raising a number to a power increases the exponent of each factor already present, but does not introduce new factors. Therefore, the resulting two numbers will still not have any common factors.

These concepts frequently appear on the SAT. It will serve the students well to practice them from time to time to keep the knowledge fresh.

Additional Exercises:

1. The ratio of the surface area of two cubes is $25 : 49$. What is the ratio of their volumes?

Answer:

$$(\sqrt{25})^3 : (\sqrt{49})^3$$
$$(5)^3 : (7)^3$$
$$125 : 343$$

1.12 Transformations

Translations

Translation or Transformation – The words translation and transformation look and sound quite similar to students at first. Emphasis their relationship. A translation is just one of the many transformations the students will be learning about in this chapter.

Point or Vector – There are two mathematical objects being use in this lesson that have extremely similar looking notation. An ordered pair is use to represent a location on the coordinate plane, and it also is used to represent movement in the form of a vector. Some texts represent the translation vector as a mapping. The vector $v = (-3, 7)$ would be written $(x, y) \rightarrow (x - 3, y + 7)$. This makes distinguishing between the two easier, but does not introduce the student to the important concept of a vector. It can be used though if the students are having a really hard time with notation.

The Power of Good Notation – There is a lot going on in these exercises. There are the points that make the preimage, the corresponding points of the image, and the vector that describes the translation. Good notation is the key to keeping all of this straight. The points of the image should be labeled with capital letters, and the prime marks should be used on the points of the image. In this way it is easy to see where each point has gone. This will be even more important when working with more complex transformations in later sections. Start good habits now. A vector should be named with a bold, lower case letter, usually from the end of the alphabet. Just writing $(9, 6)$ is a bit ambiguous, but labeling the vector $u = (9, 6)$, will make the meaning clear. In time students will be able to understand the meaning from the context, but when they are first learning good notation can avoid confusion and frustration.

Use Graph Paper and a Ruler – When making graphs of these translations by had, insist that the students use graph paper and a ruler. If students try to graph on binder paper, the result is frequently messy and inaccurate. It is beneficial for students to see that the preimage and image are congruent to reinforce the knowledge that a translation is an isometry. It is also important that students take pride in producing quality work. They will learn so much more when they take the time to do an assignment well, instead of just rushing through the work.

Translations of Sketchpad – Geometers' Sketchpad uses vectors to translate figures. The program will display the preimage, vector, and image at the same time. Students can type in the vector and can also drag points on the screen to see how the image moves when the vector is changed. It is a quick and engaging way to explore the relationships. If the students have access to Sketchpad and there is a little class time available, it is a worthwhile activity.

Matrices

Rows Then Columns – The dimensions of a matrix are stated by first stating the number of rows and then the number of columns. It may take some time for the students to remember this convention. Give them many opportunities to practice. In this lesson it is important to state the dimensions correctly because only matrices with the same dimensions can be added. The next lesson requires students to decide if two matrices can be multiplied. The order of the dimensions is critical in making that determination.

Additional Exercises:

1. What are the dimensions of any matrix that translates the vertices of a heptagon on the coordinate plane? Explain the significance of the numbers you used.

Answer: 7×2 The number of rows is seven because each of the seven vertices of the heptagon must be moved. The second number is two because each vertex has an $x-$coordinate and a $y-$coordinate that must be changed.

2. The additive identity for real numbers is the number zero because:

$a + 0 = a$, for all real numbers a.

What is the additive identity for 3×2 matrices?

Answer: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

3. If the matrix $\begin{bmatrix} 2 & -3 \\ 2 & -4 \\ 2 & -3 \end{bmatrix}$ were added to matrix a 3×2 matrix containing the vertices of a triangle would the resulting transformation be an isometry? Why or why not?

Answer: No, in an isometry each point must be moved the same distance in order to preserve size and shape. This matrix moves the second vertex farther down than it moves the other two vertices.

Reflections

Matrix Multiplication – It will take some time and practice for students to become proficient with matrix multiplication. On first inspection of the formula it is sometimes hard to see where all the numbers are coming from and where they are going. For many students a spatial representation is more useful. Here are some guidelines that will help students master matrix multiplication.

1. Use the rows of the first matrix and the columns of the second matrix.

2. Move across and down using each number only once.

3. The resulting sum of the products goes in the slot determined by the row of the first matrix and the column of the second matrix.

Think Don't Memorize – Many students will try to learn the reflection matrices using rote memory. This is difficult to begin with since the matrices are fairly similar, but it is also not a good method of learning the material because the knowledge will not last. As soon as the students stop regularly using the matrices, they will be forgotten. Instead have the students think about why the matrices produce their intended effect. When the students really understand what is happening, they will not need to memorize patterns of ones, negative ones, and zeros. The knowledge will be long lasting, and they will be able to develop new matrices that represent other types of transformation and other operations.

Additional Exercises:

1. Let matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and matrix $B = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$. Calculate AB and BA. What do you notice about the products? Will this happen with all 2×2 matrices? What is special about matrix A and matrix B that allowed this special result?

Answer: $AB = BA = \begin{bmatrix} 10 & 11 \\ 11 & 10 \end{bmatrix}$. Matrix A and B commute. This property does not hold for most matrices. These matrices commute because they are symmetric matrices. A symmetric matrix is one in which the rows and columns of the matrix are the same.

Rotation

The Trigonometry – The general rotation matrix uses the trigonometric functions, sine and cosine of the angle of rotation. Students are generally introduced to right triangle trigonometry in the third quarter of geometry. This means they understand the meaning of the trigonometric ratios sine, cosine, and tangent for acute angles in right triangles. In later courses students are introduced to the unit circle which enables them to expand the domain of the trigonometric functions to all real numbers. This is a good point to preview the upcoming material. Tell students that they will shortly learn a method for finding $\sin 90^\circ$ and $\cos 120^\circ$; let them know that it is a brilliant method used to expand these extremely useful definitions. For now though, they will have to trust the number given to them by their calculator when using the general rotation matrix. It is not practical to give them the full explanation now, but letting them know that there is an explanation, and that they will learn it soon, will avoid confusion.

Degrees or Radians – If students have followed standard high school mathematical curriculum, they have no idea what a radian is, but somehow calculators frequently end up in radian mode. Students are not familiar enough with trigonometric functions to realize that they are not getting the correct ratio, and continue with their calculations resulting in incorrect answers. Show the students how to check if their calculators are in radian or degree mode and how to change the mode. This is also a good opportunity to start students thinking about different ways to measure angles. Let them know that radians are another unit used to measure rotation. Similar to how both inches and centimeters both measure length. Making students aware of radians now, will help them avoid errors and ready their minds to learn about radians in the future.

Additional Exercises:

1. Graph $\triangle ABC$ with $A(2, 1), B(5, 3)$, and $C(4, 4)$.

2. Put the vertices of the triangle in a 3×2 matrix and use matrix multiplication to rotate the triangle 45 degrees. Graph the image of $\triangle ABC$ on the same set of axis using prime notation.

3. Use the matrix multiplication to rotate $\triangle A'B'C'$ 60 degrees. To do this, take the matrix produced in #2 and multiply it by the rotation matrix for 60 degrees. Graph the resulting triangle on the same set of axis as $\triangle A''B''C''$.

4. What single matrix could have rotated $\triangle ABC$ to $\triangle A''B''C''$ in one step? How does this matrix compare to the 45 degree and 60 degree rotation matrix?

Answers:

1.

2. $\begin{bmatrix} 2 & 1 \\ 5 & 3 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} .707 & .707 \\ -.707 & .707 \end{bmatrix} = \begin{bmatrix} .707 & 2.12 \\ 1.41 & 5.66 \\ 0 & 5.66 \end{bmatrix}$

3. $\begin{bmatrix} .707 & 2.12 \\ 1.41 & 5.66 \\ 0 & 5.66 \end{bmatrix} \begin{bmatrix} .5 & .866 \\ -.866 & .5 \end{bmatrix} = \begin{bmatrix} -1.48 & 1.67 \\ -4.20 & 4.05 \\ -4.90 & 2.83 \end{bmatrix}$

4. $\begin{bmatrix} -0.26 & 1.00 \\ -1.00 & -0.26 \end{bmatrix} = \begin{bmatrix} .707 & .707 \\ -.707 & .707 \end{bmatrix} \begin{bmatrix} .5 & .866 \\ -.866 & .5 \end{bmatrix}$

Composition

Use the Image – When first working with compositions student often try to apply both operations to the original figure. Emphasis that a composition is a two-step process 83s. The second step of which is performed on the result of the first step. Remind them of the composition of two functions if they have already learned about this topic.

Associative Property of Matrices – Students have heard about the commutative and associate properties many times during their education in mathematics. These are important properties in the study of matrices and become more meaningful for the students when applied to this new set. The fact that the commutative property does not hold for matrix multiplication is surprising at first, and is a concept that needs to be revisited. Although it has been discussed in recent lessons, it would be beneficial to go over it again here before discussing the property that is really of interest in this section, the associate property of matrix multiplication. So far the students have seen that the image of points can be found under a rotation or reflection by multiplying a matrix made up of the coordinates of the points by a matrix specific to the chosen transformation. In this section they should discover that because matrix multiplication is associative, they can multiply two or more transformation matrices together to get a matrix for the composition.

Key Exercises:

Consider the triangle with matrix representation $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \\ 4 & 4 \end{bmatrix}$.

Matrix Multiplication is NOT Commutative

1. Use matrix multiplication to rotate the triangle 90 degrees, then take the image and reflect it in the $x-$axis.

2. Use matrix multiplication to reflect the original triangle in the $x-$axis, then take the image and rotate it 90 degrees.

3. Are the results of #1 and #2 the same?

Matrix Multiplication is Associate

4. Multiply the matrix used to rotate the triangle 90 degrees by the matrix used to reflect the triangle in the line $y = x$.

5. Use the matrix found in #4 to transform the triangle. Is the result the same as that in #1?

Answers:

1. $\begin{bmatrix} 2 & 1 \\ 5 & 3 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -3 & 5 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -3 & -5 \\ -4 & -4 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 1 \\ 5 & 3 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 5 & -3 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 4 & 4 \end{bmatrix}$

3. no

4. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 1 \\ 5 & 3 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -3 & -5 \\ -4 & -4 \end{bmatrix}$, yes

Tessellations

Review Interior Angles Measures for Polygons – Earlier in the course students learned how to calculate the sum of the measures of interior angles of a convex polygon, and how to divide by the number of angles to find the measures of the interior angle of regular polygons. Now would be a good time to review this lesson. The students will need this knowledge to see which regular polygons will tessellate and the final is fast approaching.

Move Them Around – When learning about regular and semi-regular tessellation it is helpful for students to have a set of regular triangles, squares, pentagons, hexagons, and octagons that they can slide around and fit together. These shapes can be bought from a mathematics education supply company or made with paper. Exploring the relationships in this way gives the students a fuller understanding of the concepts.

Use On-Line Resources – A quick search on tessellations will produce many beautiful, artistic examples like the work of M. C. Escher and cultural examples like Moorish tiling. This bit of research will inspire students and show them how applicable this knowledge is to many areas.

Tessellation Project – A good long-term project is to have the students create their own tessellations. This is an artistic endeavor that will appeal to students that typically struggle with mathematics, and the tessellations make nice decorations for the classroom. Here are some guidelines for the assignment.

1. Fill an $8\frac{1}{2}$ by 11 inch piece of solid colored paper with a tessellation of your own creation.
2. Make a stencil from cardboard and trace it to make the figures congruent.
3. Be creative. Make your tessellation look like something.
4. Color your design to enhance the tessellation.
5. Your tessellation will be graded on complexity, creativity, and presentation.

6. Write a paragraph explaining how you made your tessellation, and why your design is a tessellation. Use vocabulary from this section.

This project could also be done on a piece of legal sized paper. The tessellation can fill the top portion and the paragraph written on the lower part.

Symmetry

360 Degrees **Doesn't Count** – When looking for rotational symmetries students will often list 360 degree rotational symmetry. When a figure is rotated 360 degrees the result is not congruent to the original figure, it is the original figure itself. This does not fit the definition of rotational symmetry. This misconception can cause error when counting the numbers of symmetries a figure has or deciding if a figure has symmetry or not.

Review Quadrilateral Classifications – Earlier in the course students learned to classify quadrilaterals. Now would be a good time to break out that Venn diagram. Students will have trouble understanding that some parallelograms have line symmetry if they do not remember that squares and rectangles are types of parallelograms. As the course draws to an end, reviewing helps students retain what they have learned past the final. It is possible to redefine the classes of quadrilaterals based on symmetry. This pursuit will make the student use and combine knowledge in different ways making what they have learned more flexible and useful.

Applications – Symmetry has numerous applications both in and outside of mathematics. Knowing some of the uses for symmetry will motivate student, especially those who are not inspired by pure mathematics, to spend their time and energy learning this material.

Biology – Most higher level animals have bilateral symmetry, starfish and flowers often have 72 degree rotational symmetry. Naturally formed nonliving structures like honeycomb and crystals have 60 degree rotational symmetry. These patterns are fascinating and can be used for classification and study.

Trigonometry – Many identities of trigonometry are based on the symmetry of a circle. In the next few years of mathematics the students will see how to simplify extremely complex expressions using these identities.

Advertising – Many company logos make use of symmetry. Ask the students to bring in examples of logos with particular types of symmetry and create a class collection. Analyze the trends. Are certain products more appropriately represented by logos that contain a specific type of symmetry? Does the symmetry make the logo more pleasing to the eye or more easily remembered?

Functions – A function can be classified as even or odd based on the symmetry of its graph. Even functions have symmetry around the $y-$axis, and odd functions have 180 degree rotational symmetry about the origin. Once a function is classified as even or odd, properties and theorems can be applied to it.

Draw – Have students be creative and create their own logos or designs with specific types of symmetry. Using these concepts in many ways will build a deeper understanding and the ability to apply the new knowledge in different situations.

Dilations

Naming Conventions – Mathematics is a language, an extremely precise method of communication. While matrices are named with uppercase letters, scalars are represented by lower case letters. Many times students do not know to look for these types of patterns. Point out these conventions when an appropriate example arises and tell the students to look for the subtle differences that have major significance when communicating with mathematics.

The Scale Factor and Area – Students frequently forget to square the scale factor when comparing the area of the original figure to the image under a dilation. This relationship has been covered several times before when studying area, similar figures, and when dilation was first introduced. Make a point of it again. This omission is quite common and the concept is often used on the SAT and other standardized tests.

Additional Exercises:

1. Does the multiplication of a scalar and a 2×2 matrix commute? If so, write a proof. If not give a counterexample.

Answer: Yes, multiplication of a scalar with a 2×2 matrix does commute.

Let k be a scalar and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then $kA = k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} = \begin{bmatrix} ak & bk \\ ck & dk \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} k = Ak$

Here a 2×2 matrix was used, but this same proof can be done with a matrix of any dimensions.

2. What scalar could be multiplied by a matrix containing the vertices of a polygon to produce an image with half the area as the original figure?

Answer: $\frac{1}{\sqrt{2}}$ since $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

3. Will a dilation followed by a reflection produce the same image if the order of the transformations is reversed? Why or why not?

Answer: The image will be the same regardless of the order in which the transformations are applied. This can be justified with the commutative property of scalar multiplication.