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Phenomenological Mass Matrices with a Democratic Warp

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Abstract. Taking into account all available data on the mass sector, we obtain unitary rotation matrices that diagonalize the quark matrices by using a specific parametrization of the Cabibbo-Kobayashi-Maskawa mixing matrix. In this way, we find mass matrices for the up- and down-quark sectors of a specific, symmetric form, with traces of a democratic texture.

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1 Introduction

The Standard Model of particle physics is flawed by the large number of free parameters, for which there is at present no explanation.

Most of these free parameters reside in flavour space, the structure of which is determined by the fermion mass matrices, i.e. by the form that the mass matrices take in the "weak basis" where mixed fermion states interact weakly. This basis differs from the mass bases, where the mass matrices are diagonal, with entries corresponding to the masses of the physical fermions.

The information content of a matrix is contained in its matrix invariants, which in the case of a $N \times N$ matrix M are the N sums and products of the eigenvalues λ_j, such as $traceM$, $detM$,

$$\begin{aligned} I_1 =& \sum\nolimits_j \lambda_j = \lambda_1 + \lambda_2 + \lambda_3 + \cdots \\ I_2 =& \sum\nolimits_{jk} \lambda_j \lambda_k = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \cdots \\ I_3 =& \sum\nolimits_{jkl} \lambda_j \lambda_k \lambda_l = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \cdots \\ &\vdots \\ I_N =& \lambda_1\lambda_2 \cdots \lambda_N \ . \end{aligned} \tag{1}$$

The search for the "right" mass matrices is based on the assumption that even if the information content of a matrix is contained in its invariants, the form of a matrix also carries important information. The hope is that the form of the mass matrices in the "weak basis" can give some hint about the origin of the fermion masses.

The crux is that we don't know which of the flavour space bases is the weak basis, we consequently don't know what form of the mass matrices have in this unknown basis. The different mass matrix ansätze found in the literature correspond to different choices, based on different assumptions, as to which flavour space basis is the weak basis.

2 Phenomenology

The Standard Model might not be a fundamental theory, but it certainly is an very successful model. In our approach, we follow the phenomenlogical track, and scrutinize all available data that are relevant for the mass sector.

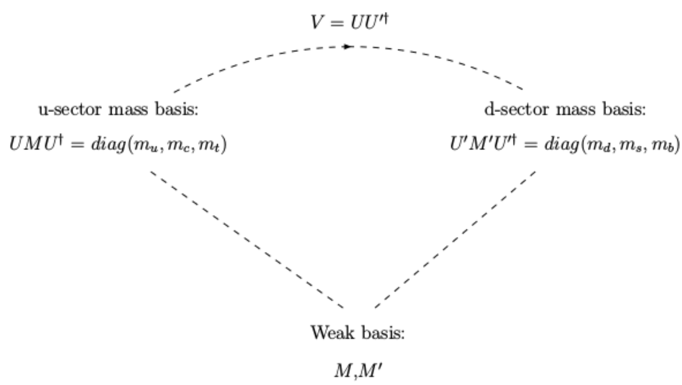
In addition to numerical mass values, there is also the mixing matrix V that appears in the flavour changing charged current Lagrangian

$$\mathcal{L}_{cc} = -\frac{g}{2\sqrt{2}}\bar{f}_L\gamma^\mu V f'_L W_\mu + h.c. \tag{2}$$

where as before f and f' are fermion fields with charges Q and $Q-1$, correspondingly. In the case of the quarks, the mixing matrix is the Cabbibo-Kobayashi-Maskawa (CKM) [1] mixing matrix.

That $V \neq \mathbf{1}$ implies that the up-sector mass basis is different from the down-sector mass basis, the CKM matrix being the bridge between the two mass bases. As we go from the weak basis to the two different mass bases by rotating the matrices by the unitary matrices U and U', respectively,

$$\begin{aligned} M &\to UMU^\dagger = D = diag(m_u, m_c, m_t) \\ M' &\to U'M'U'^\dagger = D' = diag(m_d, m_s, m_b) \end{aligned} \tag{3}$$



we have that $V = UU'^{\dagger}$. A given choice of the weak basis - i.e. of the mass matrices, thus corresponds to choosing a factorization of the mixing matrix, and since $U = U(M)$ and $U' = U'(M')$, $V = U(M)U'^{\dagger}(M') = V(M, M')$.

The charged current Lagrangian (2) can be interpreted as describing the interaction between the physical up-sector particles $\bar{\psi}_L = (\bar{u}, \bar{c}, \bar{t})_L$ with the mixed down-sector states, or equivalently as the interaction between the up-sector mixed states and the down-sector mass states $\bar{\psi'}_L = (\bar{d}, \bar{s}, \bar{b})_L$.

If we take the definition of the CKM matrix at face value, $V = UU'^{\dagger}$, it is however more natural to perceive the charged current interactions as taking place between mixed up-sector states and mixed down-sector states,

$$\mathcal{L}_{cc} = -\frac{g}{\sqrt{2}}\bar{\psi}_L\gamma^{\mu}V\psi'_L W_{\mu} + h.c. = -\frac{g}{\sqrt{2}}\bar{\varphi}_L\gamma^{\mu}\varphi'_L W_{\mu} + h.c. \qquad (4)$$

where

$$\varphi = U^{\dagger}\begin{pmatrix} u \\ c \\ t \end{pmatrix} \qquad \text{and} \qquad \varphi' = U'^{\dagger}\begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

are the fermion fields in the weak basis in flavour space, and ψ and ψ' are the corresponding mass eigenstates.

Mass eigenstates are defined as "physical", corresponding to particles with definite masses; while the weakly interacting mixings of mass states are referred to as "flavour states". Physical particles are thus identified as mass eigenstates. In the case of neutrinos the situation is however somewhat different, since neutrino mass eigenstates do not appear on stage, they merely propagate in free space. In the realm of neutral leptons it is actually the flavour states ν_e, ν_μ, ν_τ that we perceive as "physical", since they are the only neutrinos that we "see", as they appear together with the charged leptons. As the charged leptons e, μ, τ are assumed to be both weak eigenstates and mass eigenstates, the only mixing matrix that appears in the lepton sector is the Pontecorvo-Maki-Nakagawa-Sakata mixing matrix U [2], which only operates on neutrino states,

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix},$$

where (ν_1, ν_2, ν_3) are mass eigenstates, and $(\nu_e, \nu_\mu, \nu_\tau)$ are the weakly interacting "flavour states". In the lepton sector, the charged currents are thus interpreted as charged lepton flavours (e, μ, τ) interacting with the neutrino "flavour states" $(\nu_e, \nu_\mu, \nu_\tau)$.

For quarks as well as leptons, the relation between the weakly interacting fermion fields φ and the mass eigenstates ψ is determined by the unitary rotation matrix U which diagonalizes the mass matrix M,

$$\mathcal{L}_{mass} = \bar{\varphi} M \varphi = \bar{\varphi} U^\dagger (U M U^\dagger) U \varphi = \bar{\psi} \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix} \psi,$$

in the quark sector the (physical) mass eigenstates thus are the fields $\psi = U\varphi$ and $\psi' = U'\varphi'$,

$$\psi = \begin{pmatrix} u \\ c \\ t \end{pmatrix} \qquad \text{and} \qquad \psi' = \begin{pmatrix} d \\ s \\ b \end{pmatrix} .$$

The CKM matrix plays an important role in relating the mass matrices for the up- and down-sectors, since once the form of the mass matrix of one of the charge sectors is established, also the form of the mass matrix of the other charge sector is determined, via the CKM mixing matrix. Once the form of the up-sector mass matrix M is established,the unitary matrix U that diagonalizes M is determined. And since $V = UU'^\dagger$, this also determines $U' = V^\dagger U$, whereby we have $M' = U'^\dagger diag(d, s, b) U'$. In this sense M and M' are determined together.

3 Factorizing the Mixing Matrix

The Cabbibo-Kobayashi-Maskawa mixing matrix can of course be parametrized and factorized in many different ways, and different factorizations correspond to different rotation matrices U and U'. The most obvious and "symmetric" factorization of the CKM mixing matrix, following the "standard parametrization" [3] with three Euler angles α, β, 2θ,

$$V = \begin{pmatrix} c_\beta c_{2\theta} & s_\beta c_{2\theta} & s_{2\theta} e^{-i\delta} \\ -c_\beta s_\alpha s_{2\theta} e^{i\delta} - s_\beta c_\alpha & -s_\beta s_\alpha s_{2\theta} e^{i\delta} + c_\beta c_\alpha & s_\alpha c_{2\theta} \\ -c_\beta c_\alpha s_{2\theta} e^{i\delta} + s_\beta s_\alpha & -s_\beta c_\alpha s_{2\theta} e^{i\delta} - c_\beta s_\alpha & c_\alpha c_{2\theta} \end{pmatrix} = UU^{'\dagger} \tag{5}$$

is to take the diagonalizing rotation matrices for the up- and down-sectors as

$$\begin{aligned} U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & & \\ & 1 & \\ & & e^{i\gamma} \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} & \\ = \begin{pmatrix} c_\theta e^{-i\gamma} & 0 & s_\theta e^{-i\gamma} \\ -s_\alpha s_\theta e^{i\gamma} & c_\alpha & s_\alpha c_\theta e^{i\gamma} \\ -c_\alpha s_\theta e^{i\gamma} & -s_\alpha & c_\alpha c_\theta e^{i\gamma} \end{pmatrix} & \end{aligned} \tag{6}$$

and

$$U' = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & & \\ & 1 & \\ & & e^{i\gamma} \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$
$$= \begin{pmatrix} c_\beta c_\theta e^{-i\gamma} & -s_\beta & -c_\beta s_\theta e^{-i\gamma} \\ s_\beta c_\theta e^{-i\gamma} & c_\beta & -s_\beta s_\theta e^{-i\gamma} \\ s_\theta e^{i\gamma} & 0 & c_\theta e^{i\gamma} \end{pmatrix} \quad (7)$$

respectively, where α, β, θ and γ correspond to the parameters in the standard parametrization, with $\gamma = \delta/2$, $\delta = 1.2 \pm 0.08$ rad, and $2\theta = 0.201 \pm 0.011°$, while $\alpha = 2.38 \pm 0.06°$ and $\beta = 13.04 \pm 0.05°$. In this factorization scheme, α and β are rotation angles operating in the up-sector and the down-sector, respectively.

Now, with the rotation matrices U and U', we obtain the the up- and down-sector mass matrices

$$M = U^\dagger diag(m_u, m_c, m_t) U \;\text{ and }\; \mathrm{M' = U'^\dagger diag(m_d, m_s, m_b) U'},$$

such that

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} = \begin{pmatrix} X c_\theta^2 + Y s_\theta^2 & -Z s_\theta\, e^{-i\gamma} & (X-Y) c_\theta s_\theta \\ -Z s_\theta\, e^{i\gamma} & Y + 2Z\cot 2\alpha & Z c_\theta\, e^{i\gamma} \\ (X-Y) c_\theta s_\theta & Z c_\theta\, e^{-i\gamma} & X s_\theta^2 + Y c_\theta^2 \end{pmatrix}, \quad (8)$$

where $X = m_u$, $Z = (m_c - m_t)\sin\alpha\cos\alpha$ and $Y = m_t + Z\tan\alpha = m_c \sin^2\alpha + m_t\cos^2\alpha$, and

$$M' = \begin{pmatrix} M'_{11} & M'_{12} & M'_{13} \\ M'_{21} & M'_{22} & M'_{23} \\ M'_{31} & M'_{32} & M'_{33} \end{pmatrix} = \begin{pmatrix} X' s_\theta^2 + Y' c_\theta^2 & Z' c_\theta\, e^{i\gamma} & (X'-Y') c_\theta s_\theta \\ Z' c_\theta\, e^{-i\gamma} & Y' + 2Z'\cot 2\beta & -Z' s_\theta\, e^{-i\gamma} \\ (X'-Y') c_\theta s_\theta & -Z' s_\theta\, e^{i\gamma} & X' c_\theta^2 + Y' s_\theta^2 \end{pmatrix}, \quad (9)$$

where $X' = m_b$, $Z' = (m_s - m_d)\sin\beta\cos\beta$ and $Y' = m_d + Z'\tan\beta = m_d\cos^2\beta + m_s\sin^2\beta$.

The two mass matrices thus have similar textures, or forms, and there is even a relational equality,

$$M_{32}/M_{12} = M'_{12}/M'_{32} = -\cot\theta\,,$$

which is independent of the quark masses.

From $Y = m_c\sin^2\alpha + m_t\cos^2\alpha$, $Z = (m_c - m_t)\sin\alpha\cos\alpha$, $Y' = m_d\cos^2\beta + m_s\sin^2\beta$ and $Z' = (m_s - m_d)\sin\beta\cos\beta$, we moreover have

$$\begin{aligned} &m_u = X, \qquad m_c = Y + Z\cot\alpha, \qquad m_t = Y - Z\tan\alpha \\ &m_d = Y' - Z'\tan\beta, \qquad m_s = Y' + Z'\cot\beta, \qquad m_b = X' \end{aligned} \quad (10)$$

4 Numerical Matrices

Using the numerical values $\beta = 13.04°$, $\alpha = 2.38°$, $\delta = 1.2 \pm 0.08$ rad, and $2\theta = 0.201 \pm 0.011°$ for the the angles, and using the mass values (Jamin 2014) [4] for the up- and down-sectors,

$$\begin{array}{ll} m_u(M_Z) = 1.24 \text{ MeV} & m_c(M_Z) = 624 \text{ MeV} \\ m_t(M_Z) = 171550 \text{ MeV} & m_d(M_Z) = 2.69 \text{ MeV} \\ m_s(M_Z) = 53.8 \text{ MeV} & m_b(M_Z) = 2850 \text{ MeV} \end{array} \tag{11}$$

we get the numerical values for the mass matrices (8) and (9)

$$M = \begin{pmatrix} 1.767 & 12.439e^{-i\gamma} & -300.389 \\ 12.439e^{i\gamma} & 918.759 & -7091.892e^{i\gamma} \\ -300.389 & -7091.892e^{-i\gamma} & 171254.714 \end{pmatrix} \text{MeV} \tag{12}$$

and

$$M' = \begin{pmatrix} 5.299 & 11.23e^{i\gamma} & 4.99 \\ 11.23e^{-i\gamma} & 51.18 & -0.0197e^{-i\gamma} \\ 4.99 & -0.0197e^{i\gamma} & 2849.99, \end{pmatrix} \text{MeV} \tag{13}$$

where in M,

$$\begin{array}{ccc} M_{11} = m_u + \sigma, & M_{22} = m_c + Q - \sigma, & M_{33} = m_t - Q, \\ M_{22} + M_{33} = m_c + m_t - \sigma & \text{and} & |M_{33}M_{12}| \approx |M_{13}M_{32}|, \end{array}$$

with $\sigma \simeq 0.53$ MeV, $Q \simeq 295.3$ MeV.

Likewise, in M',

$$\begin{array}{ccc} M'_{11} = m_d + R, & M'_{22} = m_s + \eta - R, & M'_{33} = m_b - \eta, \\ M'_{11} + M'_{22} = m_d + m_s + \eta, & \text{and} & |M'_{33}M'_{32}| \approx |M'_{13}M'_{12}|, \end{array}$$

with $R \simeq 2.61$ MeV, $\eta \simeq 0.011$ MeV.

5 Traces of a Democratic Structure

Our factorization of the Cabbibo-Kobayashi-Maskawa mixing matrix is only one of many possible choices, in (5) we can moreover sandwich any number of unitary matrices between U and U',

$$V = UU'^{\dagger} = UO_1O_1^{\dagger}U'^{\dagger} = UO_1O_2O_2^{\dagger}O_1^{\dagger}U'^{\dagger} = ...$$

where O_j are unitary matrices such that each set of sandwiched $O_jO_j^{\dagger}$ corresponds to a new set of unitary matrices diagonalizing the mass matrices, and thus to yet another type of mass matrix texture.

In our first approach [9], the sandwich principle was used with the purpose of investigating democratic mass matrix textures. In the democratic scenario, it is assumed that both the up- and down-sector mass matrices have an initial structure of the type $M_0 = k\mathbf{N}$ and $M_0' = k'\mathbf{N}$ where

$$\mathbf{N} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

with the mass spectra $(0, 0, 3k)$, $(0, 0, 3k')$, and a mixing matrix equal to unity (i.e. no CP-violation). The flavour symmetry displayed by the matrices M_0 and M_0' is subsequently broken, whereby both mass spectra contain the three observed non-zero values, and the mixing matrix becomes the CKM matrix (with a CP-violating phase).

Our sandwiching procedure started from the factorization $V = UU'^{\dagger}$, with U and U' as in (6) and (7), and the matrix

$$U_{dem} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & -\sqrt{3} & 0 \\ 1 & 1 & -2 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix} \tag{14}$$

which diagonalizes the democratic matrix $\mathbf{N}$. When U_{dem} and its Hermitian conjugate are put into the mixing matrix,

$$V = UU'^{\dagger} \Rightarrow V = UU_{dem}U_{dem}^{\dagger}U'^{\dagger}$$

we obtain new rotation matrices UU_{dem} and $U'U_{dem}$ which indeed correspond to mass matrices with democratic textures.

In simplest case (5), without any $U_{dem}U_{dem}^{\dagger}$ or other matrices sandwiched between U and U' in $V = UU'^{\dagger}$, there is however already some interesting, democracy-like structure present, which is can be made visible by a slight reformulation of the matrices (8) and (9). Even though the matrix elements are dominated by the hierarchical family structure, which does not look very "democratic", rewriting the matrices by extracting the dimensional coefficients ρ and μ unveils this structure:

$$M = \rho \begin{pmatrix} A & Be^{-i\gamma} & -C \\ Be^{i\gamma} & H & -BCe^{i\gamma} \\ -C & -BCe^{-i\gamma} & C^2 \end{pmatrix} \tag{15}$$

and

$$M' = \mu \begin{pmatrix} A' & B'Ce^{i\gamma} & C \\ B'Ce^{-i\gamma} & H' & -B'e^{-i\gamma} \\ C & -B'e^{i\gamma} & C^2 \end{pmatrix}, \tag{16}$$

with

$$\rho = (Y-X)s_\theta^2, \quad A = (X\cot_\theta^2 + Y)/(Y-X), \quad B = Z/(Y-X)s_\theta,$$
$$H = (Y + 2Z\cot 2\alpha)/(Y-X)s_\theta^2, \quad C = \cot\theta$$
$$\mu = (X'-Y')s_\theta^2, A' = (X' + Y'\cot_\theta^2)/(X'-Y'), \quad B' = Z'/(X'-Y')s_\theta$$
$$H' = (Y' + 2Z'\cot 2\beta)/(X'-Y')s_\theta^2$$

Numerically, with the mass values (11), this corresponds to

$$\rho = 0.5269\,\text{MeV}, \quad A = 3.3533, \quad B = 23.608,$$
$$H = 1743.71, \quad C = \cot\theta \simeq 570.1$$
$$\mu = 0.00875\,\text{MeV}, \quad A' = 605.6, \quad B' = 2.2514, \quad H' = 5849.14,$$

where incidentally $H' = AH$. The up-sector mass matrix (15) can be rewritten as as

$$M = \rho\left[\begin{pmatrix}1 & & \\ & Be^{i\gamma} & \\ & & -C\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1 & & \\ & Be^{-i\gamma} & \\ & & -C\end{pmatrix} + \Lambda\right] = \rho\left[\hat{M} + \Lambda\right], \tag{17}$$

where

$$\Lambda = \begin{pmatrix}A-1 & & \\ & H-B^2 & \\ & & 0\end{pmatrix}. \tag{18}$$

Noticing that the matrix

$$\hat{M} = \begin{pmatrix}1 & & \\ & Be^{i\gamma} & \\ & & -C\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1 & & \\ & Be^{-i\gamma} & \\ & & -C\end{pmatrix} = D\mathbf{N}D^* \tag{19}$$

has only one non-zero eigenvalue, and that

$$\mathbf{N} = D^*\ \hat{M}\ D,$$

we can relate $\hat{M}$ to the democratic matrix $M_0 = k\mathbf{N}$, by equating the one non-zero eigenvalue of $\hat{M}$, $1 + B^2 + C^2 = trace(DD^*)$, to the one non-zero eigenvalue $3k$ of the democratic matrix M_0, which gives

$$3k = \rho(1 + B^2 + C^2),$$

i.e. $k = 57181.4$ MeV. Thus identifying the matrix $\hat{M}$ as having a kind of democratic texture, we determine the matrix Λ as the symmetry breaking term which finally gives the mass spectrum with the three observed non-zero masses. If we in this way interpret the mass matrix

$$M = \rho[D\mathbf{N}D^* + \Lambda]$$

as starting out as a democratic matrix $M_0 = k\mathbf{N}$, the first flavour symmetry breaking is identified as

$$M_0 \Rightarrow \hat{M} = D\mathbf{N}D^*$$

where $\hat{M}$ has the same, one non-zero eigenvalue as M_0, $3k = \rho(1 + B^2 + C^2)$, but the flavour symmetry of the fields $(\varphi_1, \varphi_2, \varphi_3)$ in the weak basis is broken. By adding Λ, with the two non-zero eigenvalues Λ_1 and Λ_2, we finally get the full mass spectrum of M.

The down-sector can be treated in a similar fashion, though here the traces of democracy are less transparent.

6 Conclusion

Without introducing any new assumptions, by just factorizing the "standard parametrization" of the CKM weak mixing matrix in a specific way, we obtain mass matrices with a specific type of democratic texture. This is a work in progress, and the implications of this democratic structure remain to be analyzed.

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