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Convergence of CR-iteration procedure for a nonlinear quasi contractive map in convex metric spaces

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Abstract

We prove that the modified CR-iteration procedure converges strongly to a fixed point of a nonlinear quasi contractive map in convex metric spaces which is the main result of this paper. The convergence of Picard-S iteration procedure follows as a corollary to our main result.

Keywords: Convex metric space, quasi contraction map, CR-iteration procedure and Picard-S iteration procedure.
2010 MSC: 47H10, 54H25.

1. Introduction and preliminaries

In 1970, Takahashi [11] introduced the concept of convexity in metric spaces as follows.

Definition 1.1. Let (X, d) be a metric space. A map $W : X \times X \times [0, 1] \to X$ is said to be a *'convex structure'* on X if

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y) \tag{1.1}$$

for $x, y, u \in X$ and $\lambda \in [0, 1]$.

A metric space (X, d) together with a convex structure W is called a *convex metric space* and we denote it by (X, d, W). We note that $W(x, y, 1) = x$ and $W(x, y, 0) = y$. A nonempty subset K of X is said to be *'convex'* if $W(x, y, \lambda) \in K$ for $x, y \in K$ and $\lambda \in [0, 1]$.

Remark 1.2. Every normed linear space $(X, ||.||)$ is a convex metric space with the convex sructure W defined by $W(x, y, \lambda) = (1 - \lambda)y + \lambda x$ for $x, y \in X$, $\lambda \in [0, 1]$. But there are convex meric spaces which are not normed linear spaces [1, 8, 11].

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In 1974, Ćirić [3] introduced quasi-contraction maps in the setting of metric spaces and proved that the Picard iterative sequence converges to the fixed point in complete metric spaces.

Definition 1.3. Let (X, d) be a metric space. A selfmap $T : X \to X$ is said to be a quasi-contraction map if there exists a real number $0 \leq k < 1$ such that

$$d(Tx, Ty) \leq kM(x, y) \tag{1.2}$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \tag{1.3}$$

for $x, y \in X$.

Let K be a nonempty convex subset of a normed linear space X and let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be sequences in $[0, 1]$. The Ishikawa iteration procedure [7] in the setting of normed linear spaces is as follows : For $x_0 \in K$,

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n T x_n \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \quad \text{for} \quad n = 0, 1, 2, \ldots . \end{aligned} \tag{1.4}$$

Ding [5] considered the Ishikawa iteration procedure in the setting of convex metric spaces as follows : Let K be a nonempty convex subset of a convex metric space (X, d, W), and let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be the sequences in $[0, 1]$. For $x_0 \in K$,

$$\begin{aligned} y_n &= W(Tx_n, x_n, \beta_n) \\ x_{n+1} &= W(Ty_n, x_n, \alpha_n) \;\; \text{for} \;\; n = 0, 1, 2, \ldots \end{aligned} \tag{1.5}$$

and proved that the Ishikawa iteration procedure (1.5) converges strongly to a unique fixed point of a quasi-contraction map in the setting of convex metric spaces, provided $\sum_{n=0}^{\infty} \alpha_n = \infty$.

In 1999, Ćirić [4] introduced a more general quasi-contraction map and proved the convergence of an Ishikawa iteration procedure in convex metric spaces to the unique fixed point and the result is the following.

Theorem 1.4. (Ćirić [4]) *Let K be a nonempty closed convex subset of a complete convex metric space X and let $T : K \to K$ be a selfmap satisfying*

$$d(Tx, Ty) \leq w(M(x, y)), \tag{1.6}$$

where $M(x, y)$ is as defined in (1.3) for $x, y \in K$ and
$w : (0, \infty) \to (0, \infty)$ is a map which satisfies (i) $0 < w(t) < t$ for each $t > 0$,
(ii) w increases, and the following conditions :

$$\lim_{t\to\infty} (t - w(t)) = \infty : \quad \text{and} \tag{1.7}$$

$$\text{either} \quad t - w(t) \quad \text{is increasing on } (0, \infty) \tag{1.8}$$

$$\text{or} \quad w(t) \text{ is strictly increasing and } \lim_{n\to\infty} w^n(t) = 0 \text{ for } t > 0. \tag{1.9}$$

Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be sequences in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. For $x_0 \in K$, the Ishikawa iteration procedure $\{x_n\}_{n=0}^{\infty}$ defined in (1.5) converges strongly to the unique fixed point of T.

Sastry, Babu and Srinivasa Rao [10] improved Theorem 1.4 by replacing (1.8) and (1.9) with a single condition, namely $0 < w(t^+) < t$ for each $t > 0$ and proved the following theorem.

Theorem 1.5. [10] *Let (X, d, W) be a complete convex metric space and $T : X \to X$ be a map that satisfies*

$$d(Tx, Ty) \leq w(M(x, y)) \tag{1.10}$$

where $M(x, y)$ is defined as in (1.3) for $x, y \in X$ and $w : (0, \infty) \to (0, \infty)$ is a map such that (i) w increases, (ii) $\lim\limits_{t\to\infty} (t - w(t)) = \infty$ (iii) $0 < w(t^+) < t$ for $t > 0$.
Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be sequences in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then for any $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by the iteration procedure (1.5) converges strongly to a unique fixed point of T.

Here we note that a map that satisfies (1.10) is said to be a nonlinear quasi contractive map on X.

Remark 1.6. (i) and (iii) of Theorem 1.5 imply that $0 < w(t) < t$ for each $t > 0$.

Remark 1.7. If $w(t) = kt$ for $t \in (0, \infty)$ and $0 \leq k < 1$ then the map T of Theorem 1.5 reduces to a quasi contraction map.

In 2012, Chugh, Kumar and Kumar [2] introduced 'CR-iteration procedure' as follows:
Let K be a nonempty convex subset of a normed linear space X, and let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ be sequences in $[0, 1]$.
For $x_0 \in K$,

$$\begin{aligned} z_n &= (1 - \gamma_n)x_n + \gamma_n T x_n \\ y_n &= (1 - \beta_n)T x_n + \beta_n T z_n, \\ x_{n+1} &= (1 - \alpha_n)y_n + \alpha_n T y_n, \quad \text{for} \quad n = 0, 1, 2, \ldots . \end{aligned} \tag{1.11}$$

By choosing $\alpha_n \equiv 1$ for all n in (1.11), we have the following.
For $x_0 \in K$,

$$\begin{aligned} z_n &= (1 - \gamma_n)x_n + \gamma_n T x_n \\ y_n &= (1 - \beta_n)T x_n + \beta_n T z_n, \\ x_{n+1} &= T y_n, \qquad \text{for} \quad n = 0, 1, 2, \ldots . \end{aligned} \tag{1.12}$$

The iteration procedure (1.12) is called the *'Picard-S iteration procedure'* [6].

In 2014, Chugh and Malik [9] introduced an anlaogue of CR-iteration procedure (1.11) in convex metric spaces as follows:

Let K be a nonempty convex subset of a convex metric space (X, d, W).
For any $x_0 \in K$,

$$\begin{aligned} z_n &= W(T x_n, x_n, \gamma_n) \\ y_n &= W(T z_n, T x_n, \beta_n) \\ x_{n+1} &= W(T y_n, y_n, \alpha_n) \end{aligned} \tag{1.13}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are in $[0, 1]$.

We call the iteration procedure $\{x_n\}$ defined in (1.13) is a *'modified CR-iteration procedure'* in convex metric spaces.
If $\alpha_n \equiv 1$ then the iteration procedure (1.13) reduces to the following which is an analogue of Picard-S iteration procedure (1.12) in a convex metric space.
For $x_0 \in K$,

$$\begin{aligned} z_n &= W(T x_n, x_n, \gamma_n) \\ y_n &= W(T z_n, T x_n, \beta_n) \\ x_{n+1} &= T y_n \end{aligned} \tag{1.14}$$

where $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are in $[0, 1]$.

We call the iteration $\{x_n\}$ defined in (1.14) is a *'modified Picard-S iteration procedure'*.

Motivated by the results of Ćirić [4] and Sastry, Babu and Srinivasa Rao [10], in Section 2 of this paper, we prove the strong convergence of modified CR-iteration procedure to a fixed point of a nonlinear quasi contractive map (Theorem 2.2) which is the main result of this paper. The convergence of modified Picard-S iteration procedure (1.14) follows as a corollary to our main result.

2. Main results

Lemma 2.1. *Let (X, d, W) be a convex metric space, and let K be a nonempty convex subset of X. Let $T : K \to K$ be a map such that*

$$d(Tx, Ty) \leq w(M(x, y)) \quad \text{for } x, y \in K, \tag{2.1}$$

where $M(x, y)$ is defined in (1.3) with $M(x, y) > 0$ and $w : (0, \infty) \to (0, \infty)$ is a map such that (i) w is increasing on $(0, \infty)$ (ii) $\lim_{t\to\infty}(t - w(t)) = \infty$, and (iii) $0 < w(t^+) < t$ for each $t > 0$. For $x_0 \in K$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences generated by the modified CR-iteration procedure (1.13). Then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{Tx_n\}$, $\{Ty_n\}$ and $\{Tz_n\}$ are bounded.

Proof. For each positive integer n, we define the set
$A_n = \{x_i\}_{i=0}^n \cup \{y_i\}_{i=0}^n \cup \{z_i\}_{i=0}^n \cup \{Tx_i\}_{i=0}^n \cup \{Ty_i\}_{i=0}^n \cup \{Tz_i\}_{i=0}^n$.
We denote the diameter of A_n by a_n. We show that $\{a_n\}_{n=1}^\infty$ is bounded. For this purpose, we define $b_n = \max\{\sup_{0\leq i\leq n} d(x_0, Tx_i), \sup_{0\leq i\leq n} d(x_0, Ty_i), \sup_{0\leq i\leq n} d(x_0, Tz_i)\}$ for $n = 1, 2, \ldots$.
We now show that $a_n = b_n$ for $n = 1, 2, \ldots$.
Clearly, $b_n \leq a_n$ for $n = 1, 2, \ldots$.
Without loss of generality, we assume that $a_n > 0$ for $n = 1, 2, \ldots$.
<u>*Case (i)*</u> : $a_n = d(Tx_i, Tx_j)$ for some $0 \leq i, j \leq n$.
Now, $a_n = d(Tx_i, Tx_j) \leq w(M(x_i, x_j)) \leq w(a_n) < a_n$,
a contradiction.
Hence, $a_n \neq d(Tx_i, Tx_j)$ for any $0 \leq i, j \leq n$.
With the similar reason, it is easy to see that $a_n \neq d(Tx_i, Ty_j)$, $a_n \neq d(Tx_i, Tz_j)$, $a_n \neq d(Ty_i, Ty_j)$, $a_n \neq d(Ty_i, Tz_j)$, and $a_n \neq d(Tz_i, Tz_j)$ for any $0 \leq i, j \leq n$.
<u>*Case (ii)*</u> : $a_n = d(y_i, Tx_j)$ for some $0 \leq i, j \leq n$.

$$\begin{aligned} a_n = d(y_i, Tx_j) = d(w(Tz_i, Tx_i, \beta_i), Tx_j) &\leq \beta_i d(Tz_i, Tx_j) + (1 - \beta_i) d(Tx_i, Tx_j) \\ &\leq \max\{d(Tz_i, Tx_j), d(Tx_i, Tx_j)\} \leq a_n \text{ so that} \end{aligned}$$

$a_n = d(Tz_i, Tx_j)$ or $a_n = d(Tx_i, Tx_j)$,
which fails to hold by *Case* (i).
Therefore $a_n \neq d(y_i, Tx_j)$ for any $0 \leq i, j \leq n$.
Similarly, it is easy to see that $a_n \neq d(y_i, Ty_j)$ and $a_n \neq d(y_i, Tz_j)$ for any $0 \leq i, j \leq n$.
<u>*Case (iii)*</u> : $a_n = d(y_i, y_j)$ for some $0 \leq i, j \leq n$.

$$\begin{aligned} a_n = d(y_i, y_j) \leq d(W(Tz_i, Tx_i, \beta_i), y_j) &\leq \beta_i d(y_j, Tz_i) + (1 - \beta_i) d(y_j, Tx_i) \\ &\leq \max\{d(y_j, Tz_i), d(y_j, Tx_i)\} \leq a_n \text{ so that} \end{aligned}$$

$a_n = d(y_j, Tz_i)$ or $a_n = d(y_j, Tx_i)$,
which fails to hold by *Case* (ii).
Therefore, $a_n \neq d(y_i, y_j)$ for any $0 \leq i, j \leq n$.
<u>*Case (iv)*</u> : $a_n = d(x_i, Tx_j)$ for some $0 \leq i, j \leq n$.
If $i > 0$ then

$$\begin{aligned} a_n = d(x_i, Tx_j) &= d(W(Ty_{i-1}, y_{i-1}, \alpha_{i-1}), Tx_j) \\ &\leq \alpha_{i-1} d(Ty_{i-1}, Tx_j) + (1 - \alpha_{i-1}) d(y_{i-1}, Tx_j) \\ &\leq \max\{d(Ty_{i-1}, Tx_j), d(y_{i-1}, Tx_j)\} \leq a_n \text{ so that} \end{aligned}$$

$a_n = d(Ty_{i-1}, Tx_j)$ or $a_n = d(y_{i-1}, Tx_j)$,
which is absurd by *Case* (i) and *Case* (ii).
Therefore $i = 0$ and hence $a_n = d(x_0, Tx_j)$ so that $a_n \leq b_n$.
<u>*Case (v)*</u> : Either $a_n = d(x_i, Ty_j)$ or $d(x_i, Tz_j)$ for some $0 \leq i, j \leq n$.
By the similar argument as in *Case* (iv), $i = 0$ and hence $a_n \leq b_n$.
<u>*Case (vi)*</u> : $a_n = d(x_i, y_j)$ for some $0 \leq i, j \leq n$.

$$\begin{aligned} a_n = d(x_i, y_j) = d(x_i, W(Tz_j, Tx_j, \beta_j)) &\leq \beta_j d(x_i, Tz_j) + (1 - \beta_j) d(x_i, Tx_j) \\ &\leq \max\{d(x_i, Tz_j), d(x_i, Tx_j)\} \leq a_n \text{ so that} \end{aligned}$$

$a_n = d(x_i, Tz_j)$ or $d(x_i, Tx_j)$. By *Case* (iv) and *Case* (v), we have

$a_n = d(x_0, Tx_j)$ or $d(x_0, Tz_j)$ so that $a_n \leq b_n$.
$\underline{Case\ (vii)}$: $a_n = d(x_i, x_j)$ for some $0 \leq i < j \leq n$.

$$\begin{aligned} a_n = d(x_i, x_j) = d(x_i, W(Ty_{j-1}, y_{j-1}, \alpha_{j-1})) &\leq \alpha_{j-1} d(x_i, Ty_{j-1}) + (1-\alpha_{j-1}) d(x_i, y_{j-1}) \\ &\leq \max\{d(x_i, Ty_{j-1}), d(x_i, y_{j-1})\} \leq a_n \end{aligned}$$

so that $a_n = d(x_i, Ty_{j-1})$ or $d(x_i, y_{j-1})$.
Hence, $a_n \leq b_n$ follows from from *Case* (v) and *Case* (vi).
$\underline{Case\ (viii)}$: $a_n = d(x_i, z_j)$ for some $0 \leq i, j \leq n$.

$$\begin{aligned} a_n = d(x_i, z_j) = d(x_i, W(Tx_j, x_j, \gamma_j)) &\leq \gamma_j d(x_i, Tx_j) + (1-\gamma_j) d(x_i, x_j) \\ &\leq \max\{d(x_i, Tx_j), d(x_i, x_j)\} \leq a_n \text{ so that} \end{aligned}$$

$a_n = d(x_i, Tx_j)$ or $d(x_i, x_j)$.
Hence, $a_n \leq b_n$ follows from *Case* (iv) and *Case* (vii).
$\underline{Case\ (ix)}$: $a_n = d(y_i, z_j)$ for some $0 \leq i, j \leq n$.

$$\begin{aligned} a_n = d(y_i, z_j) = d(y_i, W(Tx_j, x_j, \gamma_j)) &\leq \gamma_j d(y_i, Tx_j) + (1-\gamma_j) d(y_i, x_j) \\ &\leq \max\{d(y_i, Tx_j), d(y_i, x_j)\} \leq a_n \text{ so that} \end{aligned}$$

$a_n = d(y_i, Tx_j)$ or $d(y_i, x_j)$.
By *Case* (ii), $a_n \neq d(y_i, Tx_j)$.
Therefore $a_n = d(y_i, x_j)$ and hence $a_n \leq b_n$ follows from *Case* (vi).
$\underline{Case\ (x)}$: $a_n = d(z_i, Tx_j)$ for some $0 \leq i, j \leq n$.

$$\begin{aligned} a_n = d(z_i, Tx_j) = d(W(Tx_i, x_i, \gamma_i), Tx_j) &\leq \gamma_i d(Tx_i, Tx_j) + (1-\gamma_i) d(x_i, Tx_j) \\ &\leq \max\{d(Tx_i, Tx_j), d(x_i, Tx_j)\} \leq a_n \text{ so that} \end{aligned}$$

$a_n = d(Tx_i, Tx_j)$ or $d(x_i, Tx_j)$.
By *Case* (i), $a_n \neq d(Tx_i, Tx_j)$.
Therefore $a_n = d(x_i, Tx_j)$ and hence $a_n \leq b_n$ follows from *Case* (iv).
$\underline{Case\ (xi)}$: $a_n = d(z_i, z_j)$ for some $0 \leq i, j \leq n$.

$$\begin{aligned} a_n = d(z_i, z_j) = d(z_i, W(Tx_j, x_j, \gamma_j)) &\leq \gamma_j d(z_i, Tx_j) + (1-\gamma_j) d(z_i, x_j) \\ &\leq \max\{d(z_i, Tx_j), d(z_i, x_j)\} \leq a_n \text{ so that} \end{aligned}$$

$a_n = d(z_i, x_j)$ or $d(z_i, , Tx_j)$. Hence it follows from *Case* $(viii)$ and *Case* (x) that $a_n \leq b_n$.
$\underline{Case\ (xii)}$: Either $a_n = d(z_i, Ty_j)$ or $a_n = d(z_i, Tz_j)$.
In this case, clearly $a_n \leq b_n$.
Hence, by considering all the above cases, it follows that $a_n \leq b_n$ so that $a_n = b_n$ for $n = 1, 2, \ldots$.

Now for any $0 \leq i \leq n$,

$$\begin{aligned} d(x_0, Tx_i) &\leq d(x_0, Tx_0) + d(Tx_0, Tx_i) \\ &\leq A + w(M(x_0, x_i)) \\ &\leq A + w(a_n), \text{ where } A = d(x_0, Tx_0). \end{aligned}$$

Similarly, it is easy to see that
$d(x_0, Ty_i) \leq A + w(a_n)$ for $0 \leq i \leq n$ and
$d(x_0, Tz_i) \leq A + w(a_n)$ for $0 \leq i \leq n$.
Therefore $b_n \leq A + w(a_n)$ so that

$$a_n - w(a_n) \leq A \tag{2.2}$$

for $n = 1, 2, \ldots$, since $b_n = a_n$.
Since $\lim\limits_{t\to\infty} (t - w(t)) = \infty$, there exists $c > 0$ such that $t - w(t) > A$ for all $t > c$.
If $a_n > c$ for some $n \geq 1$ then $a_n - w(a_n) > A$,
a contradiction.
Thus $a_n \leq c$ for all n, i.e., the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded.
Hence the conclusion of the lemma follows. □

Theorem 2.2. *Let (X, d, W) be a complete convex metric space and K be a nonempty closed convex subset of X. Let $T : K \to K$ satisfy all the hypotheses of Lemma 2.1. Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, and $\{\gamma_n\}_{n=0}^{\infty}$ be sequences in $[0,1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}$ generated by the modified CR-iteration procedure (1.13) converges strongly to a unique fixed point of T.*

Proof. Without loss of generality, we assume that $x_n \neq Tx_n$ for any $n = 0, 1, 2, \ldots$.
For each integer $n \geq 0$, we let
$C_n = \{x_i\}_{i=n}^{\infty} \cup \{y_i\}_{i=n}^{\infty} \cup \{z_i\}_{i=n}^{\infty} \cup \{Tx_i\}_{i=n}^{\infty} \cup \{Ty_i\}_{i=n}^{\infty} \cup \{Tz_i\}_{i=n}^{\infty}$.
By Lemma 2.1, C_n is bounded. We denote the diameter of C_n by c_n.
Let $d_n = \max\{\sup_{i\geq n} d(x_n, Tx_i), \sup_{i\geq n} d(x_n, Ty_i), \sup_{i\geq n} d(x_n, Tz_i)\}$ for $n = 0, 1, 2, \ldots$.
Then it is easy to see that $c_n = d_n$ for $n = 0, 1, 2, \ldots$.
Clearly, the sequence $\{c_n\}$ is a decreasing sequence of nonnegative real numbers so that $\lim_{n\to\infty} c_n$ exists, we let it be c.
Now we prove that $c = 0$. On the contrary, we assume that $c > 0$ so that $c_n > 0$ for $n = 0, 1, 2, \ldots$.
For each positive integer n and for each $j \geq n$, we have

$$\begin{aligned} d(x_n, Tx_j) &= d(Tx_j, W(Ty_{n-1}, y_{n-1}, \alpha_{n-1})) \\ &\leq \alpha_{n-1} d(Tx_j, Ty_{n-1}) + (1 - \alpha_{n-1}) d(Tx_j, y_{n-1}) \\ &\leq \alpha_{n-1} w(M(x_j, y_{n-1})) + (1 - \alpha_{n-1}) d(Tx_j, y_{n-1}) \\ &\leq \alpha_{n-1} w(c_{n-1}) + (1 - \alpha_{n-1}) c_{n-1} \text{ so that} \end{aligned}$$

$\sup_{j\geq n} d(x_n, Tx_j) \leq \alpha_{n-1} w(c_{n-1}) + (1 - \alpha_{n-1}) c_{n-1}$.
Similarly, $\sup_{j\geq n} d(x_n, Ty_j) \leq \alpha_{n-1} w(c_{n-1}) + (1 - \alpha_{n-1}) c_{n-1}$ and
$\sup_{j\geq n} d(x_n, Tz_j) \leq \alpha_{n-1} w(c_{n-1}) + (1 - \alpha_{n-1}) c_{n-1}$ hold.
Therefore

$$d_n \leq \alpha_{n-1} w(c_{n-1}) + (1 - \alpha_{n-1}) c_{n-1} \quad \text{for} \quad n = 1, 2, \ldots .$$

Since $c_n = d_n$, we have

$$\alpha_{n-1}(c_{n-1} - w(c_{n-1})) \leq c_{n-1} - c_n \quad \text{for} \quad n = 1, 2, \ldots . \tag{2.3}$$

Let $s = \inf\{c_n - w(c_n) : n \geq 0\}$. If $s = 0$ then there exists a subsequence $\{c_{n(k)}\}$ of the sequence $\{c_n\}$ such that $\lim_{k\to\infty} (c_{n(k)} - w(c_{n(k)})) = 0$, i.e., $c - w(c^+) = 0$,
a contradiction, from (iii) of Lemma 2.1.

Therefore $s > 0$ so that there exists a real number $\eta > 0$ such that $c_n - w(c_n) \geq \eta$ for $n = 0, 1, 2, \ldots$.

It follows from the inequality (2.3) that $\eta\alpha_{n-1} \leq c_{n-1} - c_n$ for $n = 1, 2, \ldots$.
Since the sequence $\{c_n\}$ is convergent, we have the series $\sum \alpha_n < \infty$,
a contradiction.

Therefore $c = 0$ so that the sequence $\{x_n\}$ is Cauchy and hence there exists $x \in K$ such that $\lim_{n\to\infty} x_n = x$.
Since $c = 0$, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ so that $\lim_{n\to\infty} Tx_n = x$.

Now, we prove that x is a fixed point of T.
Since T satisfies the inequality (2.1), we have

$$d(Tx_n, Tx) \leq w(M(x_n, x)) \quad \text{for} \quad n = 0, 1, 2 \ldots \tag{2.4}$$

Since $M(x_n, x) \geq d(x, Tx)$ for $n = 0, 1, 2, \ldots$ and $\lim_{n\to\infty} M(x_n, x) = d(x, Tx)$, we have
$\lim_{n\to\infty} w(M(x_n, x)) = w(d(x, Tx)^+)$ so that $d(x, Tx) \leq w(d(x, Tx)^+)$.
Hence x is a fixed point of T by using (iii) of Lemma 2.1.

Now from the inequality (2.1) and Remark 1.6, clearly the uniquness of fixed point of T follows. □

If $\alpha_n \equiv 1$ in the modified CR-iteration procedure (1.13) then we have the following corollary from Theorem 2.2.

Corollary 2.3. *Let X, K, T be as in Theorem 2.2. Let $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ be sequences in $[0, 1]$. For $x_0 \in K$, let the sequence $\{x_n\}_{n=0}^{\infty}$ be generated by the modified Picard-S iteration procedure (1.14). Then $\{x_n\}_{n=0}^{\infty}$ converges to a unique fixed point of T.*

In the following, we prove that CR-iteration procedure (1.11) and Picard-S iteration procedure (1.12) converge to a unique fixed point of a quasi-contraction map under certain hypotheses in the setting of Banach spaces.

Corollary 2.4. *Let X be a Banach space, K be a nonempty closed convex subset of X, and $T : K \to K$ be a quasi contraction map. Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, and $\{\gamma_n\}_{n=0}^{\infty}$ be sequences in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. For $x_0 \in K$, let $\{x_n\}$ be the sequence generated by either CR-iteration procedure (1.11) or by Picard-S iteration procedure (1.12). Then $\{x_n\}$ converges strongly to a unique fixed point of T.*

Proof. Follows from Remark 1.7, Theorem 2.2 and Corollary 2.3. □

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