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The DEA Game Cross-Efficiency Model and Its Nash Equilibrium

Liang Liang, Jie Wu
School of Business, University of Science and Technology of China, He Fei, An Hui Province, People's Republic of China 230026
{lliang@ustc.edu.cn, jacky012@mail.ustc.edu.cn}

Wade D. Cook
Schulich School of Business, York University, Toronto, Ontario, Canada M3J 1P3,
wcook@schulich.yorku.ca

Joe Zhu
Department of Management, Worcester Polytechnic Institute, Worcester, Massachusetts 01609,
jzhu@wpi.edu

In this paper, we examine the cross-efficiency concept in data envelopment analysis (DEA). Cross efficiency links one decision-making unit's (DMU) performance with others and has the appeal that scores arise from peer evaluation. However, a number of the current cross-efficiency approaches are flawed because they use scores that are arbitrary in that they depend on a particular set of optimal DEA weights generated by the computer code in use at the time. One set of optimal DEA weights (possibly out of many alternate optima) may improve the cross efficiency of some DMUs, but at the expense of others. While models have been developed that incorporate secondary goals aimed at being more selective in the choice of optimal multipliers, the alternate optima issue remains. In cases where there is competition among DMUs, this situation may be seen as undesirable and unfair. To address this issue, this paper generalizes the original DEA cross-efficiency concept to game cross efficiency. Specifically, each DMU is viewed as a player that seeks to maximize its own efficiency, under the condition that the cross efficiency of each of the other DMUs does not deteriorate. The average game cross-efficiency score is obtained when the DMU's own maximized efficiency scores are averaged. To implement the DEA game cross-efficiency model, an algorithm for deriving the best (game cross-efficiency) scores is presented. We show that the optimal game cross-efficiency scores constitute a Nash equilibrium point.

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1. Introduction

Data envelopment analysis (DEA) has been proven an effective tool for performance evaluation and benchmarking. In the original DEA model of Charnes, Cooper, and Rhodes (CCR) (1978), the efficiency of each member of a set of n decision making units (DMUs), relative to its peers, is defined as the ratio of that member's weighted sum of outputs to weighted sum of inputs. Linear programming is used to determine a set of weights that is optimal in the sense that it results in the best efficiency score for the particular DMU under evaluation. The *cross-efficiency score* of a DMU is obtained by computing that DMU's set of n scores (using the n sets of optimal weights), and then averaging those scores. Thus, the main idea of cross efficiency is to use DEA in a peer evaluation, rather than a pure self-evaluation mode. This approach was originated by Sexton et al. (1986), and was further investigated by Doyle and Green (1994), and others. Cross efficiency provides an efficiency ordering among all the DMUs to differentiate between good and poor performers. It can eliminate the need for incorporation of additional weight restrictions into DEA, thereby avoiding unrealistic DEA weighting schemes (see Anderson et al. 2002). As pointed out by Doyle and Green (1994), cross efficiency is a democratic process with less of the arbitrariness of additional weight restrictions, as opposed to authoritarianism (externally imposed weights) or out and out egoism (self-appraisal). One can find many uses of cross efficiency, for example, R&D project selection (Oral et al. 1991), preference voting (Green et al. 1996), and others.

As noted in Doyle and Green (1994), the nonuniqueness of the DEA optimal weights possibly reduces the usefulness of cross efficiency. Specifically, cross-efficiency scores obtained from the original DEA are generally not unique. Thus, depending on which of the alternate optimal solutions to the DEA linear programs is used, it may be possible to improve a DMU's (cross-efficiency) performance rating, but generally only by worsening the ratings of others. Various secondary goals have been proposed for cross-efficiency calculation, such as those presented in Doyle and Green (1994). They developed aggressive (benevolent) model formulations to identify optimal weights that not only maximize the efficiency of a particular DMU under evaluation, but also minimize (maximize) the average efficiency of other DMUs.

In many DEA applications, some form of direct or indirect competition may exist among the DMUs under study. Certainly, in any setting where DMUs compete for scarce funds, competition is present by definition. R&D project proposals submitted by different departments in an organization can be viewed as DMUs, and subjected to a DEA analysis. These proposals are clearly competing for available funds. Candidates in a preferential election setting can be looked upon as DMUs, and competition is obviously present. An academic applying for research grants is in competition with other academics. Participants in organized sporting events such as the Olympic games constitute competitive DMUs. Arguably, the many DEA analyses of banks and of bank branches, reported in the literature, are all examples of indirect if not direct competition. Specifically, management of a large bank, in appraising the efficiency status of the branches under its direction, will be paying close attention to the worst performers with the possible intention of closing some of those, or amalgamating/merging units. These same observations apply to organizations such as hospitals and schools that operate in tight financial situations. Such organizations can be seen as competing for state or provincial funds, and a fair appraisal system is essential. Some of these examples will be discussed in more detail later.

When DMUs are viewed as players in a game, cross-efficiency scores may be viewed as payoffs, and each DMU may choose to take a noncooperative game stance to the extent that it will attempt to maximize its (worst possible) payoff. If one adopts this game-theoretic approach, it may be argued that the existing approaches to cross evaluation suffer shortcomings in regard to these common situations. This paper is aimed at rectifying this important shortcoming. Section 2 briefly reviews the original cross-efficiency concept in DEA, and introduces a generalized concept, namely, *game cross efficiency*, and its corresponding DEA model. In §3, an algorithm is developed for deriving "best" game cross-efficiency scores, and is shown to be convergent. In §4, it is proven that the best game cross efficiency is a Nash equilibrium point. Section 5 revisits the preference voting issue discussed in Green et al. (1996). It is shown that the rank-reversal problem encountered there does not occur when our approach is used. We also provide an illustrative application involving the selection of R&D projects as discussed in Oral et al. (1991). Concluding remarks are given in §6.

2. DEA Game Cross Efficiency

Adopting the conventional nomenclature of DEA, assume that there are n DMUs that are to be evaluated in terms of m inputs and s outputs. We denote the ith input and rth output for DMU_j $(j = 1, 2, \ldots, n)$ as x_{ij} $(i = 1, \ldots, m)$ and y_{rj} $(r = 1, \ldots, s)$, respectively. The efficiency rating for any given DMU_d can be computed using the CCR model (in LP format):

$$\begin{aligned} \text{Max} \quad & \sum_{r=1}^{s} \mu_r y_{rd} = \theta_d \\ \text{s.t.} \quad & \sum_{i=1}^{m} \omega_i x_{ij} - \sum_{r=1}^{s} \mu_r y_{rj} \geqslant 0, \quad j = 1, 2, \ldots, n, \\ & \sum_{i=1}^{m} \omega_i x_{id} = 1, \\ & \omega_i \geqslant 0, \quad i = 1, 2, \ldots, m, \\ & \mu_r \geqslant 0, \quad r = 1, 2, \ldots, s. \end{aligned} \tag{1}$$

For each DMU_d $(d = 1, \ldots, n)$ under evaluation, we obtain a set of optimal weights (multipliers) $\omega_{1d}^*, \ldots, \omega_{md}^*$, $\mu_{1d}^*, \ldots, \mu_{sd}^*$. Using this set, the *d-cross efficiency* for any DMU_j $(j = 1, \ldots, n)$ is then calculated as

$$E_{dj} = \frac{\sum_{r=1}^{s} \mu_{rd}^* y_{rj}}{\sum_{i=1}^{m} \omega_{id}^* x_{ij}}, \quad d, j = 1, 2, \ldots, n. \tag{2}$$

For DMU_j $(j = 1, \ldots, n)$, the average of all E_{dj} $(d = 1, \ldots, n)$, namely,

$$\bar{E}_j = \frac{1}{n} \sum_{d=1}^{n} E_{dj}, \tag{3}$$

can be used as a new efficiency measure for DMU_j, and will be referred to as the *cross-efficiency score* for DMU_j.

Note that optimal weights obtained from model (1) may not be unique. As a result, the d-cross-efficiency E_{dj} can be arbitrarily generated, depending on the optimal solution arising from the particular software in use (Despotis 2002). As mentioned above, to resolve this ambiguity, Doyle and Green (1994) introduced the aggressive and benevolent formulations of cross-efficiency calculation. Second-order models such as the benevolent model of Doyle and Green (1994) improve on the efficiency scores further by choosing an optimal bundle for DMU_d (from several possible alternates) that renders some function of the efficiency scores of the remaining $n-1$ DMUs (such as their average), as large as possible. Specifically, given a DMU_d, one version of their model seeks to find a multiplier bundle that maximizes the average of the efficiency ratios of the other $n-1$ DMUs with the constraint that the ratio for DMU_d stays at or above its predetermined optimal level.

In this paper, we view DMUs from the perspective of a noncooperative game. As discussed above, in many DEA settings, DMUs can be looked upon as being in competition with one another, and as such each may argue that its multiplier bundle should be chosen with a view to how that bundle impacts the implied performance of the other DMUs (should that bundle be used to evaluate each of those others). Conventional DEA, as per model (1), does this in a narrow sense by restricting the choice of bundles to those that keep the efficiency scores of all DMUs at or below unity. Cross efficiency goes a step further, providing for a measure of efficiency of a DMU_d in terms of not only the best multiplier bundle for d, as derived from model (1), but as well in terms of the best bundles for all the other DMUs as well.

In the model described here, we adopt what might be regarded as a form of a generalized benevolent approach. The difference between our approach and conventional cross efficiency has to do with what will be taken as the "optimal" rating for DMU_d. Rather than looking at the situation from the perspective of DMU_d, and finding a multiplier bundle that optimizes say the average rating for the other $n-1$ DMUs, while restricting the score for DMU_d to be at or above its ideal level (from model (1)), we instead look at the problem from the point of view of each of the competitors j. For each competing DMU_j, a multiplier bundle is determined that optimizes the efficiency score for j, with the additional constraint that the resulting score for d should be at or above d's estimated best performance, in a cross-efficiency sense. One can view the conventional benevolent model as looking at the problem from the perspective of the collection of $n-1$ DMUs, and attempting to find the best score for the average of this collection, while guaranteeing that the ideal score for DMU_d is not violated. In our case, rather than using the ideal score for DMU_d, we strive to use a score which will actually be representative of its final measure of performance. The problem, of course, arises that we will not know this best performance score for d until the best performances of all other DMUs are known as well. To combat this "chicken and egg" phenomenon, we adopt an iterative approach that we shall prove leads to an equilibrium.

To make these ideas more concrete, suppose that in a game sense, one player DMU_d is given an efficiency score α_d, and that another player DMU_j then tries to maximize its own efficiency, subject to the condition that α_d cannot be decreased. We define the *game cross efficiency* for DMU_j relative to DMU_d as

$$\alpha_{dj} = \frac{\sum_{r=1}^{s} \mu_{rj}^{d} y_{rj}}{\sum_{i=1}^{m} \omega_{ij}^{d} x_{ij}}, \quad d = 1, 2, \ldots, n, \tag{4}$$

where μ_{rj}^{d} and ω_{ij}^{d} are optimal weights in the following model (5). The subscript dj is intended to indicate that DMU_j is permitted only to choose weights that will not deteriorate the currently estimated efficiency of DMU_d. The difference between (2) and (4) is that weights in (4) are not necessarily an optimal, but rather are a feasible solution to the CCR model. Such a definition allows DMUs to choose (negotiate) a set of weights (hence a form of cross-efficiency scores), that are best for all of the DMUs. So, in this sense, we adopt a noncooperative game approach.

To calculate the game d-cross efficiency defined in (4), we consider the following mathematical programming problem for each DMU_j:

$$\begin{aligned}
\text{Max} \quad & \sum_{r=1}^{s} \mu_{rj}^{d} y_{rj} \\
\text{subject to} \quad & \sum_{i=1}^{m} \omega_{ij}^{d} x_{il} - \sum_{r=1}^{s} \mu_{rj}^{d} y_{rl} \geqslant 0, \quad l = 1, 2, \ldots, n, \\
& \sum_{i=1}^{m} \omega_{ij}^{d} x_{ij} = 1, \\
& \alpha_d \times \sum_{i=1}^{m} \omega_{ij}^{d} x_{id} - \sum_{r=1}^{s} \mu_{rj}^{d} y_{rd} \leqslant 0, \\
& \omega_{ij}^{d} \geqslant 0, \quad i = 1, \ldots, m, \\
& \mu_{rj}^{d} \geqslant 0, \quad r = 1, \ldots, s,
\end{aligned} \tag{5}$$

where $\alpha_d \leqslant 1$ is a parameter. In the algorithm to be developed, this α_d initially takes the value given by the average original cross efficiency of DMU_d. When the algorithm converges, this α_d becomes the best (average) game-cross efficiency score. (If one wished to draw a comparison of this model to the benevolent model of Doyle and Green (1994), we would replace the objective function by the average of the efficiency ratios for the $n-1$ DMUs j, and α_d would be the ideal rating (from model (1)) for DMU_d.) We refer to model (5) as the *DEA game d-cross-efficiency model*. Note that model (5) maximizes the efficiency of DMU_j under the condition that the efficiency of a given DMU_d, namely, $\sum_{r=1}^{s} \mu_{rj}^{d} y_{rd} / \sum_{i=1}^{m} \omega_{ij}^{d} x_{id}$, is not less than a given value (α_d). Thus, the efficiency of DMU_j is further constrained by the requirement that the ratio efficiency of DMU_d is not less than its original average cross efficiency.

For each DMU_j, model (5) is solved n times, once for each $d = 1, \ldots, n$. Note that for each d, at optimality, $\sum_{i=1}^{m} \omega_{ij}^{d} x_{ij} = 1$ holds for DMU_j ($j = 1, 2, \ldots, n$). Therefore, for each DMU_j, the optimal value to model (5) actually represents a game cross efficiency with respect to DMU_d (d-game cross efficiency), as defined in (4). We have

Definition 1. Let $\mu_{rj}^{d*}(\alpha_d)$ be an optimal solution to model (5). For each DMU_j,

$$\alpha_j = \frac{1}{n} \sum_{d=1}^{n} \sum_{r=1}^{s} \mu_{rj}^{d*}(\alpha_d) y_{rj}$$

is called the *average game cross-efficiency* for that DMU.

Note that the average game cross efficiency no longer represents a regular DEA cross-efficiency value.

We now present a procedure for determining the best average game cross efficiency for DMU_j.

3. Optimal Average Game Cross-Efficiency Scores

In this section, we present an iterative procedure for deriving average game cross-efficiency scores, and prove that these converge. The basic idea of the algorithm is to begin with the conventional cross-efficiency score as developed in (3), and for each DMU d, solve model (5) for each j, using this as the initial α_d. This process is repeated for every d, and the average of the objective function values of (5) becomes the new α_d. When consecutive values of α_d converge to within ε of one another, the algorithm terminates. The specifics follow.

Algorithm

Step 1. Solve model (1) and obtain a set of original average DEA cross-efficiency scores defined in (3). Let $t = 1$ and $\alpha_d = \alpha_d^1 = \bar{E}_d$.

Step 2. Solve model (5). Let

$$\alpha_j^2 = \frac{1}{n}\sum_{d=1}^{n}\sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha_d^1)y_{rj},$$

or in a general format,

$$\alpha_j^{t+1} = \frac{1}{n}\sum_{d=1}^{n}\sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha_d^t)y_{rj}, \tag{6}$$

where $\mu_{rj}^{d*}(\alpha_d^t)$ represents optimal value of μ_{rj}^d in model (5) when $\alpha_d = \alpha_d^t$.

Step 3. If $|\alpha_j^{t+1} - \alpha_j^t| \geqslant \varepsilon$ for some j, where ε is a specified small positive value, then let $\alpha_d = \alpha_d^{t+1}$ and go to Step 2. If $|\alpha_j^{t+1} - \alpha_j^t| < \varepsilon$ for all j, then stop. α_j^{t+1} is the best average game cross efficiency given to DMU_j.

Remarks. In Step 1, the $\bar{E}_d$ represent traditional (average) cross-efficiency scores for DMU_d, $d = 1, 2, \ldots, n$, and are the initial values for α_d (denoted as α_d^1) in model (5). Although traditional cross-efficiency scores may not be unique, from the proof of convergence of the algorithm, it follows that any initial values for α_d (or any traditional cross-efficiency scores), will lead to unique game cross-efficiency scores. When the algorithm stops, because $\sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha_d^t)y_{rj}$ is the optimal value to model (5), $\alpha_j^{t+1} = (1/n)\sum_{d=1}^{n}\sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha_d^t)y_{rj}$, $t \geqslant 1$, is unique. Also, the notation $\alpha_d = \alpha_d^t$, $t \geqslant 1$, given in Step 2, means that in model (5), α_d is replaced with α_d^t. Step 3 is used to indicate when to terminate the process of executing model (5).

Convergence of the Algorithm

The following theorem indicates that (i) all the data points α_j are bounded between α_j^1 and α_j^2, (ii) all the even data points are nonincreasing, and (iii) all the odd data points are nondecreasing. This ensures that the above algorithm converges.

Theorem 1. *Let α_j^1 be the regular (average) DEA cross efficiency defined in (3). With model (5), for any $t = 2, 3, 4, \ldots,$ and $j = 1, 2, \ldots, n$, we have*

(i) $\alpha_j^1 \leqslant \alpha_j^t$,

(ii) $\alpha_j^2 \geqslant \alpha_j^4 \geqslant \cdots \geqslant \alpha_j^{2t-2} \geqslant \alpha_j^{2t} \geqslant \alpha_j^{2t-1} \geqslant \alpha_j^{2t-3} \geqslant \cdots \geqslant \alpha_j^3 \geqslant \alpha_j^1$.

Proof. (i) Let $\alpha_d = \alpha_d^{\text{CCR}}$ in model (5), where α_d^{CCR} is the CCR efficiency for DMU_d. Note that weights $\omega_{1d}^*, \ldots, \omega_{md}^*$, $\mu_{1d}^*, \ldots, \mu_{sd}^*$ obtained from model (1) when DMU_d is under evaluation, are feasible solutions to (5). Therefore, when $\alpha_d = \alpha_d^{\text{CCR}}$, we have $(1/n)\sum_{d=1}^{n}\sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha_d^{\text{CCR}})y_{rj} \geqslant (1/n)\sum_{d=1}^{n}E_{dj} = \alpha_j^1$, where the $\mu_{rj}^{d*}(\alpha_d^{\text{CCR}})$ represent optimal values of μ_{rj}^d in model (5), when $\alpha_d = \alpha_d^{\text{CCR}}$. Because α_d^{CCR} is the maximum efficiency that DMU_d can achieve, i.e., $\alpha_d^t \leqslant \alpha_d^{\text{CCR}}$, then the feasible region of (5) will not be reduced when $\alpha_d = \alpha_d^{\text{CCR}}$ is replaced with $\alpha_d = \alpha_d^t$. Thus, the optimal value to (4) will always be at least as large as $\sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha_d^{\text{CCR}})y_{rj}$, meaning that $\alpha_j^t \geqslant (1/n)\sum_{d=1}^{n}\sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha_d^{\text{CCR}})y_{rj}$ for all $t > 1$. Therefore, $\alpha_j^t \geqslant \alpha_j^1$ for all t.

(ii) Let us first look at the relations among α_j^1, α_j^2, α_j^3, and α_j^4 for all j. Based upon (i), we know that $\alpha_j^2 \geqslant \alpha_j^1$. Therefore, when $t = 2$ and $\alpha_d = \alpha_d^1$ is replaced with $\alpha_d = \alpha_d^2$, the feasible region of model (5) reduces. Thus, $\sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha_d^2)y_{rj} \leqslant \sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha_d^1)y_{rj}$ for $d = 1, \ldots, n$, where $\mu_{rj}^{d*}(\alpha_d^2)$ and $\mu_{rj}^{d*}(\alpha_d^1)$ represent optimal values of μ_{rj}^d in (5) associated with α_d^2 and α_d^1, respectively. We then have $\alpha_j^3 = (1/n)\sum_{d=1}^{n}\sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha_d^2)y_{rj} \leqslant (1/n)\sum_{d=1}^{n}\sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha_d^1)y_{rj} = \alpha_j^2$ for all j. When $t = 3$ and $\alpha_d = \alpha_d^2$ is replaced with $\alpha_d = \alpha_d^3$, the feasible region of model (5) increases. As a result, $\sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha_d^3)y_{rj} \geqslant \sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha_d^2)y_{rj}$ for all d, indicating that $\alpha_j^4 \geqslant \alpha_j^3$ for all j. Similarly, because $\alpha_j^3 \geqslant \alpha_j^1$ is true for all j, if $\alpha_d = \alpha_d^1$ is replaced with $\alpha_d = \alpha_d^3$ in model (5), then $\sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha_d^3)y_{rj} \leqslant \sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha_d^1)y_{rj}$ for all d, indicating that $\alpha_j^4 \leqslant \alpha_j^2$. Therefore, we have $\alpha_j^2 \geqslant \alpha_j^4 \geqslant \alpha_j^3 \geqslant \alpha_j^1$ for all j.

We next prove that for $t \geqslant 2$:

(A) $\alpha_j^{2a} \geqslant \alpha_j^{2a-1}$, $j = 1, 2, \ldots, n$; $a = 1, 2, 3, \ldots$.

(B) $\alpha_j^{2a} \geqslant \alpha_j^{2a+2}$, $j = 1, 2, \ldots, n$; $a = 1, 2, 3, \ldots$.

(C) $\alpha_j^{2a+1} \geqslant \alpha_j^{2a-1}$, $j = 1, 2, \ldots, n$; $a = 1, 2, 3, \ldots$.

Note that $\alpha_j^2 \geqslant \alpha_j^1$. Proceeding by induction, suppose that for $a = \Delta$, we have $\alpha_j^{2\Delta} \geqslant \alpha_j^{2\Delta-1}$, $j = 1, 2, \ldots, n$. We further have that the feasible region of (5), when $\alpha_d = \alpha_d^{2\Delta}$, is not larger than when $\alpha_d = \alpha_d^{2\Delta-1}$. Thus, when $\alpha_d = \alpha_d^{2\Delta}$, we have $\alpha_j^{2\Delta} \geqslant \alpha_j^{2\Delta+1}$. Furthermore, when $\alpha_d = \alpha_d^{2\Delta}$ is replaced with $\alpha_d = \alpha_d^{2\Delta+1}$, the feasible region of model (5) will not reduce. Therefore, $\alpha_j^{2\Delta+2} \geqslant \alpha_j^{2\Delta+1}$. Thus, by induction, (A) is true for all a.

In case (B), note that $\alpha_j^2 \geqslant \alpha_j^4 \geqslant \alpha_j^3 \geqslant \alpha_j^1$, i.e., $\alpha_j^2 \geqslant \alpha_j^4$, $j = 1, 2, \ldots, n$. Let Θ_t denote the feasible region for model (5) when $\alpha_d = \alpha_d^t$. Suppose when $a = \Delta$, (B) is true. i.e., $\alpha_j^{2\Delta} \geqslant \alpha_j^{2(\Delta+1)}$, $j = 1, 2, \ldots, n$, indicating $\Theta_{2\Delta+2} \supseteq \Theta_{2\Delta}$. From (A), we have $\alpha_j^{2(\Delta+1)} \geqslant \alpha_j^{2\Delta+1}$, and $\Theta_{2\Delta+1} \supseteq \Theta_{2\Delta+2}$. Thus, $\Theta_{2\Delta+1} \supseteq \Theta_{2\Delta+2} \supseteq \Theta_{2\Delta}$. Because $\alpha_j^{2\Delta+3}$ and $\alpha_j^{2\Delta+2}$ are the optimal values based upon $\Theta_{2\Delta+2}$ and $\Theta_{2\Delta+1}$, respectively, then $\alpha_j^{2\Delta+2} \geqslant \alpha_j^{2\Delta+3}$, indicating $\Theta_{2\Delta+3} \supseteq \Theta_{2\Delta+2}$. Next,

Table 1. Numerical Example 1.

	X_1	X_2	X_3	Y_1	Y_2
DMU1	7	7	7	4	4
DMU2	5	9	7	7	7
DMU3	4	6	5	5	7
DMU4	5	9	8	6	2
DMU5	6	8	5	3	6

suppose that $\alpha_j^{2\Delta+4} > \alpha_j^{2\Delta+2}$. We know that $\alpha_j^{2\Delta+3}$ is the optimal value for $\Theta_{2\Delta+2}$, and $\alpha_j^{2\Delta+2} \geqslant \alpha_j^{2\Delta+3}$. Because $\alpha_j^{2\Delta+4}$ is the optimal value in $\Theta_{2\Delta+3}$, if $\alpha_j^{2\Delta+4} > \alpha_j^{2\Delta+2}$, this indicates that optimal solutions for obtaining $\alpha_j^{2\Delta+4}$ are also feasible in $\Theta_{2\Delta+2}$. Given (A), we have that $\alpha_j^{2\Delta+4} \geqslant \alpha_j^{2\Delta+3}$. This is a contradiction of the fact that $\alpha_j^{2\Delta+3}$ is the optimal value based upon $\Theta_{2\Delta+2}$. Therefore, $\alpha_j^{2\Delta+2} \geqslant \alpha_j^{2\Delta+4}$. This shows that (B) is true when $a = \Delta + 1$, and by induction for all a.

We finally prove (C) is true. From (B), we have $\alpha_j^{2a} \geqslant \alpha_j^{2a+2}$. Note that α_j^{2a+1} and α_j^{2a+3} are obtained based upon feasible regions Θ_{2a} and Θ_{2a+2}. Note also that $\Theta_{2a+2} \supseteq \Theta_{2a}$. Therefore, $\alpha_j^{2a+3} \geqslant \alpha_j^{2a+1}$, and (C) is true.

From (A), (B), and (C), we have

$$\alpha_j^2 \geqslant \alpha_j^4 \geqslant \cdots \geqslant \alpha_j^{2t-2} \geqslant \alpha_j^{2t} \geqslant \alpha_j^{2t-1} \geqslant \alpha_j^{2t-3} \geqslant \cdots \geqslant \alpha_j^3 \geqslant \alpha_j^1. \quad \text{Q.E.D.}$$

Numerical Example

To illustrate the game-efficiency model and the proposed algorithm, we consider a simple numerical example given in Table 1 involving five DMUs, with three inputs X_1, X_2, X_3 and two outputs Y_1, Y_2. In the algorithm, we use the regular cross efficiency as the starting point for our game cross-efficiency scores. Cross efficiency is not unique here, and secondary goals can be imposed. For example, we can use an aggressive strategy which not only obtains the maximum DEA efficiency for a DMU as the primary goal, but also as a secondary goal, minimizes the other DMUs' cross efficiencies (Sexton et al. 1986). We can also use a benevolent strategy which not only obtains the maximum DEA efficiency, but also maximizes the other DMUs' cross efficiencies (Doyle and Green 1994). The cross efficiency calculated without imposing the secondary goal is referred to as an *arbitrary strategy*, as defined in (2).

Table 2. CCR efficiency, cross efficiencies, and game cross efficiency for Example 1.

	CCR efficiency	Game cross efficiency*	DEA cross efficiency		
			Arbitrary	Aggressive	Benevolent
DMU1	0.6857	0.6384	0.4743	0.4473	0.5845
DMU2	1	0.9766	0.8793	0.8629	0.9295
DMU3	1	1	0.9856	0.9571	1
DMU4	0.8571	0.7988	0.5554	0.54	0.71
DMU5	0.8571	0.667	0.5587	0.4971	0.6386

*In the algorithm, we set $\varepsilon = 0.001$.

The results of the cross-efficiency evaluation under three strategies are reported in the last three columns of Table 2. The game cross efficiency is shown in the third column. Figure 1 shows the solution process for DMU1. Three different traditional cross-efficiency scores (arbitrary, aggressive, benevolent) are used. All these cross-efficiency scores lead to the same game cross-efficiency scores. In the next section, we will show that this solution is a Nash equilibrium. If one views the DMUs as competitive, it is noted that in a (noncooperative) game sense, each "player" has an improved score over that which it received under the usual cross-efficiency models (except in the case of the 100% efficient DMU3).

Figure 2 shows that after 11 iterations, the proposed algorithm finds the game cross-efficiency scores for the five DMUs. As per Theorem 1, it can be seen that the game cross-efficiency scores increase when t takes on even numbered increasing values, and decrease for increasing odd numbered values.

Figure 1. Achieving the best game cross efficiency for DMU1 in Table 1.

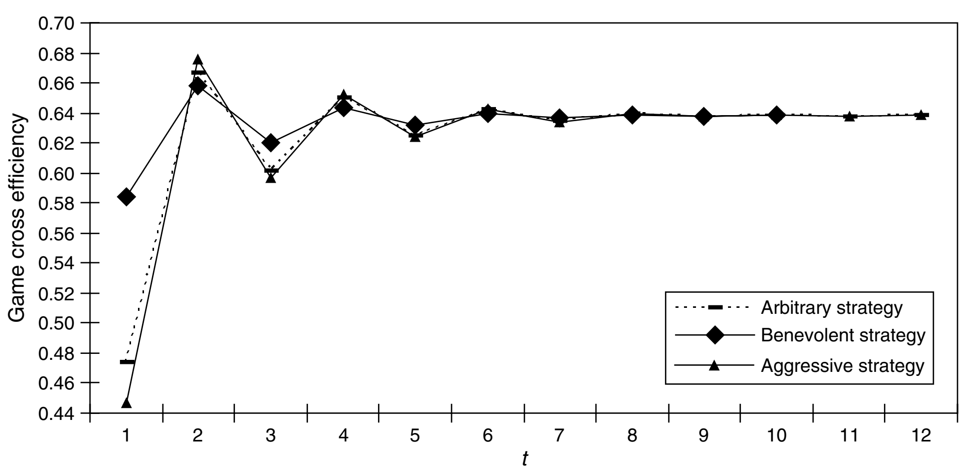
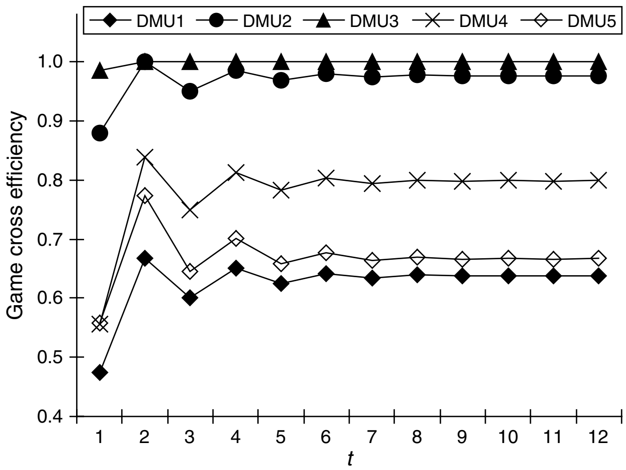


Figure 2. Game cross-efficiency calculation for DMUs in Table 1.



4. Nash Equilibrium

In this section, we show that the DEA game, with the game cross efficiencies as the payoffs, has a Nash equilibrium given by the solution obtained from the proposed algorithm given above.

From the proof of Theorem 1, the following is true:

Lemma 1. *If $\alpha_d^{CCR} \geqslant \alpha_d \geqslant \bar{E}_d$, $d = 1, 2, \ldots, n$, where $\bar{E}_d$ is the original average DEA cross efficiency of DMU_d defined in* (3), *and α_d^{CCR} is the CCR efficiency of DMU_d, then model* (5) *is feasible.*

Definition 2. A *DEA game* is defined as

$$\Gamma = \left\langle N, (S_j)_{j\in N}, \left(\frac{1}{n}\sum_{d=1}^{n}\sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha_d)y_{rj}\right)_{j\in N}\right\rangle,$$

where $N = \{1, 2, \ldots, n\}$ is the set of n players (DMUs), and $S_j = \{$the constraints of model (5) and $\alpha_d \in [\bar{E}_d, \alpha_d^{\text{CCR}}]\}$ represents the strategy set of DMU_j, $j = 1, 2, \ldots, n$.

Lemma 2. *The S_j, $j = 1, 2, \ldots, n$, of Definition* 2, *are nonempty convex sets.*

Proof. By Lemma 1, we know that S_j is nonempty. Now, assume both $(\omega'_{1d}, \ldots, \omega'_{md}, \mu'_{1d}, \ldots, \mu'_{sd})$, α'_d and $(\omega''_{1d}, \ldots, \omega''_{md}, \mu''_{1d}, \ldots, \mu''_{sd})$, $\alpha''_d \in S_j$, $j, d \in N$. For any $\lambda \in [0, 1]$, we have

$$[\lambda\omega'_{id} + (1-\lambda)\omega''_{id}, i = 1, 2, \ldots, m;$$
$$\lambda\mu'_{rd} + (1-\lambda)\mu''_{rd}, r = 1, 2, \ldots, s; \lambda\alpha'_d + (1-\lambda)\alpha''_d] \in S_j.$$

Therefore, S_j, $j = 1, 2, \ldots, n$ is convex. Q.E.D.

Lemma 3. $(1/n)\sum_{d=1}^{n}\sum_{r=1}^{s}\mu_{rj}^{d}(\alpha_d)y_{rj}$ *is a continuous semiconcave function of α_d.*

Proof. (i) In regard to the continuity of

$$\frac{1}{n}\sum_{d=1}^{n}\sum_{r=1}^{s}\mu_{rj}^{d}(\alpha_d)y_{rj}$$

with respect to α_d, Lemma 1 indicates that model (5) becomes the CCR model if $\alpha_d < \bar{E}_d$, and is infeasible if $\alpha_d > \alpha_d^{\text{CCR}}$. If $\alpha_d \in [\bar{E}_d, \alpha_d^{\text{CCR}}]$, model (5) arises by adding a constraint $\alpha_d \times \sum_{i=1}^{m}\omega_{ij}^{d}x_{id} - \sum_{r=1}^{s}\mu_{rj}^{d}y_{rd} \leqslant 0$, into the CCR model of DMU_j, $j = 1, 2, \ldots, n$. Using arguments based on sensitivity analysis of linear programming, it can be shown that $\{\sum_{r=1}^{s}\mu(\alpha_d)_{rj}^{d}y_{rj}, d = 1, 2, \ldots, n\}$ are continuous functions of α_d if $\alpha_d \in [\bar{E}_d, \alpha_d^{\text{CCR}}]$. (See the appendix for a detailed proof.)

(ii) To show that $(1/n)\sum_{d=1}^{n}\sum_{r=1}^{s}\mu_{rj}^{d}(\alpha_d)y_{rj}$ is semiconcave, assume that α'_d, $\alpha''_d \in [\bar{E}_d, \alpha_d^{\text{CCR}}]$, $\lambda \in [0, 1]$, and suppose that $\alpha'_d > \alpha''_d$. Then, $\alpha'_d \geqslant \lambda\alpha'_d + (1-\lambda)\alpha''_d \geqslant \alpha''_d$. The feasible region of model (5) with $\alpha_d = \lambda\alpha'_d + (1-\lambda)\alpha''_d$ will not become smaller when $\alpha_d = \alpha'_d$ and will not become larger when $\alpha_d = \alpha''_d$. Therefore, we have $\sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha'_d)y_{rj} \leqslant \sum_{r=1}^{s}\mu_{rj}^{d*}(\lambda\alpha'_d + (1-\lambda)\alpha''_d)y_{rj} \leqslant \sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha'_d)y_{rj}$. This means $(1/n)\sum_{d=1}^{n}\sum_{r=1}^{s}\mu_{rj}^{d}(\alpha_d)y_{rj}$ is semiconcave. Q.E.D.

Based on Lemmas 2 and 3, Debreu (1952), and Glicksberg (1952), we have

Theorem 2. *The DEA game* $\Gamma = \langle N, (S_j)_{j\in N}, ((1/n) \cdot \sum_{d=1}^{n}\sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha_d)y_{rj})_{j\in N}\rangle$ *has at least one Nash equilibrium strategy portfolio.*

We next show that the solution obtained from the proposed algorithm is such a Nash equilibrium point. Let $f_j(\alpha^{t-1})$ be defined as

$$f_j(\alpha^{t-1}) = \alpha_j^t = \frac{1}{n}\sum_{d=1}^{n}\sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha_d^{t-1})y_{rj}, \tag{7}$$

and let $\vec{\alpha}^t = [\alpha_1^t, \alpha_2^t, \ldots, \alpha_n^t]^T$, $t = 2, 3, 4 \ldots$.

Now define

$$F(\vec{\alpha}^{t-1}) = [f_1(\alpha^{t-1}), f_2(\alpha^{t-1}), \ldots, f_n(\alpha^{t-1})]^T. \tag{8}$$

We have $\vec{\alpha}^t = F(\vec{\alpha}^{t-1})$, $t \geqslant 2$.

Theorem 3. *For the $F()$ defined in* (8), *there must exist $\vec{\alpha}^*$, such that $\vec{\alpha}^* = F(\vec{\alpha}^*)$, i.e., there must exist a fixed point of $\vec{\alpha}^* = [\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*]^T$.*

Proof. Note that $\omega_{ij}^{d*}(\alpha_d^{t-1})$, $\mu_{rj}^{d*}(\alpha_d^{t-1})$, $\alpha_d^{t-1} \in S_j$, $j = 1, 2, \ldots, n$. From Lemma 2, the Cartesian product S of S_j (i.e., $S = S_1 \times S_2 \times \cdots \times S_n$) is a nonempty, compact, convex set. Further, from this lemma, $(1/n)\sum_{d=1}^{n}\sum_{r=1}^{s}\mu_{rj}^{d*}(\alpha_d^{t-1})y_{rj}$, $\alpha_d^{t-1} \in S_d$, $d = 1, 2, \ldots, n$, $j \in N$ is continuous. Therefore, $F()$: $S \to S$ is a continuous function from a nonempty, compact, convex set $S \subset R^n$ into itself. From Brouwer's fixed-point theorem (Brouwer 1911), we know that there must exist $\vec{\alpha}^* \in S$, such that $\vec{\alpha}^* = F(\vec{\alpha}^*)$, where $\vec{\alpha}^* = [\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*]^T$. Q.E.D.

Recall that the algorithm terminates when $|\vec{\alpha}^t - \vec{\alpha}^{t-1}| = |F(\alpha^{t-1}) - \vec{\alpha}^{t-1}| < \varepsilon$. Therefore, the smaller the ε, the closer the solution is to the fixed point. Such a fixed point is a Nash equilibrium (Becker and Chakrabarti 2005).

5. Applications to Preference Voting and R&D Projects Selection

Preference Voting

Cook and Kress (1990) developed a DEA-type model to rank order the candidates in a preferential election. The candidates are allowed to choose the most favorable weights to be applied to his/her standings (first place, second place, etc. votes). Green et al. (1996) consider this type of weighting illusory and propose using cross efficiency to maximize discrimination between the candidates. One can argue that in such a setting, competition exists among the players (candidates). Therefore, it is appropriate to apply the game cross-efficiency approach.

Green et al. (1996) consider a case of 20 voters, each of whom is asked to rank four out of six candidates on a ballot. The voting outcomes are given in Table 3. For example, candidate "a" receives 3 first, 3 second, 4 third, and 3 fourth-placed votes. The data shown in Table 3 are used as four outputs. We suppose that each candidate or DMU has a single input of one.

Table 3. Votes achieved by candidates a–f.

	Standing			
Candidate	1	2	3	4
a	3	3	4	3
b	4	5	5	2
c	6	2	3	2
d	6	2	2	6
e	0	4	3	4
f	1	4	3	3

The following weight restrictions used in Cook and Kress (1990) are imposed in our algorithm:

$$w_{ij} \geqslant 0, \quad w_{ij} - w_{i,\,j+1} \geqslant d(j, \delta), \quad j = 1, 2, \ldots, k-1,$$
$$d(\cdot, \delta) = \delta.$$

These additional restrictions mean that the weight for a jth place vote should be more than that for a $j+1$st place vote by some amount.

Table 4. Raw data of 37 R&D projects on five outputs and cost.

R&D project	Indirect economic contribution	Direct economic contribution	Technical contribution	Social contribution	Scientific contribution	Budget
1	67.53	70.82	62.64	44.91	46.28	84.20
2	58.94	62.86	57.47	42.84	45.64	90.00
3	22.27	9.68	6.73	10.99	5.92	50.20
4	47.32	47.05	21.75	20.82	19.64	67.50
5	48.96	48.48	34.90	32.73	26.21	75.40
6	58.88	77.16	35.42	29.11	26.08	90.00
7	50.10	58.20	36.12	32.46	18.90	87.40
8	47.46	49.54	46.89	24.54	36.35	88.80
9	55.26	61.09	38.93	47.71	29.47	95.90
10	52.40	55.09	53.45	19.52	46.57	77.50
11	55.13	55.54	55.13	23.36	46.31	76.50
12	32.09	34.04	33.57	10.60	29.36	47.50
13	27.49	39.00	34.51	21.25	25.74	58.50
14	77.17	83.35	60.01	41.37	51.91	95.00
15	72.00	68.32	25.84	36.64	25.84	83.80
16	39.74	34.54	38.01	15.79	33.06	35.40
17	38.50	28.65	51.18	59.59	48.82	32.10
18	41.23	47.18	40.01	10.18	38.86	46.70
19	53.02	51.34	42.48	17.42	46.30	78.60
20	19.91	18.98	25.49	8.66	27.04	54.10
21	50.96	53.56	55.47	30.23	54.72	74.40
22	53.36	46.47	49.72	36.53	50.44	82.10
23	61.60	66.59	64.54	39.10	51.12	75.60
24	52.56	55.11	57.58	39.69	56.49	92.30
25	31.22	29.84	33.08	13.27	36.75	68.50
26	54.64	58.05	60.03	31.16	46.71	69.30
27	50.40	53.58	53.06	26.68	48.85	57.10
28	30.76	32.45	36.63	25.45	34.79	80.00
29	48.97	54.97	51.52	23.02	45.75	72.00
30	59.68	63.78	54.80	15.94	44.04	82.90
31	48.28	55.58	53.30	7.61	36.74	44.60
32	39.78	51.69	35.10	5.30	29.57	54.50
33	24.93	29.72	28.72	8.38	23.45	52.70
34	22.32	33.12	18.94	4.03	9.58	28.00
35	48.83	53.41	40.82	10.45	33.72	36.00
36	61.45	70.22	58.26	19.53	49.33	64.10
37	57.78	72.10	43.83	16.14	31.32	66.40

When $d(j, \delta) = 0$, there is a weak ordering of weights $w_{i1} \geqslant w_{i2} \geqslant w_{i3} \geqslant w_{i4}$. Our game cross efficiencies and the order of the candidates are given as follows:

$$\text{b}(1) = \text{d}(1) > \text{c}(0.9147) > \text{a}(0.8062) > \text{f}(0.6704) > \text{e}(0.6603).$$

From this result, we can see that the order of the candidates is the same as those obtained by the benevolent cross efficiency in the paper of Green et al. (1996).

Green et al. (1996) also considered the situation when there were two extra candidates, g and h, each receiving one third-place vote. They found that there was a reversal in the positions of candidates b and d on the introduction of the two lowly-rated candidates g and h. To mitigate this effect, those authors relax the assumption that each candidate be accorded a weight of $1/m$ in the establishment of the overall ratings, and suggested that each candidate applied a weight in proportion to his/her overall rating rather than uniformly $1/m$, i.e., a form of "weighted voting."

Now, we examine whether such a reversal will occur under our game cross-efficiency structure. The following shows the results:

$$\text{b}(1.0000) > \text{d}(0.9704) > \text{c}(0.9011) > \text{a}(0.7983) > \text{f}(0.6047) > \text{e}(0.5893) > \text{g}(0.0662) = \text{h}(0.0662).$$

We find that the order of the candidates does not change after adding two lowly-rated candidates. We do not claim, however, that rank reversal would not occur in some cases under our method.

R&D Project Selection

We finally apply our approach to a data set of 37 project proposals relating to the Turkish iron and steel industry (see Oral et al. 1991). Each project is characterized by five output measures: direct economic contribution, indirect economic contribution, technological contribution, scientific contribution, and social contribution. Again, it may be argued that this is a case where there is competition among DMUs. The single input is the cost. Table 4 reports the data.

Table 5 shows the results along with Green et al. (1996) DEA cross-efficiency scores reported in column 3. Based upon the Green et al. (1996) project selection rule, which chooses projects by decreasing values of DEA cross-efficiency scores, until the budget for the program is exhausted (the budget cannot exceed 1,000), the same 17 projects are selected, with two exceptions, namely, projects 15 and 29. There are substantial ranking differences between the two approaches. For example, DMU6 is ranked 18th by the game cross efficiency, whereas it is ranked 21st by the DEA cross efficiency. In all, 13 of the 37 projects are ranked differently on the game cross-efficiency approach versus on the conventional method.

Table 5. A comparison of results to Green et al. (1996).

Project no.	Game cross-efficiency score ($\varepsilon = 0.0001$)	Green et al. score	Green et al. selection	Our selection	Budget
35	1	1	yes	yes	36
17	0.9987	0.975	yes	yes	32.1
31	0.9078	0.866	yes	yes	44.6
16	0.8162	0.78	yes	yes	35.4
36	0.7671	0.759	yes	yes	64.1
34	0.7373	0.699	yes	yes	28
18	0.7373	0.715	yes	yes	46.7
27	0.7287	0.712	yes	yes	57.1
37	0.7050	0.684	yes	yes	66.4
23	0.6696	0.655	yes	yes	75.6
26	0.6504	0.632	yes	yes	69.3
1	0.6332	0.614	yes	yes	84.2
14	0.6292	0.611	yes	yes	95
32	0.6209	0.606	yes	yes	54.5
21	0.5843	0.565	yes	yes	74.4
15	0.5829	0.537	no	yes	83.8
29	0.5710	0.559	yes	no	72
6	0.5609	0.528	no	no	
11	0.5542	0.544	no	no	
30	0.5446	0.538	no	no	
12	0.5422	0.53	yes	yes	47.5
10	0.5353	0.525	no	no	
2	0.5343	0.519	no	no	
19	0.5015	0.484	no	no	
24	0.4905	0.476	no	no	
13	0.4898	0.466	no	no	
22	0.4895	0.472	no	no	
4	0.4860	0.457	no	no	
5	0.4788	0.457	no	no	
9	0.4735	0.444	no	no	
7	0.4639	0.436	no	no	
33	0.4188	0.404	no	no	
8	0.4168	0.409	no	no	
25	0.3799	0.359	no	no	
28	0.3412	0.331	no	no	
20	0.3294	0.307	no	no	
3	0.2388	0.259	no	no	
Budget sum			982.9	994.7	

As a final example, and to point out the extent to which rank positions can differ between methods, we refer to Tables 6 and 7. Table 6 displays data on 10 DMUs and the efficiency scores under our method, and the various ver-

Table 6. Numerical Example 2.

	X_1	X_2	Y_1	Y_2	Y_3
DMU1	0.37589	0.19389	0.62731	0.71654	0.11461
DMU2	0.0098765	0.90481	0.69908	0.51131	0.66486
DMU3	0.41986	0.56921	0.39718	0.7764	0.36537
DMU4	0.75367	0.63179	0.41363	0.48935	0.14004
DMU5	0.79387	0.23441	0.65521	0.1859	0.56677
DMU6	0.91996	0.54878	0.83759	0.70064	0.82301
DMU7	0.84472	0.93158	0.37161	0.98271	0.67395
DMU8	0.36775	0.3352	0.42525	0.80664	0.99945
DMU9	0.6208	0.65553	0.59466	0.70357	0.96164
DMU10	0.73128	0.3919	0.56574	0.48496	0.058862

Table 7. Results for numerical Example 2.

	CCR efficiency	Rank	Arbitrary	Rank	Game cross efficiency	Rank	Benevolent	Rank	Aggressive	Rank
DMU1	1.0000	1	0.8253	1	1.0000	1	0.9928	1	0.7576	2
DMU2	1.0000	1	0.7574	3	0.9811	3	0.8945	3	0.7146	3
DMU3	0.7590	5	0.4518	**7**	0.6554	**5**	0.6172	5	0.4228	6
DMU4	0.3099	10	0.2299	10	0.3026	10	0.2914	10	0.2148	10
DMU5	1.0000	1	0.4553	**6**	0.7520	**4**	0.6374	4	0.3932	7
DMU6	0.7155	6	0.4572	5	0.6409	6	0.5882	6	0.4342	5
DMU7	0.5062	8	0.3093	9	0.4374	8	0.4123	9	0.2852	9
DMU8	1.0000	1	0.8081	2	0.9999	2	0.9587	2	0.7584	1
DMU9	0.6608	7	0.4598	**4**	0.6306	**7**	0.5802	7	0.4356	4
DMU10	0.4594	9	0.3270	8	0.4323	9	0.4158	8	0.2917	8

sions of the conventional cross-efficiency approaches are provided. A comparison of the game cross-efficiency results (column 7 in Table 7) with the "arbitrary" cross-efficiency model (column 5) reveals that six of the 10 DMUs occupy different rank positions in one than in the other. Granted, in the benevolent model, only two rank positions differ from those in the game analysis, but the aggressive model ranks eight of the 10 DMUs differently from what they are ranked at in the game model. This example illustrates that not only is it true that the game model can rank DMUs very differently from the ranks arising from other cross-efficiency approaches, but as well, there are significant differences in rankings within the various "conventional" cross-efficiency approaches (arbitrary, benevolent, and aggressive models).

6. Conclusions

In many, if not most situations, DMUs can be viewed as being in (at least) indirect, if not direct competition with one another. Because efficiency ratings arising from models such as DEA provide targets that DMUs must achieve, to become "efficient," each unit is thus in a form of competition with its peers. Cross efficiency, as developed by Sexton et al. (1986), and elaborated by Doyle and Green (1994) and others, offers the opportunity to arrive at efficiency ratings that provide a form of joint or coordinated strategy for the DMUs involved. These ratings yield targets that are intended to be best, in a coordinated sense. From a managerial perspective, the cross-efficiency idea has appeal in that one can view the score for a given DMU as having been the result of not just a single set of (possibly unacceptable) multipliers, but rather arising from the application of the multipliers of that DMU's peers. As has been recognized in the literature, the arbitrariness of choice of optimal multipliers in model (1), when alternate optima exist, can often result in nonunique final scores. To partially dampen this affect, various models that implement second-level goals can be applied. Even here, however, alternate optima can still exist, meaning that still uniqueness may not materialize. This paper represents a significant extension to the conventional cross-efficiency method. It attempts to rectify some shortcomings of the earlier approach by providing three important features: (1) it retains the essential peer-evaluation concept of cross evaluation, but at the same time capitalizes on the CCR approach via model (5) to find the best possible score, in a coordinated sense, for each DMU; (2) the iterative approach provided converges to a unique point (score); and (3) this unique point is shown to be a Nash equilibrium point. This latter important link to game theory lends credibility to this new (game cross-efficiency) concept.

We point out that while we have concentrated herein on the CCR model, further research is to extend the ideas to other DEA models, such as the VRS (variable returns to scale) DEA model. As pointed out by one reviewer, cross evaluation is problematic in the VRS model. This is due to the fact that the "free in sign" variable in the VRS model can lead to negative cross efficiencies. Tests of our approach in several numerical examples reveal that negative game cross-efficiency scores do not occur even when scores for traditional cross efficiency are negative. We point out, however, that there is as yet no proof that this outcome will always occur. An investigation of a VRS version of the DEA game cross-efficiency model will be the subject of later research.

Appendix

$\{\sum_{r=1}^{s} \mu(\alpha_d)_{rj}^d y_{rj},\ d = 1, 2, \ldots, n\}$ are continuous functions of α_d if $\alpha_d \in [\bar{E}_d, \alpha_d^{\mathrm{CCR}}]$.

Proof. Let $X_l = [x_{il}, i = 1, 2, \ldots, m]^T$, $Y_l = [y_{rl}, r = 1, 2, \ldots, s]^T$, $l = 1, 2, \ldots, n$. For the convenience of proof, we rewrite the CCR model and model (5) in matrix format as

$$\begin{aligned} \text{Max}\quad & Y_j^T \mu_j^d \\ \text{s.t.}\quad & -X_l^T \omega_j^d + Y_l^T \mu_j^d \leqslant 0, \\ & X_l^T \omega_j^d \leqslant 1, \\ & -X_l^T \omega_j^d \leqslant -1, \\ & \omega_j^d = [\omega_{ij}^d, i = 1, 2, \ldots, m]^T \geqslant 0, \\ & \mu_j^d = [\mu_{rj}^d, r = 1, 2, \ldots, s]^T \geqslant 0, \end{aligned} \qquad (9)$$

and

$$\begin{aligned}\text{Max}\quad & Y_j^T \mu_j^d\\ \text{s.t.}\quad & -X_l^T \omega_j^d + Y_l^T \mu_j^d \leqslant 0,\\ & X_l^T \omega_j^d \leqslant 1,\\ & -X_l^T \omega_j^d \leqslant -1,\\ & \alpha_d \times X_d^T \omega_j^d - Y_d^T \mu_j^d \leqslant 0,\\ & \omega_j^d = [\omega_{ij}^d, i = 1, 2, \ldots, m]^T \geqslant 0,\\ & \mu_j^d = [\mu_{rj}^d, r = 1, 2, \ldots, s]^T \geqslant 0.\end{aligned} \tag{10}$$

Further, we let $C = (0, Y_j^T)$, $X = [\omega_j^d, \mu_j^d]^T$,

$$A = \begin{bmatrix} -X_1^T, & Y_1^T \\ \cdots & \\ -X_n^T, & Y_n^T \\ X_j^T, & 0 \\ -X_j^T, & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0. \\ \cdots \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \text{and}$$

$A' = (\alpha_d \times X_d^T, -Y_d^T)$. Then, (9) and (10) become

$$\begin{cases}\text{Max} & CX\\ \text{s.t.} & AX \leqslant b,\\ & X \geqslant 0,\end{cases} \tag{11}$$

$$\begin{cases}\text{Max} & CX\\ \text{s.t.} & AX \leqslant b,\\ & A'X \leqslant 0,\\ & X \geqslant 0.\end{cases} \tag{12}$$

Note that if we add $A'X \leqslant 0$ into (11), then (11) becomes (12).

Let B be the basis for (11) and X_B be its corresponding basic feasible solution. Then, the initial basis for (12) can be written as

$$B' = \begin{pmatrix} B, & 0_q \\ A'_B, & 1 \end{pmatrix},$$

where A'_B is obtained after adjusting the components' positions in A' so that it corresponds to B for (11). $q = n + 2$ represents the number of constraints except for nonnegativity conditions ($X \geqslant 0$), and 0_q is a q-dimensional vector of zeros.

Note that

$$(B')^{-1} = \begin{pmatrix} B^{-1}, & 0_q \\ -A'_B B^{-1}, & 1 \end{pmatrix}.$$

Let $C'_B = (C_B, 0)$, where C_B represents the coefficients in the objective function relative to basis B. We then have

$$z' = (C_B B^{-1} \;\; 0) \cdot (b \;\; 0)^T$$

as the value to the objective function of model (12).

Denote $\sigma'_j = C'_B (B')^{-1} P'_j$, $j \in I_N$, where I_N is the set of indices of nonbasic variables and P'_j is a coefficient vector for nonbasic variables in

$[A \;\; A']^T$.

The simplex tableau (B' as the basis) is given by

$C_B B^{-1} A - C$	$C_B B^{-1}$	0	$C_B B^{-1} b$
$B^{-1} A$	B^{-1}	0_q	$B^{-1} b$
$-A'_B B^{-1} A + A'$	$-A'_B B^{-1}$	1	$-\tau$

where $\tau = A'_B X_B$ are slack variables in $A'X \leqslant 0$.

We next show that $\tau > 0$. Suppose that $\tau \leqslant 0$ (i.e., $-\tau \geqslant 0$). Then, optimal solutions to (12) can be obtained from the optimal solutions to (11), and the slack variables in $A'X \leqslant 0$. Note that (11) is the CCR model, and the objective function in (12) has nothing to do with $-\tau$. Therefore, the optimal value to (12) is the CCR efficiency score. This indicates that $A'X \leqslant 0$ (or $\alpha_d \times X_d^T \omega_j^d - Y_d^T \mu_j^d \leqslant 0$) are redundant—a contradiction. Thus, $\tau > 0$. This indicates that $[B^{-1} b, -\tau]^T$ is not a basic feasible solution to (12).

Now, based upon the dual simplex method, the simplex tableau of (12) is obtained by adding a new row d_{q+1} into the simplex tableau of (11). Suppose that B' now becomes $\bar{B}$. For (12), choose one of d_{q+1}'s component $d_{q+1,k} < 0$, and get the related new basic variable as $\bar{x}_{q+1,0} = -(\tau / d_{q+1,k}) > 0$. Other basic variables are

$$\bar{x}_{p0} = x_{p0} - \bar{x}_{q+1,0} d_{p,k} = x_{p0} + \frac{\tau \times d_{p,k}}{d_{q+1,k}}, \quad p = 1, \ldots, q,$$

where x_{p0}, $p = 1, 2, \ldots, q$ are basic solutions related to B.

If $\bar{x}_{p0} \geqslant 0$, $p = 1, 2, \ldots, q+1$, this means that $\bar{B}$ is the optimal basis for (12). If some $\bar{x}_{p'0} < 0$, $1 \leqslant p' \leqslant q$, then we need more iterations. We thus discuss two cases.

Case 1. $\bar{x}_{p0} \geqslant 0$, $p = 1, 2, \ldots, q+1$. Then, $\bar{x}_{p0} \geqslant 0$, $p = 1, 2, \ldots, q+1$ are basic solutions and the optimal value to (12) is

$$\bar{z} = z' - \bar{x}_{q+1,0} \sigma'_k = z' + \tau \times \frac{\sigma'_k}{d_{q+1,k}} = z' + A' X_B \times \frac{\sigma'_k}{d_{q+1,k}},$$

where σ'_k is the z_j related to $d_{q+1,k} < 0$ in the simplex tableau.

Because the objective function CX does not contain α_d, σ'_k does not contain α_d. However, $X_j^T \omega_j^d = 1$ is a constraint. Therefore, the optimal basis must contain components from ω_j^d. (Otherwise, $X_j^T \omega_j^d = 1$ cannot be satisfied.) Thus, $A'_B X_B$ must have α_d, and is a linear function of α_d.

Note that $d_{q+1,k}$ is a linear function of α_d. $\bar{z}$ is a ratio of two functions that are linear with respect to α_d. $\bar{z}$ is continuous when $d_{q+1,k} \neq 0$. Note also that $d_{q+1,k} < 0$. Therefore, when $\alpha_d \in [\bar{E}_d, \theta_d^*]$, $\bar{z}$ is a continuous function of α_d.

Case 2. $\bar{x}_{p'0} < 0$, $1 \leqslant p' \leqslant q$. Based upon the dual simplex method, in row $h_{p'}$ of the simplex tableau, with $\bar{B}$ as the basis, we take $h_{p'k'} < 0$, and we have $\bar{x}'_{p'0} = \bar{x}_{p'0} / h_{p'k'} > 0$. Then, the new objective function value $\bar{z}' = \bar{z} - \bar{\sigma}_{k'} \bar{x}'_{p'0}$, where $\bar{\sigma}_{k'}$ is the z_j related to $h_{p'k'} < 0$ in the simplex tableau.

Similar to the discussion in Case 1, $\bar{\sigma}_{k'}$ does not contain α_d, and $\bar{z}$ is a continuous function of α_d. Whether $\bar{z}'$ is a continuous function of α_d depends on whether $\bar{x}'_{p'0}$ is continuous with respect to α_d.

Note that

$$\bar{x}'_{p'0} = \bar{x}_{p'0}/h_{h'k'} = \left(x_{p'0} + \frac{\tau \times d_{p',k}}{d_{q+1,k}}\right) \Big/ h_{p'k'}$$
$$= \left(x_{p'0} + \frac{A'_B X_B \times d_{p',k}}{d_{q+1,k}}\right) \Big/ h_{p'k'}.$$

Because $d_{q+1,k} < 0$ and $h_{p'k'} < 0$, $\bar{z}'$ is a continuous function of α_d if $\alpha_d \in [\bar{E}_d, \theta_d^*]$, and note that Case 1 indicates that $A'_B X_B$ is a linear function of α_d, when $\alpha_d \in [\bar{E}_d, \theta_d^*]$, and $\bar{x}'_{p'0}$ is a continuous function of α_d. Therefore, $\bar{z}'$ is a continuous function of α_d.

If there still exist negative basic variables, we can repeat the above procedure until all $\bar{x}_{p0} \geqslant 0$, $p = 1, 2, \ldots, q+1$. Case 1 has already indicated that the objective function of model (12) is a continuous function of α_d.

Therefore, based upon Cases 1 and 2, we have that if $\alpha_d \in [\bar{E}_d, \theta_d^*]$, then $\sum_{r=1}^{s} \mu_{rj}^d(\alpha_d) y_{rj}$, $d = 1, 2, \ldots, n$ are continuous functions of α_d.

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